# Resistor Networks in a Punctured Disk

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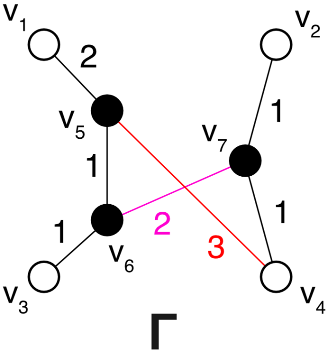
# Overview

1. Background Definitions and Results
   - Resistor Networks and Inverse Problem
   - Known Results: Circular Planar Resistor Networks
2. Resistor Networks on a Punctured Disk
3. Conjectures

# Resistor Networks

**Definition**

A *resistor network* is a finite graph $(V, E)$ with a specified set $B \subseteq V$ of *boundary vertices* and a real non-negative conductance $c_e$, for each $e \in E$. The remaining vertices, $I = V - B$, are called *internal vertices*.



# Kirchoff Matrix

## Definition

The *Kirchoff Matrix* $K(\Gamma)$ of a resistor network $\Gamma$ is the unique matrix with $K(\Gamma)_{ij}$ equal to the sum of conductances of edges between $i$ and $j$ and row sums equal to 0.

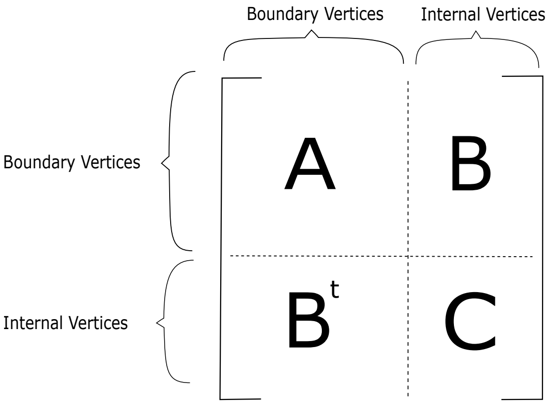
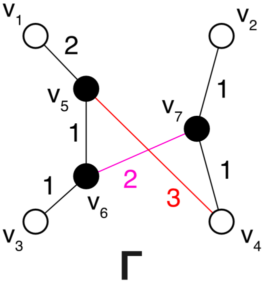



Figure: We divide the Kirchoff Matrix into 4 submatrices

# Example




$$K(\Gamma) = \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 & 3 & 0 & 1 \\ 2 & 0 & 0 & 3 & -6 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 2 \\ 0 & 1 & 0 & 1 & 0 & 2 & -4 \end{bmatrix}$$

# Response Matrix

**Definition**

A potential function assignment to the boundary vertices of $\Gamma$ induces a net current at boundary vertices. This may be represented by the *response matrix* of $\Gamma$, $\Lambda(\Gamma)$.

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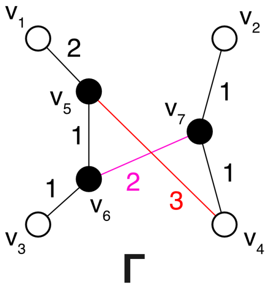
A potential function assignment to the boundary vertices of $\Gamma$ induces a net current at boundary vertices. This may be represented by the *response matrix* of $\Gamma$, $\Lambda(\Gamma)$.

The Response Matrix can be calculated in terms of the Kirchoff matrix:

$$\Lambda = A - BC^{-1}B^t$$

# Example




$$K(\Gamma) = \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 & 3 & 0 & 1 \\ 2 & 0 & 0 & 3 & -6 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 2 \\ 0 & 1 & 0 & 1 & 0 & 2 & -4 \end{bmatrix}$$

$$\Lambda(\Gamma) = \begin{bmatrix} -\frac{22}{17} & \frac{1}{17} & \frac{2}{17} & \frac{19}{17} \\ \frac{1}{17} & -\frac{7}{17} - \frac{1}{4} & \frac{3}{17} & \frac{3}{17} + \frac{1}{4} \\ \frac{2}{17} & \frac{3}{17} & -\frac{11}{17} & \frac{6}{17} \\ \frac{19}{17} & \frac{3}{17} + \frac{1}{4} & \frac{6}{17} & -\frac{28}{17} - \frac{1}{4} \end{bmatrix}$$

# Inverse Problem

**Inverse Problem**

Given a resistor network $\Gamma$ without labeled conductances and $\Lambda(\Gamma)$, when are we able to uniquely recover its conductances?

# Circular Planar Resistor Networks

Curtis, Ingerman, and Morrow solved the inverse problem for a special class of graphs known as circular planar resistor networks (cprns)

## Definition

A *circular planar resistor network* is a resistor network that can be embedded in a disk so that it is planar with all boundary vertices are on the boundary of the disk.

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## Example

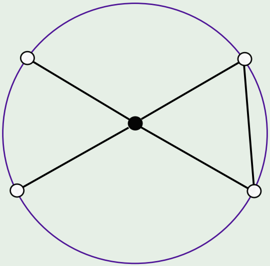
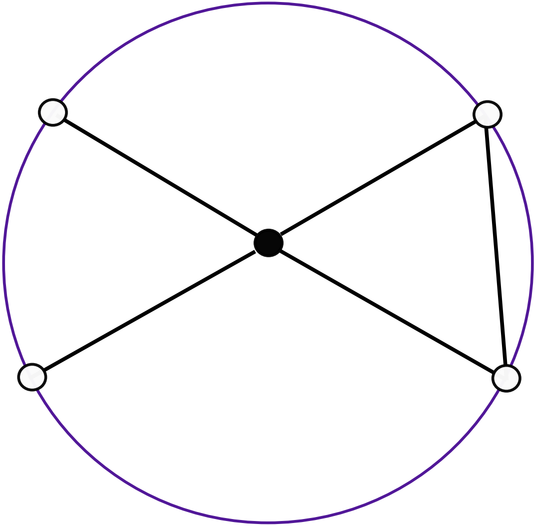


Figure: A cprn

# Constructing the Medial Graph



# Constructing the Medial Graph

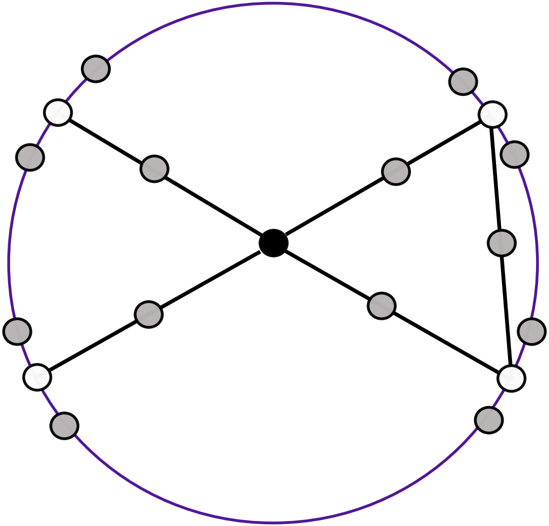


Figure: Add in new vertices

# Constructing the Medial Graph

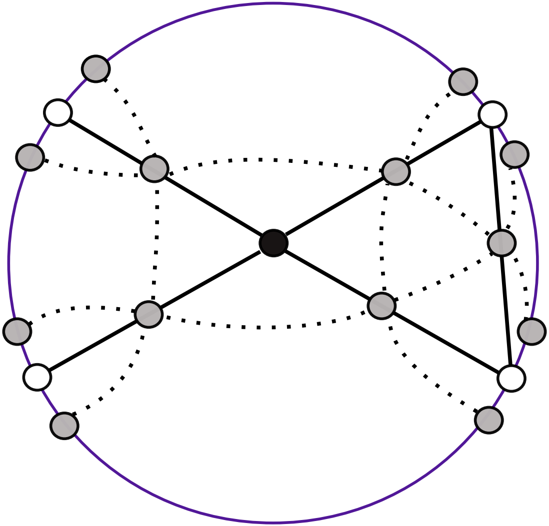


Figure: Connect Edges

# Constructing the Medial Graph

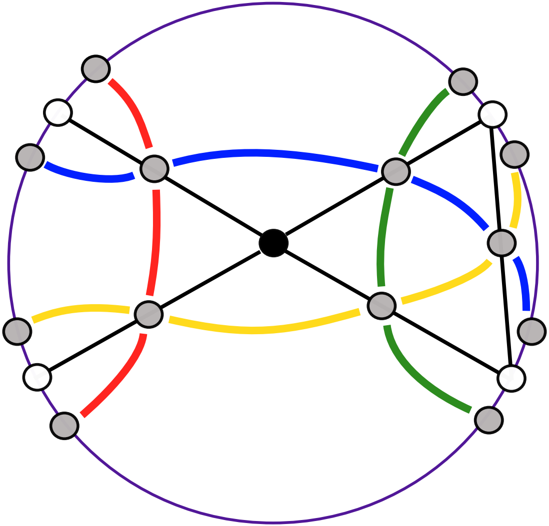


Figure: 4 Strands of the Medial Graph

# Z-Sequence

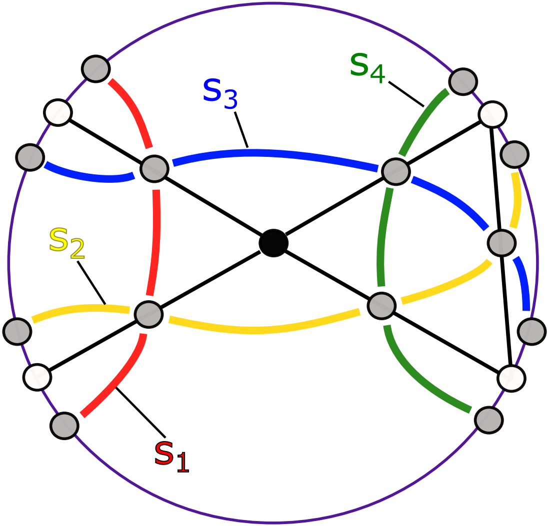



Figure: The $z$-sequence of this network is 1 2 3 1 4 2 3 4

**Definition**

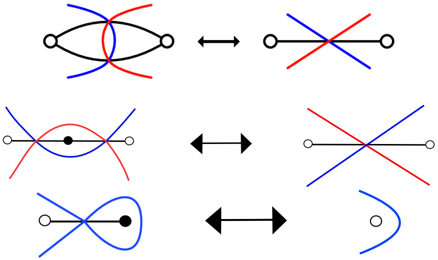
Call two resistor networks $\Gamma$ and $\Gamma'$ *electrically equivalent* if the following holds:

- For every assignment of conductances to $\Gamma$, there exists an assignment of conductances to $\Gamma'$ such that $\Lambda(\Gamma) = \Lambda(\Gamma')$.
- For every assignment of conductances to $\Gamma'$, there exists an assignment of conductances to $\Gamma$ such that $\Lambda(\Gamma) = \Lambda(\Gamma')$.

# Local Transformations

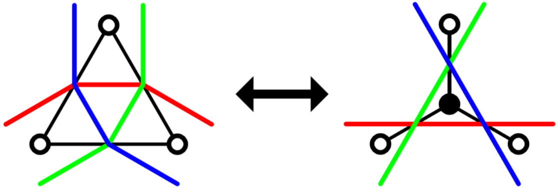
The following transformations can be done without affecting the response matrix:

1. Parallel Reduction
2. Series Reduction
3. Pendant Removal



# Local Transformations (continued)

- $Y - \Delta$ moves



# Critical cprns

**Defnition**

Call a cprn *critical* if it is not electrically equivalent to any graph with fewer edges.

## Theorem (Curtis, Ingerman, Morrow)

*A cprn is critical if and only if it satisfies the following medial graph conditions:*

- *No medial strands form closed loops.*
- *No medial strands self-intersect.*
- *No two medial strands intersect more than once.*

*Furthermore, For two critical circular planar resistance networks $\Gamma_1$ and $\Gamma_2$, the following conditions are equivalent:*

- $\Gamma_1$ *and* $\Gamma_2$ *are electrically equivalent.*
- $\Gamma_1$ *and* $\Gamma_2$ *are related by* $Y - \Delta$ *moves.*
- $\Gamma_1$ *and* $\Gamma_2$ *share a z-sequence.*

# Answer to Inverse Problem: CPRN Case

**Theorem**

*We can uniquely recover the conductances of a cprn if and only if it is critical. Additionally, every cprn can be transformed to a critical cprn through a sequence of the defined local moves.*

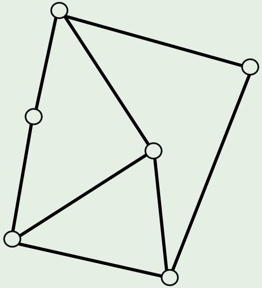
# Resistor Networks on a Punctured Disk

We worked towards expanding on Curtis, Ingerman, and Morrow's results by examining a new class of resistor networks.

## Definition

A *Resistor Network on a Punctured Disk* (rnpd) is a resistor network that can be embedded in a disk so that it is planar, and all boundary vertices but one (the *interior boundary vertex*) are on the boundary of the disk.

## Example



# The Medial Graph and $Z$-sequences for RNPDs

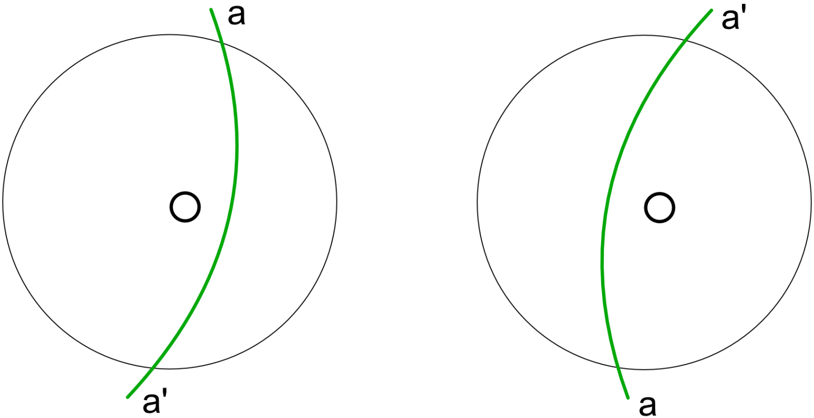
## Definition

The *medial graph for an rnpd* is the medial graph of the cprn that results from treating the interior boundary vertex as internal.

## Definition

The *z-sequence for an rnpd* is defined similarly as for cprns, with a slight modification. In the medial graph, we label one endpoint of each strand $s$ with an $s'$, such from the perspective of the interior boundary vertex the strand travels clockwise from $s$ to $s'$. Additionally, if a strand $s$ contains a self-intersection, underline $s'$.

# $Z$-sequence Illustration



# Example

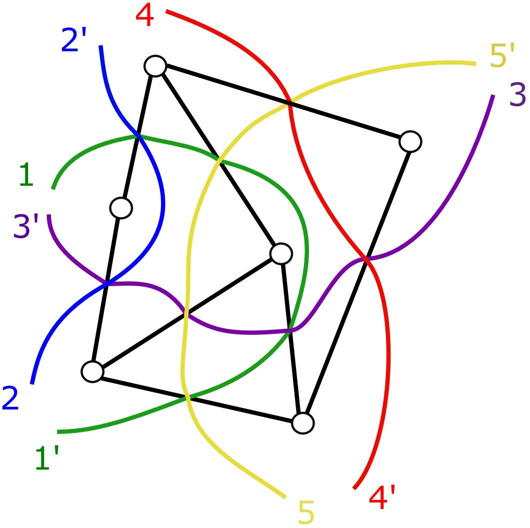



Figure: Z-Sequence: $1' \quad 2 \quad 3' \quad 1 \quad 2' \quad 4 \quad 5' \quad 3 \quad 3 \quad 4' \quad 5$

# Irreducible RNPD results

## Definition

We call an rnpd *irreducible* if it is not electrically equivalent to an rnpd with fewer edges.

## Theorem

In any irreducible rnpd,

- No medial strand is a closed circle.
- Every medial lens and medial loop contains the interior boundary vertex.
- Every strand intersects itself at most once.
- At most one strand contains a self-intersection.
- Every pair of strands intersects at most twice.

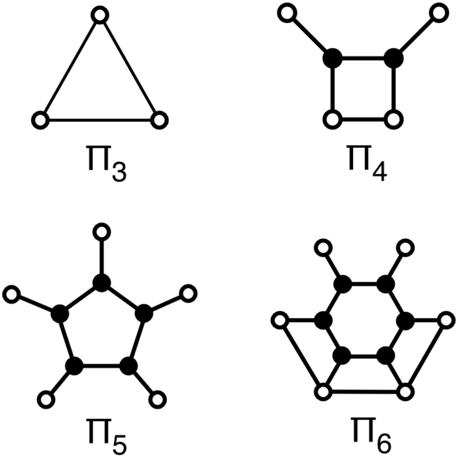
# Irreducible RNPD results

**Theorem**

Two irreducible rnpds share a $z$-sequence if and only if they are related by $Y - \Delta$ moves

# 4-Periodic Graphs

We define an infinite family of cprns called 4-*periodic graphs*



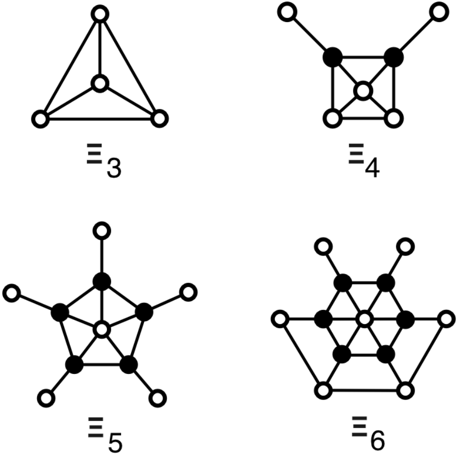
# 4-Periodic Graphs

Properties of 4-periodic graphs:

- Critical cprns with $z$-sequence $1 \quad 2 \quad \cdots \quad n \quad 1 \quad 2 \quad \cdots \quad n$ (Electrically equivalent to special network in cprn case: $\Sigma_n$)
- Maximal critical cprns

# Spider Graphs

We construct a new family of graphs known as *spider graphs* from 4-periodic graphs



# Spider Graphs

**Theorem**

*Spider Graphs are recoverable*

# Spider Graphs

## Theorem

*Spider Graphs are recoverable*

## Proof Idea

Terms:

- *Boundary Edge*: An edge connecting two boundary vertices
- *Boundary Spike*: An edge connecting an internal vertex to a boundary vertex of degree 1
- We say $P, Q \subseteq B$ form a *connection* $(P, Q)$ if $|P| = |Q| = k$ and there exist $k$ disjoint paths through internal vertices connecting each $p \in P$ to a $q \in Q$

# Spider Graphs

## Theorem

*Spider Graphs are recoverable*

## Proof Idea

*Known for cprns*: If deleting or contracting a boundary edge or spike breaks some connection, we can recover the conductance of that edge or spike from the response matrix.

We generalized this result for rnpds by restricting $P$ and $Q$ to not contain the interior boundary vertex.

## Theorem

*Spider Graphs are recoverable*

## Proof Idea

Deleting any boundary edge or boundary spike in our spider graph results in a broken connection (because 4-periodic graphs are critical).

We can delete and contract boundary edges and spikes one by one, until we are left with a *star graph*, which is trivially recoverable.

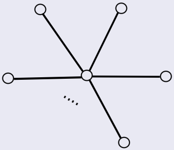


Figure: Star Graph

# Sufficient Condition for Recoverability

We can use the same process to generalize our result for Spider Graphs to a much larger family of rnpds:

**Theorem**

*Let $\Gamma$ be any critical cprn. Let $\Gamma'$ be the result of inserting a star graph into one of the faces of $\Gamma$. Then, $\Gamma'$ is a recoverable rnpd.*

# Necessary Condition for Recoverability of RNPDs

- Algorithm:
  - For rnpd $\Gamma$: Remove interior boundary vertex and change all its neighbors to boundary vertices. Repeat process for each newly created interior boundary vertex until you get a cprn. If the original rnpd was recoverable, then the resulting cprn is.

# Additional Local Moves for RNPDs

The following moves can be done in a way that does not affect the response matrix:

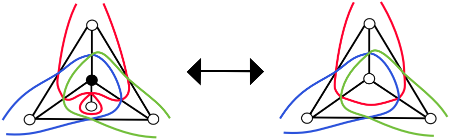


Figure: Antenna Absorption

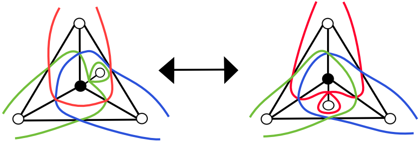


Figure: Antenna Jumping

# Additional Local Moves for RNPDs

The following are local move equivalences that alter $z$-sequences.

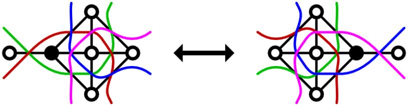


Figure: Square Jump

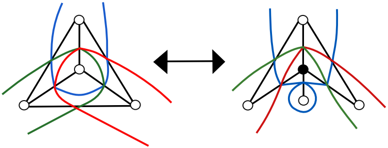


Figure: Generalized Antenna Absorption

# Conjectures

- An rnpd is recoverable if and only if it is irreducible (in which case we'd have a natural definition of *critical*).
- The moves described in the talk are sufficient to describe all electrical equivalences of rnpds.

# References

E.B. Curtis, D. Ingerman, J.A. Morrow (1998)
Circular Planar Graphs and Resistor Networks
*Linear Algebra and its Applications* 283, pgs 115 - 150

# Questions?