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Some Combinatorial and Algebraic Properties of Coxeter Complexes and Tits Buildings

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INTRODUCTION

With Coxeter groups and groups with a BN-pair are associated certain combinatorial structures known as *Coxeter complexes* and *Tits buildings*. Their theory, which is intimately related to the properties of such groups, has been developed mainly by Tits [39, 40]. In recent years a general theory of *Cohen–Macaulay complexes* has taken shape, which relates combinatorial structures with ring theory. The main architects of this development are Hochster [18] and Stanley [34–37]. The notion of a *shellable complex* provides a constructive approach to Cohen–Macaulay complexes in the finite case and has several interesting features of its own. In this paper Coxeter complexes and Tits buildings will be studied from the point of view of shellable and Cohen–Macaulay complexes.

It follows from the Solomon–Tits homology computation [31] via the work of Reisner [24] that a finite building is Cohen–Macaulay. However, neither the Stanley–Reisner ring of a building nor its shellability seem to have been explicitly considered before. It turns out that Coxeter complexes and buildings are very naturally shellable and that shellability provides quick and elementary access to their basic ring-theoretic and topological properties. On the ring-theoretic side this means mainly that with each finite group of Lie (resp. Coxeter) type is canonically associated a certain Cohen–Macaulay (resp. Gorenstein) ring $\mathbf{k}[x_1, x_2, \ldots, x_n]/I$ on which the group acts. This ring is provided with a canonical characteristic-free basis over a certain natural system of parameters. Here $\mathbf{k}$ is a field, the indeterminates x_i correspond bijectively to the left cosets of a system of maximal parabolic subgroups, and the ideal I is generated by all monomials $x_i x_j$ corresponding to pairs of disjoint such cosets. The topological properties of Coxeter complexes and buildings are better known. The principal advantage of the present combinatorial approach is that it leads to a simple and unified treatment which simultaneously includes all type-selected subcomplexes as well.

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The contents of the paper are as follows:

In Section 1 we give a condensed presentation of the basic properties of shellable balanced complexes. This material is included mainly to make the paper reasonably self-contained, but also because of some new results and other improvements. It complements Section 5 of [7] in the sense that simple proofs are here provided for all the topological and ring-theoretic properties of Bruhat order referred to there.

In Section 2 it is shown that the weak ordering of a Coxeter group gives rise to a class of shellings of the associated Coxeter complex. As a byproduct one obtains an intrinsic proof that the main classes of Coxeter complexes are piecewise linearly homeomorphic to spheres or Euclidean spaces, without the usual reliance on a geometric representation as the convex cone chamber complex of a Euclidean reflection group. A similar class of shellings is constructed for buildings in Section 4. This construction relies on certain partial orderings of the coset space G/B, G being a group with BN-pair, which when G is a finite Chevalley group over $GF(q)$ can be viewed as "q-analogues" of ordinary weak and Bruhat order. The necessary properties of these orderings are derived in Section 3.

The shellings of a building Δ constructed in Section 4 have enough flexibility to allow certain deletion arguments which are important for the study of the Stanley–Reisner ring $\mathbf{k}[\Delta]$. Specifically, let Δ be the building of a Chevalley group $G=(G; B, N)$ over $GF(q)$. We show that any q vertices (and all proper faces incident with them) can be deleted from Δ without losing shellability. Using results of Baclawski [1, 2] and Hochster [18] one can from this deduce

(i) that the Cohen–Macaulay ring $\mathbf{k}[\Delta]$ is of type $|B: B\cap w_0Bw_0^{-1}| = q^{l(w_0)}$ (more generally, it is in principle possible to express the last q Betti numbers of $\mathbf{k}[\Delta]$ in terms of indices $[B: B\cap wBw^{-1}]$, $w\in W=N/(B\cap N)$), and

(ii) that the canonical module of $\mathbf{k}[\Delta]$ is isomorphic with the ideal generated by the elementary cycles of all apartments.

When comparing buildings with some other well-known classes of Cohen–Macaulay complexes one notices an interesting feature which seems common in geometric structures of this kind. This is the existence of a sufficiently rich family of "degenerate" substructures, be it the apartments of a building, the Boolean sublattices of a geometric lattice, or the bases of a matroid. These substructures provide cycles, in terms of which homology can be described. In Section 5 type-selected subcomplexes Δ_J, $J\subseteq S$, of Coxeter complexes and Tits buildings are studied from this point of view. They possess a natural class of "apartments" and are therefore also "geometric." Some combinatorial properties of such apartments are described leading to the construction of bases for the homology of Δ_J.

A finite Coxeter group or group with a BN-pair acts on each type-selected subcomplex Δ_J of its associated complex. This simplicial action on Δ_J, $J \subseteq S$, induces an action on homology $\tilde{H}_{|J|-1}(\Delta_J, \mathbb{C})$, i.e., a complex representation β_J. By a familiar argument based on the Hopf trace formula the characters β_J can be related to the characters induced from principal characters of parabolic subgroups. In Section 6 we first review this general construction and then explore some special properties of the homology characters β_J. In particular, we are interested in whether these can be realized by matrices having all entries equal to 0, +1, or −1. The irreducible β_J-characters of symmetric groups were observed to have this property already by Schur in 1908 [27]. We show that the $\{0, +1, -1\}$-property holds also in some other cases, e.g., for the Steinberg character. Finally, a connection between the homology characters β_J of a finite Coxeter group and the Kazhdan–Lusztig characters [20] is mentioned.

In the Appendix we show, following a suggestion by Tits, how the shellability arguments of Sections 2 and 4 can be adapted to so-called "weak buildings."

1. Shellable and Balanced Complexes

The purpose for this section is to establish those parts of the theory of shellable balanced complexes which are relevant for the rest of the paper.

The set-theoretic notation used is standard. However, it should be mentioned that $A = \dot{\bigcup}_{i \in I} B_i$ stands for *disjoint union*, i.e., $A = \bigcup_{i \in I} B_i$ and $B_i \cap B_j = \emptyset$ when $i \neq j$, $i, j \in I$. Also, the cardinality of a set A is denoted by either card A or $|A|$. For a given set S, fixed and determined by context, complements of subsets $J \subseteq S$ and elements $s \in S$ will be denoted $\hat{J} = S - J$ and $(s) = S - \{s\}$, respectively.

A *linear extension* of a partially ordered set $(P, \leqslant)$ is a well-ordering $(P, \preccurlyeq)$ such that $x < y$ implies $x \prec y$ for all $x, y \in P$. If P is countably infinite we will usually require of a linear extension also that it is a isomorphic as an ordered set to the natural numbers. This extra requirement is not essential, but is adopted for the sake of simplicity.

By a *complex* (or, abstract simplicial complex) Δ on vertex set V is meant a collection Δ of finite subsets of V, called *faces*, such that $x \in V$ implies $\{x\} \in \Delta$ and $F \subseteq F' \in \Delta$ implies $F \in \Delta$. We allow also the empty complex $\Delta = \emptyset$. If $\Delta \neq \emptyset$, then $\emptyset \in \Delta$. If F, $F' \in \Delta$, let $[F, F'] = \{F'' \in \Delta \mid F \subseteq F'' \subseteq F'\}$. As usual, *dimension* is defined by $\dim F = \operatorname{card} F - 1$ and $\dim \Delta = \sup_{F \in \Delta} \dim F$. Assume from now on that Δ is *pure d-dimensional*, i.e., every face is contained in some d-dimensional face. In this case the d-dimensional faces are called *chambers* and the $(d-1)$-dimensional faces *walls*. The collection of all chambers is denoted by $\mathscr{C}\!\mathscr{h}(\Delta)$. Two chambers C

and C' are *adjacent* if $\dim(C \cap C') = d-1$. Δ is said to be a *pseudomanifold* if (i) every wall borders exactly two chambers and (ii) any two chambers are linked by a finite sequence of successively adjacent chambers, called a *gallery*. If $F \in \Delta$, let $\bar{F}$ be the *simplex* $\{E \mid E \subseteq F\}$. Finally, $\|\Delta\|$ will denote the (topological) *space* of Δ (cf. Spanier [32, p. 111]).

(A) *Shellable Complexes*

Throughout this section let Δ be a pure d-dimensional complex of at most countable cardinality. We will be considering linear orderings $C_1, C_2, C_3, \ldots$ of $\mathscr{C}\!\mathscr{h}(\Delta)$. Given such an ordering let $\Delta_k = \bar{C}_1 \cup \bar{C}_2 \cup \cdots \cup \bar{C}_k$ for $k \geqslant 1$, and let $\Delta_0 = \varnothing$.

1.1. Definition. Δ is said to be *shellable* if its chambers can be arranged in linear order $C_1, C_2, C_3, \ldots$ in such a way that $\Delta_{k-1} \cap \bar{C}_k$ is pure $(d-1)$-dimensional for $k = 2, 3, \ldots$. Such an ordering of $\mathscr{C}\!\mathscr{h}(\Delta)$ is called a *shelling*.

In this definition and in the sequel it is tacitly understood that if Δ is finite, having say t chambers, then statements like $k = 2, 3, \ldots$ or $k \geqslant 2$ mean $k = 2, 3, \ldots, t$. The restriction to countable cardinality in this definition of shellability can be relaxed in different ways, cf. Remark 4.21.

Given a shelling, define the *restriction* of chamber C_k by

$$\mathscr{R}(C_k) = \{x \in C_k \mid C_k - \{x\} \in \Delta_{k-1}\}.$$

Shellings and their restriction maps have several useful characterizations. Condition (iv) below was observed by Wachs.

1.2. Proposition. *Given an ordering $C_1, C_2, \ldots$ of $\mathscr{C}\!\mathscr{h}(\Delta)$ and a map $\mathscr{R} : \mathscr{C}\!\mathscr{h}(\Delta) \to \Delta$, the following are equivalent:*

(i) *$C_1, C_2, \ldots$ is a shelling and $\mathscr{R}$ its restriction map,*
(ii) *if $F \subseteq C_k$, $k \geqslant 1$, then $F \in \Delta_{k-1} \Leftrightarrow F \not\supseteq \mathscr{R}(C_k)$,*
(iii) *$\Delta_k = \bigcup_{i=1}^{k} [\mathscr{R}(C_i), C_i]$, for all $k \geqslant 1$,*
(iv) (α) *$\Delta = \bigcup_{i \geqslant 1} [\mathscr{R}(C_i), C_i]$, and*
(β) *$\mathscr{R}(C_i) \subseteq C_j \Rightarrow i \leqslant j$, for all $i, j \geqslant 1$.*

Proof. (i) $\Rightarrow$ (ii): $F \in \Delta_{k-1} \Leftrightarrow F \subseteq C_k - \{x\} \in \Delta_{k-1} \cap \bar{C}_k$, for some $x \in C_k \Leftrightarrow F \not\ni x$, for some $x \in \mathscr{R}(C_k)$.

(ii) $\Leftrightarrow$ (iii): Both conditions imply $\mathscr{R}(C_k) \subseteq C_k$ and are equivalent with $\Delta_k - \Delta_{k-1} = [\mathscr{R}(C_k), C_k]$, for all $k \geqslant 1$.

(iii) $\Rightarrow$ (iv): $\Delta = \dot{\bigcup}_{i \geqslant 1} (\Delta_i - \Delta_{i-1}) = \dot{\bigcup}_{i \geqslant 1} [\mathscr{R}(C_i), C_i]$, and $\mathscr{R}(C_i) \in \Delta_j \Leftrightarrow i \leqslant j$.

(iv) $\Rightarrow$ (i): First, $C_k \in \Delta \Rightarrow \mathscr{R}(C_k) \subseteq C_k$, by ($\alpha$). Suppose $F \in \Delta_{k-1} \cap \bar{C}_k$. Then $F \not\supseteq \mathscr{R}(C_k)$, by ($\beta$). Thus, $F \subseteq C_k - \{x\}$ for some $x \in \mathscr{R}(C_k)$. By (α), $\mathscr{R}(C_i) \subseteq C_k - \{x\} \subseteq C_i$, and then by ($\beta$) $i < k$. Hence, $C_k - \{x\} \in \Delta_{k-1} \cap \bar{C}_k$. Finally, $x \in \mathscr{R}(C_k) \Leftrightarrow C_k - \{x\} \in \Delta_{k-1}$. ∎

Let us say that Δ is *shellable of characteristic* h, if for some shelling, $h = \operatorname{card}\{C \in \mathscr{C}\!h(\Delta) \mid \mathscr{R}(C) = C\}$. It will soon appear that h depends only on Δ and not on the particular shelling.

1.3. THEOREM. *Let Δ be a shellable d-dimensional complex of characteristic h. Then $\|\Delta\|$ has the homotopy type of a wedge of h d-spheres. In particular, Δ is $(d-1)$-connected.*

Proof. For some fixed shelling of Δ, let $\Delta^* = \Delta - \{C \in \mathscr{C}\!h(\Delta) \mid \mathscr{R}(C) = C\}$. Then Δ^* is shellable, it inherits shelling order and a restriction map from Δ. If C_k is the kth chamber of Δ^*, then since $\mathscr{R}(C_k) \neq C_k$, C_k has at least one free wall in Δ_k^* (i.e., a wall $\in \Delta_k^* - \Delta_{k-1}^*$). Thus, Δ_k^* can be collapsed back onto Δ_{k-1}^*, and $\|\Delta_{k-1}^*\|$ is a strong deformation retract of $\|\Delta_k^*\|$. Since $\Delta^* = \bigcup_{k \geqslant 1} \Delta_k^*$ it follows that $\|\Delta^*\|$ is contractible.

Now use the fact that smashing a contractible subcomplex does not alter homotopy type: $\|\Delta\| \simeq \|\Delta\|/\|\Delta^*\|$ (cf. [32, Corollary 5, p. 118]). The space $\|\Delta\|$ is obtained from $\|\Delta^*\|$ by attaching the remaining h d-cells $\|\bar{C}\|$ along their entire boundary. Thus, when $\|\Delta^*\|$ is smashed, $\|\Delta\|$ is deformed into a wedge of h d-spheres. ∎

1.4. COROLLARY.

$$\tilde{H}_i(\Delta, \mathbb{Z}) = \mathbb{Z}^h, \qquad i = d,$$
$$= 0, \qquad i \neq d.$$

Here and in the sequel $\tilde{H}_i(\Delta, G)$ denotes reduced simplicial homology with coefficients in the group G. Note that if Δ is finite $h = (-1)^d \tilde{\chi}(\Delta)$, so that the characteristic considered here is the absolute value of the reduced Euler characteristic.

If every vertex of Δ lies in only finitely many chambers, Δ is said to be *locally finite.* The space $\|\Delta\|$ can then be linearly embedded in $\mathbb{R}^{2d+1}$ [32, p. 120], and we can apply the notions of piecewise linear (p.l.) topology (cf. Hudson [19] or Rourke and Sanderson [26]). Let $\mathbb{B}^d$ and $\mathbb{S}^d$ denote the standard p.l. d-ball and d-sphere, i.e., a geometric d-simplex and the boundary of a geometric $(d+1)$-simplex, respectively. Part (i) of the following result is due to Danaraj and Klee [15, p. 444].

1.5. THEOREM. *Let Δ be a shellable d-dimensional pseudomanifold.*

(i) *If Δ is finite, then $\|\Delta\|$ is* p.l. *homeomorphic with the d-sphere $\mathbb{S}^d$.*

(ii) *If Δ is infinite and locally finite, then $\|\Delta\|$ is* p.l. *homeomorphic with Euclidean space $\mathbb{R}^d$.*

(iii) *If Δ is infinite, then $\|\Delta\|$ is contractible.*

Proof. Let $C_1, C_2, \ldots$ be a shelling, and assume that $\mathscr{R}(C_i) \neq C_i$ for $i \leqslant k$. An induction shows that then $\|\Delta_k\|$ is p.l. homeomorphic with the standard d-ball $\mathbb{B}^d$ (in simpler language: $\|\Delta_k\|$ is a p.l. d-ball). Clearly, $\|\Delta_1\| = \|\bar{C}_1\| \cong \mathbb{B}^d$. Suppose $\|\Delta_{k-1}\| \cong \mathbb{B}^d$. $\mathscr{R}(C_k) \neq C_k$ implies that $\|\Delta_{k-1} \cap \bar{C}_k\| \cong \mathbb{B}^{d-1}$, and since each wall is incident with exactly two chambers, $\Delta_{k-1} \cap \bar{C}_k$ lies on the boundary of Δ_{k-1} as well as of $\bar{C}_k$. The claim then follows from the general fact that if two p.l. d-balls intersect along a p.l. $(d-1)$-ball lying in the boundary of each then their union is a p.l. d-ball ([19, p. 39] or [26, p. 36]).

(i): If Δ is finite, having say t chambers, then $\mathscr{R}(C_n) = C_n$ for some $1 < n \leqslant t$. If this were not so, then $\|\Delta\| = \|\Delta_t\| \cong \mathbb{B}^d$ by the preceding paragraph, and any wall on the nonempty boundary of Δ would border only one chamber. Suppose that $\mathscr{R}(C_i) \neq C_i$ for all $i < n$. Then $\|\Delta_{n-1}\| \cong \mathbb{B}^d$, and $\|\Delta_{n-1} \cap \bar{C}_n\| = \|\mathrm{bd}(\bar{C}_n)\| \cong \mathbb{S}^{d-1}$. Also, $\mathrm{bd}(\bar{C}_n) \subseteq \mathrm{bd}(\Delta_{n-1})$ and $\|\mathrm{bd}(\Delta_{n-1})\| \cong \mathbb{S}^{d-1}$, and so $\mathrm{bd}(\bar{C}_n) = \mathrm{bd}(\Delta_{n-1})$. Hence, the two p.l. d-balls Δ_{n-1} and $\bar{C}_n$ are glued together along their entire boundary to get Δ_n, consequently $\|\Delta_n\| \cong \mathbb{S}^d$. No more chambers can now be added to Δ_n without forcing some wall into three chambers, hence $n = t$.

(ii) and (iii): If Δ is infinite, then $\mathscr{R}(C_i) \neq C_i$ for all $i \geqslant 1$, as the preceding argument shows. Hence, $\|\Delta\|$ is contractible by Theorem 1.3. Now, suppose Δ is locally finite. Then one can select a sequence $1 \leqslant n_1 < n_2 < n_3 < \cdots$ of integers such that $\|\Delta_{n_i}\|$ is contained in the interior of $\|\Delta_{n_{i+1}}\|$ for all $i \geqslant 1$. Since all $\|\Delta_{n_i}\|$ are p.l. balls, the annulus property ([19, p. 74] or [26, p. 36]) gives that $\mathrm{cl}(\|\Delta_{n_{i+1}}\| - \|\Delta_{n_i}\|) \cong \mathrm{bd}\,\|\Delta_{n_i}\| \times I$. Now, choose a nested sequence of p.l. d-balls $\mathbb{B}_1 \subset \mathbb{B}_2 \subset \mathbb{B}_3 \subset \cdots$ in $\mathbb{R}^d$ such that $\mathbb{R}^d = \bigcup_{i \geqslant 1} \mathbb{B}_i$ and $\mathbb{B}_i \subset \mathrm{int}\,\mathbb{B}_{i+1}$. Then a p.l. homeomorphism from $\|\Delta\|$ to $\mathbb{R}^d$ can be pieced together by extending a homeomorphism $\|\Delta_{n_i}\| \cong \mathbb{B}_i$ across an annulus to $\|\Delta_{n_{i+1}}\| \cong \mathbb{B}_{i+1}$ successively for $i = 1, 2, \ldots$. ∎

(B) *Balanced Complexes*

A *balanced* complex Δ on vertex set V is by definition a pure d-dimensional complex Δ with a partition $V = \dot{\bigcup}_{s \in S} V_s$ such that $|C \cap V_s| = 1$ for all $C \in \mathscr{C}\!\mathscr{h}(\Delta)$ and all $s \in S$. It is convenient to think of S as a set of colors, the condition being that every chamber has exactly one vertex of each color. Clearly, $|S| = d + 1$. (A balanced complex is called "numbered" by

Lannér [23] and Bourbaki [8, p. 42] and "completely balanced" by Stanley [35].)

Let Δ be a balanced complex as above. For a face $F \in \Delta$, define its *type* $\tau(F) = \{s \in S \mid F \cap V_s \neq \varnothing\}$. Then for $J \subseteq S$, let $\Delta_J = \{F \in \Delta \mid \tau(F) \subseteq J\}$. The *type-selected subcomplex* Δ_J is pure $(|J| - 1)$-dimensional.

1.6. THEOREM. *Suppose that Δ is balanced and shellable. Fix $J \subseteq S$, and let $h_J = \operatorname{card}\{C \in \mathscr{C}\!\mathscr{h}(\Delta) \mid \tau(\mathscr{R}(C)) = J\}$. Then Δ_J is shellable of characteristic h_J.*

Proof. If $\tau(F) = J$, then by Proposition 1.2 there exists a unique $C_F \in \mathscr{C}\!\mathscr{h}(\Delta)$ such that $\mathscr{R}(C_F) \subseteq F \subseteq C_F$. Define $\mathscr{R}_J(F) = \mathscr{R}(C_F)$. The map $\mathscr{C}\!\mathscr{h}(\Delta_J) \to \mathscr{C}\!\mathscr{h}(\Delta)$ defined by $F \mapsto C_F$ is injective, so $\mathscr{C}\!\mathscr{h}(\Delta_J)$ inherits a linear ordering from the shelling order of $\mathscr{C}\!\mathscr{h}(\Delta)$. One easily verifies via Proposition 1.2(iv) that this is a shelling of Δ_J with restriction map $\mathscr{R}_J$. Furthermore, $\mathscr{R}_J(F) = F \Leftrightarrow \mathscr{R}(C_F) = F$, hence $\operatorname{card}\{F \in \mathscr{C}\!\mathscr{h}(\Delta_J) \mid \mathscr{R}_J(F) = F\} = h_J$. ∎

(C) *Algebraic Properties*

Suppose from now on that Δ is a finite balanced complex with partition $V = \bigcup_{s \in S} V_s$, and that $V = \{x_1, x_2, \ldots, x_n\}$. Let $\mathbf{R}$ be a commutative ring with unit, $\mathbf{A} = \mathbf{R}[x_1, x_2, \ldots, x_n]$, and define $\mathbf{R}[\Delta] = \mathbf{A}/I_\Delta$, where $I_\Delta = (x_{i_1} x_{i_2} \cdots x_{i_k} \mid \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \notin \Delta)$. The ring $\mathbf{R}[\Delta]$ is commonly called the *Stanley–Reisner ring* or *face ring* of Δ. Notice that the ideal I_Δ is generated by monomials of degree two if and only if every set of pairwise incident vertices of Δ is a face, i.e., exactly when Δ is a flag complex in the sense of Tits [39].

For each $s \in S$, let $\theta_s = \sum_{x_i \in V_s} x_i$. The elements θ_s are algebraically independent, so $\mathbf{R}[\theta] = \mathbf{R}[\theta_s \mid s \in S]$ is a polynomial subring of $\mathbf{R}[\Delta]$. If $F = \{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \in \Delta$, let $x(F) = x_{i_1} x_{i_2} \cdots x_{i_k} \in \mathbf{R}[\Delta]$. We do not distinguish notationally between elements of $\mathbf{A}$ and their classes in $\mathbf{R}[\Delta]$. The following result is due to Garsia [17, p. 250] and to Kind and Kleinschmidt [21, p. 175].

1.7. THEOREM. *Suppose Δ is shellable. Fix a shelling order $C_1, C_2, \ldots, C_t$ of $\mathscr{C}\!\mathscr{h}(\Delta)$. Then $\mathbf{R}[\Delta]$ is a free $\mathbf{R}[\theta]$-module with basis $\{x(\mathscr{R}(C_i)) \mid 1 \leqslant i \leqslant t\}$.*

Proof. Let $\delta_i = x(\mathscr{R}(C_i))$ for $1 \leqslant i \leqslant t$. It is to be shown that every $\gamma \in \mathbf{R}[\Delta]$ has a unique expression of the form $\gamma = \sum_{i=1}^{t} \delta_i p_i(\theta)$, $p_i(\theta) \in \mathbf{R}[\theta]$.

(i) *Existence.* We may assume that $\gamma = x(F)$ for some $F \in \Delta$, because all other elements of $\mathbf{R}[\Delta]$ are reached by multiplying such γ's by appropriate θ's and adding. By Proposition 1.2 there exists a unique $C_i \in \mathscr{C}\!\mathscr{h}(\Delta)$ such that $\mathscr{R}(C_i) \subseteq F \subseteq C_i$. Let $J = \tau(F - \mathscr{R}(C_i))$. Then $\delta_i \prod_{s \in J} \theta_s = x(F) + \sum x(F')$, where the latter summation extends over all

$F' \neq F$ such that $\tau(F') = \tau(F)$ and $F' \supseteq \mathscr{R}(C_i)$. If $\mathscr{R}(C_j) \subseteq F' \subseteq C_j$, then condition ($\beta$) of Proposition 1.2(iv) implies $i < j$. However, we may by induction assume that if $F' \in [\mathscr{R}(C_j), C_j]$ with $j > i$ then $x(F')$ can be correctly expressed, since the previous argument for $i = t$ shows that this is the case when $F \in [\mathscr{R}(C_t), C_t]$.

(ii) *Uniqueness.* Suppose that $\gamma = \sum_{i=1}^{t} \delta_i p_i(\theta)$, $p_i(\theta) \in \mathbf{R}[\theta]$, and that $p_1 = p_2 = \cdots = p_{g-1} = 0$ and $p_g \neq 0$. All terms of $\delta_g p_g(\theta)$ are of the form $r x_{i_1}^{e_1} x_{i_2}^{e_2} \cdots x_{i_k}^{e_k}$, $r \in \mathbf{R} - \{0\}$, $e_j \geqslant 1$, $\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} = F \in \Delta$, and $F \supseteq \mathscr{R}(C_g)$, and among them there is at least one for which $F \subseteq C_g$. That term cannot be cancelled by a term coming from a later product $\delta_i p_i(\theta)$, $i > g$, because then $\mathscr{R}(C_i) \subseteq F \subseteq C_g$, contradicting property 1.2(ivβ). Hence, $\gamma \neq 0$. ∎

From now on let $\mathbf{R} = \mathbf{k}$ be a field. The ring $\mathbf{k}[\Delta]$ has a standard grading induced by setting $\deg x_i = 1$, $1 \leqslant i \leqslant n$. Such a ring is said to be *Cohen–Macaulay* if it is a free and finitely generated $\mathbf{k}[\pi]$-module for some, and equivalently all, homogeneous system(s) of parameters π. Since $\theta = \{\theta_s \mid s \in S\}$ is clearly a system of parameters, i.e., $\dim_{\mathbf{k}} \mathbf{k}[\Delta]/(\theta) < \infty$, we deduce the following result, first obtained by Hochster.

1.8. Corollary. *If Δ is shellable, then $\mathbf{k}[\Delta]$ is Cohen–Macaulay.*

Now consider the Hilbert series $F(\mathbf{k}[\Delta], z) = \sum_{i \geqslant 0} \dim_{\mathbf{k}} \mathscr{H}_i(\mathbf{k}[\Delta])\, z^i$, where $\mathscr{H}_i(\mathbf{k}[\Delta]) = \{\gamma \in \mathbf{k}[\Delta] \mid \deg \gamma = i\}$. If $\mathbf{k}[\Delta]$ is Cohen–Macaulay, say $\mathbf{k}[\Delta] = \bigoplus_{i=1}^{t} \eta_i \mathbf{k}[\theta]$ for some homogeneous basis $\eta = \{\eta_1, \eta_2, \ldots, \eta_t\}$ over $\mathbf{k}[\theta]$, then η is a $\mathbf{k}$-basis for $\mathbf{k}[\Delta]/(\theta)$. Hence, if $F(\mathbf{k}[\Delta]/(\theta), z) = h_0 + h_1 z + \cdots + h_{d+1} z^{d+1}$, $d + 1 = |S|$, then $F(\mathbf{k}[\Delta], z) = (1 - z)^{-(d+1)} (h_0 + h_1 z + \cdots + h_{d+1} z^{d+1})$. Now, by straightforward counting one finds that $\dim_{\mathbf{k}} \mathscr{H}_i(\mathbf{k}[\Delta]) = \sum_{j=0}^{d} f_j \binom{i-1}{j}$ if $i > 0$ and $=1$ if $i = 0$, where f_j is the number of j-dimensional faces of Δ (cf. Stanley [34]). Comparing terms in the two expressions for $F(\mathbf{k}[\Delta], z)$ one deduces that $h_i = (-1)^{i-1} \sum_{j=-1}^{d} (-1)^j f_j \binom{d-j}{d+1-i}$, for $i = 0, 1, \ldots, d+1$. In particular, $h_{d+1} = (-1)^d \tilde{\chi}(\Delta)$. Notice that if Δ is shellable, then $h_i = \sum h_J$, the sum extending over all subsets $J \subseteq S$ of cardinality i.

Suppose that $\mathbf{k}[\Delta]$ is Cohen–Macaulay and let m be the maximal ideal generated by the indeterminates x_i, $1 \leqslant i \leqslant n$. Let $\mathbf{Q} = \mathbf{k}[\Delta]/(\theta)$, $\bar{m} = m/(\theta)$, and define the *socle* of $\mathbf{Q}$ by $\operatorname{soc} \mathbf{Q} = \{q \in \mathbf{Q} \mid q\bar{m} = 0\}$. The integer $\dim_{\mathbf{k}} \operatorname{soc} \mathbf{Q}$ is known as the *type* of $\mathbf{k}[\Delta]$. A *Gorenstein* ring is by definition a Cohen–Macaulay ring of type one.

1.9. Theorem. *Suppose that $\mathbf{k}[\Delta]$ is Cohen–Macaulay and Δ a d-dimensional pseudomanifold such that $\tilde{\chi}(\Delta) = (-1)^d$. Then $\mathbf{k}[\Delta]$ is Gorenstein.*

Proof. The homogeneous graded component $\mathscr{H}_{d+1}(\mathbf{Q})$ is the linear span in $\mathbf{Q}$ of the (classes of) monomials $x(C)$, $C \in \mathscr{C}\!h(\Delta)$. If $C \in \mathscr{C}\!h(\Delta)$ and $x_i \in V_s$, then either $x_i \notin C$, in which case $x(C)\,x_i = 0$ in $\mathbf{k}[\Delta]$, or $x_i \in C$, in which case $x(C)\,x_i = x(C)\,\theta_s$. Consequently, $\mathscr{H}_{d+1}(\mathbf{Q}) \subseteq \operatorname{soc} \mathbf{Q}$.

Since $\dim_{\mathbf{k}} \mathscr{H}_{d+1}(\mathbf{Q}) = h_{d+1} = (-1)^d \tilde{\chi}(\Delta) = 1$, it remains only to prove the reverse inclusion.

Suppose that $C, C' \in \mathscr{C}\!h(\Delta)$ are adjacent. Then $C - C' = \{x_i\}$, $C' - C = \{x_k\}$, and $x_i, x_k \in V_s$ for some $s \in S$. Since Δ is a pseudomanifold, the wall $C \cap C'$ lies in no other chambers, hence $x(C \cap C')\,\theta_s = x(C) + x(C')$. Since any two chambers can be connected by a gallery, it follows that $[x(C)] = \pm [x(C')]$ in $\mathbf{Q}$ for arbitrary chambers C and C' (brackets denote classes $\operatorname{mod}(\theta)$). In particular, $[x(C)] \neq 0$ for all $C \in \mathscr{C}\!h(\Delta)$, and consequently $[x(F)] \neq 0$ for all $F \in \Delta$. It follows that $\operatorname{soc} \mathbf{Q} \subseteq \mathscr{H}_{d+1}(\mathbf{Q})$. ∎

1.10. COROLLARY. *If Δ is a shellable pseudomanifold, then $\mathbf{k}[\Delta]$ is Gorenstein.*

1.11. *Remark.* The preceding algebraic results have been developed specifically for the kind of complexes encountered in this paper, viz., shellable balanced complexes and pseudomanifolds. For the general theory of *Cohen–Macaulay* and *Gorenstein complexes* (i.e., complexes Δ such that $\mathbf{k}[\Delta]$ has the respective properties), and in particular for the ring-theoretic aspects of the subject, the reader is referred to the papers of Baclawski, Garsia, Hochster, and Stanley [1, 3, 17, 18, 34, 35, 37], and the references therein. From the characterization of Gorenstein complexes [37, p. 75] a converse to Theorem 1.9 is known: if Δ is nonacyclic and Gorenstein then Δ is a pseudomanifold and $\tilde{\chi}(\Delta) = (-1)^{\dim \Delta}$.

2. COXETER COMPLEXES

Throughout this section (W, S) will denote a *Coxeter group* ("système de Coxeter"), and the set S of distinguished generators will be assumed finite. The notation and terminology of Bourbaki [8, pp. 9–22] will be adopted. In particular, for $J \subseteq S$ let W_J be the subgroup (called *parabolic*) generated by J. Also, for $s \in S$ recall the notation $(s) = S - \{s\}$.

The *weak ordering* of W is defined as follows. For $w, w' \in W$ let $w \leqslant w'$ mean that there exist $s_1, s_2, \ldots, s_k \in S$ such that $l(ws_1 s_2 \cdots s_i) = l(w) + i$ for $1 \leqslant i \leqslant k$ and $ws_1 s_2 \cdots s_k = w'$. Here $l(w)$ denotes the length function on W. The more familiar *Bruhat ordering* (or, *strong ordering*) of W is defined in the same way except that one allows $s_1, s_2, \ldots, s_k \in \{wsw^{-1} \mid w \in W, s \in S\}$. Except for in Sections 5 and 6 the Bruhat ordering will be of little use to us. The weak ordering has many linear extensions since $\{w \in W \mid l(w) = j\}$ is finite for all $j \geqslant 0$. In fact, it can be shown that every linear ordering of W

which contains the Bruhat ordering is well-ordered, but the same is not in general true for the weak ordering. Further information about the two orderings of Coxeter groups can be found in [6] and [16].

For $w \in W$, define the *descent set* $\mathscr{D}(w) = \{s \in S \mid ws < w\}$, and for $J \subseteq S$ define the *descent class* $\mathscr{D}_J = \{w \in W \mid \mathscr{D}(w) = J\}$. Also, let $W^J = \{w \in W \mid ws > w$ for all $s \in J\} = \bigcup_{I \subseteq J} \mathscr{D}_I$. It is well known that each $w \in W$ can be uniquely factored $w = uv$, $u \in W^J$, $v \in W_J$, and then $l(w) = l(u) + l(v)$ [8, p. 37]. It follows that W^J is a system of distinct coset representatives modulo W_J. Furthermore, $u \in W^J$ is the unique element in uW_J of minimal length, and if $w \in uW_J$ then $u \leqslant w$.

The *Coxeter complex* $\Delta(W, S)$ is by definition the simplicial complex on vertex set $V = \bigcup_{s \in S} W/W_{(s)}$ with chambers $C_w = \{wW_{(s)} \mid s \in S\}$, $w \in W$. Equivalently, $\Delta(W, S)$ is the nerve of the covering of W by left cosets of maximal parabolic subgroups.

A few initial observations are in order. More details can be found in Bourbaki [8, pp. 40–44] and Tits [39, Chap. 2]. $\Delta = \Delta(W, S)$ is a pure $(|S| - 1)$-dimensional complex and it is naturally balanced $V = \dot{\bigcup}_{s \in S} V_s$ with $V_s = W/W_{(s)}$. The group W acts on Δ by left translation $w\colon vW_{(s)} \mapsto wvW_{(s)}$, and this action is *type-preserving*, i.e., $\tau(w(F)) = \tau(F)$ for all $F \in \Delta$. The action of W on $\{F \in \Delta \mid \tau(F) = J\}$ is transitive for all $J \subseteq S$, and if $F \subseteq C_e$ then $\mathrm{Stab}(F) = W_{S - \tau(F)}$. In particular, the fundamental chamber C_e is stabilized by $\{e\}$, hence $w \mapsto C_w$ is a bijection $W \leftrightarrow \mathscr{Ch}(\Delta)$. Two chambers C_w and $C_{w'}$ are adjacent if and only if $w' = ws$ for some $s \in S$. Consequently, Δ is a pseudomanifold.

2.1. THEOREM. *Let (W, S) be a Coxeter group, $|S| < \infty$. Then any linear extension of the weak ordering of W assigns a shelling order to the chambers of $\Delta = \Delta(W, S)$. As a consequence, for all $J \subseteq S$, the type-selected subcomplex Δ_J is shellable of characteristic $|\mathscr{D}_J|$.*

Proof. Since S is finite, W is at most countable. Suppose that $\mathscr{Ch}(\Delta)$ is linearly ordered in such a way that $w_1 < w_2$ implies $C_{w_1} < C_{w_2}$. Let C_w be the kth chamber in this ordering, and let Δ_{k-1} as usual denote the subcomplex generated by the $k - 1$ first chambers. Suppose that $F \subseteq C_w$. Then

$$
\begin{aligned}
F \in \Delta_{k-1} &\Leftrightarrow F \subseteq C_{w'}, \qquad C_{w'} < C_w, \\
&\Leftrightarrow wW_{(s)} = w'W_{(s)} \qquad \text{for all } s \in \tau(F),\ C_{w'} < C_w, \\
&\Leftrightarrow w^{-1}w' \in \bigcap_{s \in \tau(F)} W_{(s)} = W_{S - \tau(F)}, \qquad C_{w'} < C_w, \\
&\Leftrightarrow wW_{S - \tau(F)} = w'W_{S - \tau(F)}, \qquad C_{w'} < C_w, \\
&\Leftrightarrow w \notin W^{S - \tau(F)} \\
&\Leftrightarrow \mathscr{D}(w) \nsubseteq \tau(F).
\end{aligned}
$$

In view of Proposition 1.2(ii) this shows that the ordering of $\mathscr{C}\!\mathscr{h}\,(\Delta)$ is a shelling whose restriction map $\mathscr{R}$ sends C_w to its face of type $\mathscr{D}(w)$. The claim for Δ_J then follows from Theorem 1.6. ∎

2.2. Corollary. (i) *If W is finite, then $\|\Delta(W, S)\|$ is* p.l. *homeomorphic with the $(|S|-1)$-sphere.*

(ii) *If W is infinite, then $\|\Delta(W, S)\|$ is* (p.l.) *homeomorphic with Euclidean space $\mathbb{R}^{|S|-1}$ if and only if all maximal parabolic subgroups are finite.*

(iii) *If W is infinite, then $\|\Delta(W, S)\|$ is contractible.*

Proof. This follows from Theorem 1.5. Notice that $\Delta(W, S)$ is locally finite if and only if $|W_{(s)}| < \infty$ for all $s \in S$, and that any triangulation of $\mathbb{R}^{|S|-1}$ must be locally finite. ∎

The topological facts arrived at in the Corollary are well known, see Bourbaki [8, p. 133], Coxeter and Moser [11, Chap. 9], Lannér [23, Chap. 5], Serre [28, p. 108], and Tits [39, p. 22]. They are usually obtained as follows. Every finite Coxeter group (W, S) can be realized as a reflection group in Euclidean space, and the reflecting hyperplanes cut the unit sphere into spherical simplices which together form a realization of $\Delta(W, S)$. The infinite groups with finite maximal parabolic subgroups are the affine Weyl groups, classified by Coxeter and Witt, and the groups of compact hyperbolic type, classified by Lannér. They can be realized as groups of isometries of Euclidean space and hyperbolic space, respectively, from which the topological type of their complexes can be seen.

The theorem contains topological information about the type-selected subcomplexes as well, and in this connection it becomes of interest to study the possible descent classes. Recall that if W is finite there exists a unique element $w_0 \in W$ such that $w \leqslant w_0$ for all $w \in W$ [8, p. 43]. Furthermore, if (W, S) is a general Coxeter group and $w \in W$, then $\mathscr{D}(w) = S \Leftrightarrow W$ is finite and $w = w_0$. Suppose for $J \subseteq S$ that W_J is finite and denote by $w_0(J)$ the top element of W_J. Then, clearly, every coset uW_J has a unique representative of maximal length, viz., $uw_0(J)$ (taking $u \in W^J$). Also, ${}^JW = \{w \in W \mid \mathscr{D}(w) \supseteq J\}$ is the set of maximal coset representatives.

Now, let $\Delta = \Delta(W, S)$, $J \subseteq S$, and consider the type-selected subcomplex Δ_J.

2.3. Corollary. *The following are equivalent for all $J \subseteq S$:*

(i) *W_J is infinite,*

(ii) $\mathscr{D}_J = \varnothing$,

(iii) *$\|\Delta_J\|$ is contractible.*

Proof. The equivalence of (ii) and (iii) follows from Theorems 1.3 and 2.1. If W_J is finite then $w_0(J) \in \mathscr{D}_J$. Conversely, if $w \in \mathscr{D}_J$ and $w = u \cdot v$, where $u \in W^J$ and $v \in W_J$, then $\mathscr{D}(v) = J$, and hence W_J is finite. ∎

2.4. COROLLARY. *If W is finite, then for all $J \subseteq S$:*

$$\operatorname{rank} \tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z}) = \sum_{I \supseteq J} (-1)^{|I-J|} [W : W_I].$$

Proof. Recall from Theorems 1.3 and 2.1 that $\operatorname{rank} \tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z}) = |\mathscr{D}_J|$. Hence, the formula follows by Möbius inversion [25] from

$$[W : W_J] = |{}^J W| = \sum_{I \supseteq J} |\mathscr{D}_I|. \quad ∎$$

Suppose now that W is finite. The map $\theta : w \mapsto w_0 w$ is known to be an involutory antiautomorphism of weak and Bruhat order, i.e., $w \leqslant w' \Leftrightarrow \theta w \geqslant \theta w'$, and $\theta^2 = \mathrm{id}$ [8, p. 43]. In particular, $\mathscr{D}(\theta w) = S - \mathscr{D}(w)$ for all $w \in W$. Consequently, $|\mathscr{D}_J| = |\mathscr{D}_{\hat{J}}|$ for all $J \subseteq S$, or equivalently,

$$\operatorname{rank} \tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z}) = \operatorname{rank} \tilde{H}_{|\hat{J}|-1}(\Delta_{\hat{J}}, \mathbb{Z}). \tag{2.5}$$

This formula can also be deduced from Alexander duality since $\|\Delta\|$ is a sphere. By Corollary 2.4 it is equivalent to the following formula of Solomon [29]:

$$\sum_{I \supseteq J} (-1)^{|I-J|} [W : W_I] = \sum_{I \supseteq \hat{J}} (-1)^{|I-\hat{J}|} [W : W_I]. \tag{2.6}$$

2.7. *Remark.* Danaraj and Klee have studied restrictions that can be made on shelling order for shellings of the boundary of convex polytopes [15, p. 449]. The shellings of spherical Coxeter complexes which are induced by weak or Bruhat order admit similar but not quite as far-reaching restrictions. One can prove the following. Let $\Delta = \Delta(W, S)$ be a finite Coxeter complex, and let $\varnothing = K_{-1} \subset K_0 \subset \cdots \subset K_{|S|-1}$ be a sequence of faces of Δ with $\dim K_i = i$. For $0 \leqslant i \leqslant |S|$ let $\mathscr{F}_i$ denote the set of all chambers C of Δ such that $C \supseteq K_{|S|-i-1}$ but $C \not\supseteq K_{|S|-i}$. Then Δ admits a Bruhat shelling in which the chambers appear in the order $\mathscr{F}_0, \mathscr{F}_1, \ldots, \mathscr{F}_{|S|}$. (Without loss of generality one could assume that $K_{|S|-1} = C_e$.) Also, the reverse of a Bruhat shelling order is a shelling. Thus, all the chambers which contain any preassigned nonempty face can be arranged to come last in a shelling. It follows that if any vertex and all faces containing it are removed, the remaining complex is still shellable. More generally, by the same method as in Theorem 4.8 below one can show that all faces which have nonempty intersection with some fixed nonempty face of Δ can be deleted from Δ

without losing shellability or diminishing dimension. In particular, it follows that all finite Coxeter complexes are 2-Cohen–Macaulay in the sense defined after Corollary 4.9.

3. BN-Pairs: Orderings of G/B

Let G be a *group with a* BN-*pair* ("système de Tits") in the sense of Bourbaki [8, pp. 22–32] and Tits [39]. Recall that $W = N/B \cap N$ (the *Weyl group*) is a Coxeter group with a uniquely determined set S of Coxeter generators. The cardinality of S, called the *rank* of G, will always be assumed finite. For each $J \subseteq S$ there is a *standard parabolic subgroup* $G_J = BW_JB = \bigcup_{w \in W_J} BwB$. It is a fundamental fact that $G = \dot{\bigcup}_{w \in W} BwB$ (*Bruhat decomposition*).

Suppose that $X \subseteq G$ is a subset such that $XB \subseteq X$. Then X is a union of left cosets gB, and we write $X/B = \{gB \mid g \in X\}$. This applies in particular to double cosets BwB, and for $w \in W$ define the *index* $q_w = \text{card}(BwB/B)$. The following facts (i)–(iv) can be found more or less explicitly formulated in Bourbaki [8, pp. 54–55], for (v) see Carter [10, p. 121].

3.1. Lemma. (i) $q_w = |B : B \cap wBw^{-1}|$,

(ii) $q_w = q_{s_1} q_{s_2} \cdots q_{s_k}$, *if* $w = s_1 s_2 \cdots s_k$, $l(w) = k$, $s_i \in S$,

(iii) $q_s \geqslant 2$, *for all* $s \in S$,

(iv) $q_s = q_{s'}$, *if* $s, s' \in S$ *are conjugate*,

(v) $q_w = q^{l(w)}$, *if* G *is a finite Chevalley group over* $GF(q)$.

The purpose for this section is to define useful partial orderings of the coset space G/B. Note that every coset gB can be represented in the form $gB = bwB$, with $b \in B$ and unique $w \in W$, because of Bruhat decomposition.

3.2. Definition. For $g, g' \in G$, let $gB \leqslant g'B$ mean that there exist $g = g_0, g_1, \ldots, g_k = g'$ such that for $i = 1, 2, \ldots, k$ there are representations $g_{i-1}B = b_i w_{i-1} B$ and $g_i B = b_i w_i B$ with $b_i \in B$ and $w_{i-1} \leqslant w_i$ in the weak ordering of W. This ordering will be called the *weak ordering* of G/B.

If in this definition one uses instead the Bruhat ordering of W, one gets a corresponding *strong ordering* of G/B. One sees that in these partial orderings B is the unique minimal element, and if W is finite there are q_{w_0} maximal elements. Furthermore, if $gB = bwB$ then all maximal chains from B to gB have length $l(w)$. For finite Chevalley groups over $GF(q)$ these orderings could be regarded as "q-analogues" of Coxeter weak and Bruhat order, since each $w \in W$ splits into $q^{l(w)}$ points in G/B.

As an example take $G = GL_3(\mathbb{Z}_2)$, and let B and N be the subgroups of

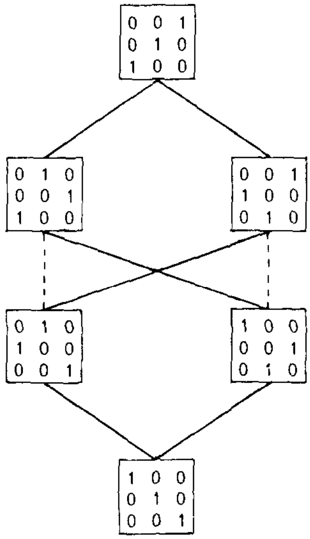


FIGURE 1

upper triangular and monomial matrices, respectively. Then W is the symmetric group S_3, whose weak ordering is depicted in Fig. 1. The additional covering relations in the Bruhat ordering are dotted. Fig. 2 shows the weak ordering of G/B. Again, the extra edges for the strong ordering are dotted. In this drawing the left cosets which belong to a common double coset are grouped together, and the groups are distributed in correspondence with Fig. 1. The symbol

$$\boxed{\begin{matrix}1 & \varepsilon \\ & \delta\end{matrix}}$$

stands for the coset

$$\begin{pmatrix}1 & 1 & \varepsilon \\ 0 & 1 & \delta \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1\end{pmatrix}=bwB, \qquad \varepsilon, \delta \in \{0, 1\},$$

and the symbol

$$\boxed{\begin{matrix}\varepsilon & 1 \\ & 0\end{matrix}}$$

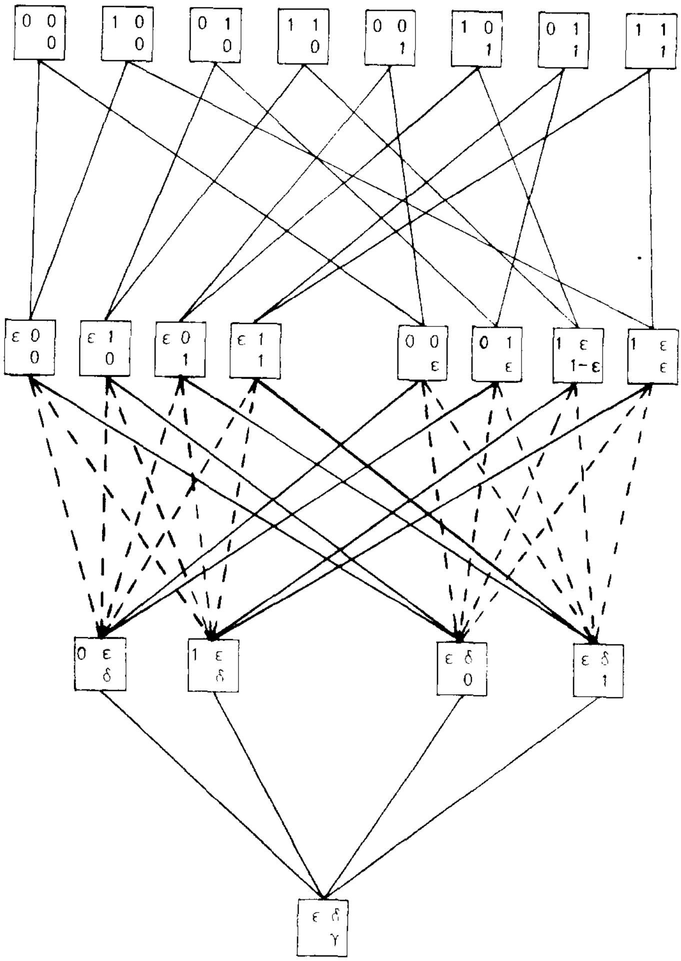


FIGURE 2

stands for the coset

$$\begin{pmatrix}1 & \varepsilon & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & * & *\\ 0 & 1 & *\\ 0 & 0 & 1\end{pmatrix}=b'w'B, \qquad \varepsilon\in\{0,1\},$$

etc.

For $J\subseteq S$, let $G^J=BW^JB=\dot{\bigcup}_{w\in W^J}BwB$. The crucial property we will need is that G^J/B acts as a set of minimal coset representatives modulo G_J.

3.3. PROPOSITION. *Fix $J\subseteq S$. Given $gB\in G/B$ there exists a unique $g'B\in G^J/B$ such that $g'B\subseteq gG_J$.* Furthermore, then $g'B\leqslant gB$.

Proof. Suppose $gB = bwB$, $b \in B$, $w \in W$. Factor $w = uv$, $u \in W^J$, $v \in W_J$, and let $g'B = buB$. Then, $buB = bwv^{-1}B \subseteq bwG_J = gG_J$. In addition, $u \leqslant w$ implies that $buB \leqslant bwB$. Now, suppose that also $b'u'B \subseteq gG_J$, $b' \in B$, $u' \in W^J$. Then $buG_J = b'u'G_J$, i.e., $b^{-1}b'u' \subseteq uG_J = uBW_JB$, which implies that

$$b^{-1}b'u' \subseteq uBv'B \qquad \text{for some} \quad v' \in W_J. \tag{3.4}$$

Since $BuBv'B = Buv'B$ (from $l(uv') = l(u) + l(v')$) one concludes that $b^{-1}b'u' \subseteq Buv'B$, and then by Bruhat decomposition that $u' = uv'$. Hence, $u' = u$ and $v' = e$. Finally, (3.4) gives $b^{-1}b'u' \subseteq uB$, i.e., $b'u'B = buB$. ∎

4. Buildings

Let G be a group with a BN-pair of finite rank. Following Tits [39] or Bourbaki [8, pp. 49–53] one defines the *building* $\Delta(G; B, N)$ to be the simplicial complex on the vertex set $V = \bigcup_{s \in S} G/G_{(s)}$ with chambers $C_g = \{gG_{(s)} \mid s \in S\}$, $g \in G$. The formal analogy with Coxeter complexes is evident, and the introductory comments of Section 2 can be repeated almost word by word. Thus, $\Delta = \Delta(G; B, N)$ is a pure $(|S| - 1)$-dimensional balanced complex. G acts on Δ by left translation and this action is type-preserving. The action of G is transitive on $\{F \in \Delta \mid \tau(F) = J\}$ for all $J \subseteq S$, and if $F \subseteq C_g$ then $\mathrm{Stab}(F) = gG_{S - \tau(F)}g^{-1}$. In particular, C_e is stabilized by B, so $g \mapsto C_g$ determines a bijection $G/B \leftrightarrow \mathscr{C}\!h\,(\Delta)$. Two chambers C_g and $C_{g'}$ are adjacent if and only if $g^{-1}g' \in BsB$ for some $s \in S$. It follows that a wall $C_g - \{gG_{(s)}\}$ lies in exactly $q_s + 1$ chambers.

In Section 1 shellability was for convenience defined only for countable complexes. The extension to arbitrary cardinality is straightforward, cf. Remark 4.21 at the end of this section.

4.1. Theorem. *Let G be a group with* BN*-pair of finite rank and with building $\Delta = \Delta(G; B, N)$. Then any linear extension of the weak ordering of G/B assigns a shelling order to $\mathscr{C}\!h\,(\Delta)$. It follows for each $J \subseteq S$ that Δ_J is shellable of characteristic $\sum_{w \in \mathscr{D}_J} q_w$.*

Proof. Suppose that $\mathscr{C}\!h\,(\Delta)$ is well-ordered in such a way that $g_1B < g_2B$ implies $C_{g_1} < C_{g_2}$. Let C_{bw}, $b \in B$, $w \in W$, be the αth chamber and let $\Delta_{<\alpha}$ denote the subcomplex generated by its predecessors. The argument is now formally analogous to that for Coxeter complexes, the key point here, however, being Proposition 3.3. Suppose that $F \subseteq C_{bw}$. Then

$$\begin{aligned}F \in \Delta_{<\alpha} &\Leftrightarrow F \subseteq C_{b'w'}, \quad C_{b'w'} < C_{bw},\\ &\Leftrightarrow bwG_{(s)} = b'w'G_{(s)} \quad \text{for all } s \in \tau(F),\ C_{b'w'} < C_{bw},\\ &\Leftrightarrow w^{-1}b^{-1}b'w' \subseteq \bigcap_{s \in \tau(F)} G_{(s)} = G_{S-\tau(F)}, \quad C_{b'w'} < C_{bw},\\ &\Leftrightarrow bwG_{S-\tau(F)} = b'w'G_{S-\tau(F)}, \quad C_{b'w'} < C_{bw},\\ &\Leftrightarrow bwB \notin G^{S-\tau(F)}/B\\ &\Leftrightarrow \mathscr{D}(w) \nsubseteq \tau(F).\end{aligned}$$

Consequently, the ordering is a shelling under which the restriction of chamber C_{bw} is its face of type $\mathscr{D}(w)$. Finally, for each $w \in W$ there are q_w distinct chambers C_{bw}, $b \in B$. ∎

4.2. COROLLARY. (i) *$\|\Delta_J\|$ is contractible if and only if W_J is infinite.*

(ii) *If W_J is finite $\|\Delta_J\|$ has the homotopy type of a wedge of $\sum_{w \in \mathscr{D}_J} q_w$ $(|J|-1)$-spheres.*

The preceding follows from Theorem 1.3 and Corollary 2.3. These facts are well known for $J = S$ (cf. Tits [40, p. 215]). Also, for finite groups G and $J \neq S$ the homology of Δ_J implied by part (ii) was previously determined by Bromwich [9]. In the finite case there is the following alternative expression.

4.3. COROLLARY. *If G is finite and $J \subseteq S$, then*

$$\operatorname{rank} \tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z}) = \sum_{I \supseteq \hat{J}} (-1)^{|I-\hat{J}|}\,|G\colon G_I|.$$

Proof. $|G\colon G_J| = |G^J/B| = \sum_{w \in W^J} q_w = \sum_{I \subseteq \hat{J}} \sum_{w \in \mathscr{D}_I} q_w$, from where the formula is obtained by Möbius inversion. ∎

Taking $J = S$ one gets the following formula of Solomon [29, p. 379]:

$$q_{w_0} = \sum_{I \subseteq S} (-1)^{|I|}\,|G\colon G_I|. \tag{4.4}$$

Assume from now on that $\Delta = \Delta(G; B, N)$ is a *spherical* building, i.e., that W is finite. Let Σ_0 be the subcomplex generated by the chambers C_w, $w \in W$. The map $wW_{(s)} \mapsto wG_{(s)}$ induces an isomorphism with the Coxeter complex $\Delta(W, S) \cong \Sigma_0$. Subcomplexes of the form $g\Sigma_0$, $g \in G$, are called *apartments*, and we denote the collection of them by $\mathscr{A}p(\Delta)$. Clearly, all apartments are mutually isomorphic triangulations of the $(|S|-1)$-sphere.

Suppose that Δ is somehow oriented, e.g., by linearly ordering the set S and letting faces receive the induced orientation. Let ∂ be the corresponding boundary operator. If $\Sigma \in \mathscr{A}p(\Delta)$, pick one $g \in G$ such that $\Sigma = g\Sigma_0$ and then define $\rho_\Sigma = \sum_{w \in W} (-1)^{l(w)}\, C_{gw}$. Thus, ρ_Σ is uniquely determined by Σ

up to sign. Also, $\partial\rho_\Sigma = 0$, so $\rho_\Sigma \in \tilde{H}_{|S|-1}(\Delta, \mathbb{Z})$. Let us call cycles of the form ρ_Σ *elementary*. Solomon [31, p. 215] observed that the elementary cycles containing the fundamental chamber C_e form a basis of the free group $\tilde{H}_{|S|-1}(\Delta, \mathbb{Z}) \cong \mathbb{Z}^{q_{w_0}}$. We will add a minor observation.

4.5. PROPOSITION. *Let $C \in \mathscr{C}\!h(\Delta)$, and let $A_C = \{\Sigma \in \mathscr{A}\!p(\Delta) \mid C \in \Sigma\}$. Then the elementary cycles $\{\rho_\Sigma \mid \Sigma \in A_C\}$ form a basis of $\tilde{H}_{|S|-1}(\Delta, \mathbb{Z})$. Furthermore, for any $\Sigma' \in \mathscr{A}\!p(\Delta)$ if $\rho_{\Sigma'} = \sum_{\Sigma \in A_C} n_\Sigma \rho_\Sigma$ then $n_\Sigma \in \{0, +1, -1\}$ for all $\Sigma \in A_C$.*

Proof. Without loss of generality we may take $C = C_e$. Let $Bw_0B/B = \{b_i w_0 B \mid i \in I\}$ (equivalently, $\{b_i\}_{i\in I}$ is a system of distinct coset representatives of $B \cap w_0 B w_0^{-1}$ in B), and write $\Sigma_i = b_i \Sigma_0$ for all $i \in I$. Then $\{\rho_\Sigma \mid \Sigma \in A_C\} = \{\rho_{\Sigma_i} \mid i \in I\}$ and $\rho_{\Sigma_i} = \sum_{w \in W} (-1)^{l(w)} C_{b_i w}$. Now observe for $i, j \in I$ that

$$\pm C_{b_j w_0} \text{ is a term in } \rho_{\Sigma_i} \qquad \text{if and only if } i = j. \tag{4.6}$$

Linear independence and the $\{0, +1, -1\}$-property follow immediately from this. If $\rho \in \tilde{H}_{|S|-1}(\Delta, \mathbb{Z})$, then (4.6) shows that for some integers n_i, only finitely many $\neq 0$, $\rho - \sum n_i \rho_{\Sigma_i}$ lacks terms which are nonzero multiples of $C_{b_i w_0}$. So $\rho - \sum n_i \rho_{\Sigma_i}$ is a cycle in the contractible subcomplex $\Delta^* = \Delta - \{C_{b_i w_0} \mid i \in I\}$, hence equals zero. ∎

Two chambers C_g and $C_{g'}$ in a spherical building are said to be *opposite* if $C_g = g''(C_e)$ and $C_{g'} = g''(C_{w_0})$ for some $g'' \in G$. For instance, in the terms of the preceding proof $C_{b_i w_0}$, $i \in I$, are the chambers opposite to C_e. Oppositeness is clearly a symmetric relation.

4.7. LEMMA. *Let $k \leqslant \min_{s \in S} q_s$, k finite. Then any k chambers in Δ have a common opposite.*

Proof. This can be shown by extending Tits's proof [39, p. 55] that any 2 chambers have a common opposite. Here is a sketch of the argument. Define a distance on $\mathscr{C}\!h(\Delta)$ by letting $d(C_g, C_{g'})$ be the shortest length of a gallery starting in C_g and ending in $C_{g'}$. Then C_g and $C_{g'}$ are opposite if and only if $d(C_g, C_{g'}) = l(w_0) = \operatorname{diam}(\Delta)$. The crux of the argument is the following sublemma:

Suppose $d(C_g, C_{g'}) = \delta$ and $F = C_g - \{gG_{(s)}\}$, and let $\mathscr{F} = \{C \in \mathscr{C}\!h(\Delta) \mid C \supseteq F,\ C \neq C_g\}$. Then *either*

(i) $d(C, C_{g'}) = \delta + 1$ for all $C \in \mathscr{F}$, *or*

(ii) $d(C, C_{g'}) = \delta - 1$ for one $C \in \mathscr{F}$ and $d(C, C_{g'}) = \delta$

for the remaining $q_s - 1$ chambers $C \in \mathscr{F}$.

Furthermore, if $0 < \delta < l(w_0)$ then both cases occur for suitably chosen walls F and F' of C_g.

One proves the sublemma either combinatorially using Lemma 3.19.7 of Tits [39, p. 51], or directly in terms of the ambient group. In the latter case one takes $C_{g'} = C_e$ and $C_g = C_{bw}$, $b \in B$, $w \in W$, and shows that the first case is equivalent to $ws > w$ and the second to $ws < w$. (Remark: The sublemma can be regarded as a geometric interpretation of the basic relations of the Hecke algebra $\mathscr{H}(G, B)$.)

Now, suppose the chambers $C^1, C^2, \dots, C^j$, $1 \leqslant j < k$, have a common opposite. Among all such pick one C_g which maximizes $d(C_g, C^{j+1}) = \delta$. If $\delta < l(w_0)$ there exists a wall $F = C_g - \{gG_{(s)}\}$ such that $d(C, C^{j+1}) = \delta + 1$ for all $C \in \mathscr{F} = \{C \in \mathscr{Ch}(\Delta) \mid C \supseteq F,\ C \neq C_g\}$. For $i = 1, 2, \dots, j$, $d(C, C^i) = d(C_g, C^i) = l(w_0)$ for all $C \in \mathscr{F}$ except one. Since $|\mathscr{F}| = q_s$ it is possible to select a chamber $C^* \in \mathscr{F}$ which is simultaneously opposite to $C^1, C^2, \dots$, and C^j. But then $d(C^*, C^{j+1}) = \delta + 1$, contradicting the choice of C_g. Hence, $C^1, C^2, \dots, C^{j+1}$ have the common opposite C_g. ∎

Recall that for $J \subseteq S$ the parabolic subgroup G_J has the BN-pair (B, N_J), $N_J = N \cap G_J$, with corresponding Weyl group (W_J, J). For every $w \in W_J$ the index q_w is clearly the same in G and in G_J.

For $F \in \Delta$ define the *star* $\mathrm{st}_\Delta F = \{E \in \Delta \mid E \supseteq F\}$ and the *link* $\mathrm{lk}_\Delta F = \{E - F \mid E \in \mathrm{st}_\Delta F\}$. Then for nonempty faces $F_1, F_2, \dots, F_k$ let $\Delta(F_1, F_2, \dots, F_k) = \Delta - \{E \in \Delta \mid E \cap F_i \neq \varnothing \text{ for some } i\}$. In other words, $\Delta(F_1, F_2, \dots, F_k) = \Delta - \bigcup_{i=1}^k \bigcup_{v \in F_i} \mathrm{st}_\Delta v$.

4.8. THEOREM. *Let $\Delta = \Delta(G; B, N)$ be a spherical building, and let $k \leqslant \min_{s \in S} q_s$, k finite. Suppose that $F_1, F_2, \dots, F_k$ are nonempty faces. Then $\Delta(F_1, F_2, \dots, F_k)$ is shellable and* $\dim \Delta(F_1, F_2, \dots, F_k) = \dim \Delta$.

Proof. Let us first show that $\Delta(F) = \Delta(F_1, F_2, \dots, F_k)$ is pure $(|S| - 1)$-dimensional. If $|S| = 1$ this is so because Δ is then a 0-dimensional complex on $q_s + 1$ vertices. Hence we may inductively assume that the statement is true for buildings of rank less than $|S|$. Let E be a nonempty face of $\Delta(F)$. Then,

$$\mathrm{lk}_{\Delta(F)} E = (\mathrm{lk}_\Delta E)(\hat{F}_1, \hat{F}_2, \dots, \hat{F}_k), \text{ where } \hat{F}_i = \{x \in F_i \mid \{x\} \in \mathrm{lk}_\Delta E\}.$$

To see that $\hat{F}_i \in \mathrm{lk}_\Delta E$ one uses Proposition 3.16 of Tits [39]. Now, $\mathrm{lk}_\Delta E \cong \Delta(G_J; B, N_J)$, where $J = S - \tau(E)$ [39, p. 47] and $k \leqslant \min_{s \in J} q_s$, so by the induction hypothesis $\mathrm{lk}_{\Delta(F)} E$ is pure $(|J| - 1)$-dimensional. It follows that the maximal members of $\mathrm{st}_{\Delta(F)} E$ are $(|S| - 1)$-dimensional.

Now, select chambers $C^i \in \mathscr{Ch}(\Delta)$ so that $F_i \subseteq C^i$ for $i = 1, 2, \dots, k$. By Lemma 4.7 these have a common opposite, which without loss of generality can be assumed to be C_e. Hence, $C^i = C_{b_i w_0}$, $b_i \in B$, for $i = 1, 2, \dots, k$. Let $F_1 \cup F_2 \cup \cdots \cup F_k = \{v_1, v_2, \dots, v_r\}$. For each v_i, $1 \leqslant i \leqslant r$, take one F_{j_i} such that $v_i \in F_{j_i}$, suppose $\tau(v_i) = s_i$ and define $g_i B \in G^{(s_i)}/B$ to be the minimal representative in the coset $b_{j_i} w_0 G_{(s_i)}$ (cf. Proposition 3.3). We claim that

$$v_i \in C_g \Leftrightarrow g_i B \leqslant gB.$$

Taking this momentarily for granted, it is clear how to finish the proof. Take a linear extension of the weak ordering of G/B such that the elements in $\{gB \mid gB \geqslant g_i B \text{ for some } i\}$ come last. By Theorem 4.1 this corresponds to a shelling order of $\mathscr{CK}(\Delta)$, and in this order the chambers in $\bigcup_{i=1}^{r} \mathrm{st}_\Delta v_i$ come last. Deleting these chambers we get a shelling of $\Delta(F)$.

To prove the claim one argues

$$v_i \in C_g \Leftrightarrow b_{j_i} w_0 G_{(s_i)} = gG_{(s_i)} \Leftrightarrow g_i B \leqslant gB.$$

The only part which needs further motivation is the backward direction of the last arrow. By definition $g_i B = b_{j_i} w_0^{(s_i)} B$, where $w_0^{(s_i)}$ is the unique greatest element of $W^{(s_i)}$. The statement $g_i B \leqslant gB$ means that there exists a sequence $g_i B = b_1' w_0^{(s_i)} B < b_1' w_1 B = b_2' w_1 B < b_2' w_2 B = b_3' w_2 B < \cdots < b_n' w_n B = gB$, with $b_h' \in B$ and $w_0^{(s_i)} < w_1 < w_2 < \cdots < w_n$ in W. Then $w_0((s_i)) = w_0 w_0^{(s_i)} > w_0 w_1 > w_0 w_2 > \cdots > w_0 w_n$, and from, e.g., the subword property (cf. [7, 2.3] or [16, Theorem 1.1(III)]) one concludes that $w_0 w_h \in W_{(s_i)}$ for $h = 1, 2, \ldots, n$. A consequence is that $w_h^{-1} w_{h-1} \in W_{(s_i)}$ for $h = 2, 3, \ldots, n$, and $w_1^{-1} w_0^{(s_i)} \in W_{(s_i)}$. Hence, $b_{j_i} w_0 G_{(s_i)} = g_i G_{(s_i)} = b_1' w_0^{(s_i)} G_{(s_i)} = b_1' w_1 (w_1^{-1} w_0^{(s_i)}) G_{(s_i)} = b_2' w_1 G_{(s_i)} = b_2' w_2 (w_2^{-1} w_1) G_{(s_i)} = b_3' w_2 G_{(s_i)} = \cdots = b_n' w_n G_{(s_i)} = gG_{(s_i)}$. ∎

4.9. Corollary. *If Δ is finite and $\bigcup_{i=1}^{k} F_i = \{v_1, v_2, \ldots, v_r\}$, then $\Delta(F_1, F_2, \ldots, F_k)$ is shellable of characteristic* $\geqslant q_{w_0} - \sum_{i=1}^{r} q_{w_0(S - \tau(v_i))}$, equality holding if and only if $\mathrm{st}_\Delta v_i \cap \mathrm{st}_\Delta v_j = \varnothing$ for all $i \neq j$.

Proof. The preceding proof shows that $\Delta(F)$ is shellable of characteristic $\mathrm{card}(Bw_0B/B - \{gB \mid gB \geqslant g_i B$ for some $i = 1, 2, \ldots, r\})$. Now $\mathrm{card}(Bw_0 B/B) = q_{w_0}$, and one sees for $i = 1, 2, \ldots, r$ that $\mathrm{card}(Bw_0 B/B \cap \{gB \mid gB \geqslant g_i B\}) = q_{w_0(S - \tau(v_i))}$ as follows. Let as before $w_0^{(s_i)}$ and $w_0((s_i))$ denote the unique greatest elements of $W^{(s_i)}$ and $W_{(s_i)}$, respectively, where $(s_i) = S - s_i$ and $s_i = \tau(v_i)$. Assume that $Bw_0((s_i))B/B = \{v_p B \mid p = 1, 2, \ldots, q_{w_0((s_i))}\}$. Then $g_i v_p \in B w_0^{(s_i)} B w_0((s_i)) B = Bw_0 B$ and $v_p \in G_{(s_i)}$. Hence, $\{g_i v_p B \mid p = 1, 2, \ldots, q_{w_0((s_i))}\} \subseteq (Bw_0 B/B \cap \{gB \mid gG_{(s_i)} = g_i G_{(s_i)}\}) = (Bw_0 B/B \cap \{gB \mid gB \geqslant g_i B\})$. The reverse inclusion is easily established by a counting argument using that $g_i v_p B \neq g_i v_{p'} B$ when $p \neq p'$ and that $q_w q_{w'} = q_{w_0}$ when $w = w_0^{(s_i)}$ and $w' = w_0((s_i))$ (cf. Lemma 3.1(ii)). ∎

The preceding method can easily be adjusted to prove other similar results. For instance, under the hypotheses of Theorem 4.8 the complex $\Delta' = \Delta - \bigcup_{i=1}^{k} \mathrm{st}_\Delta F_i$ is pure $(\dim \Delta)$-dimensional and shellable, and if Δ is finite Δ' is of characteristic $\geqslant q_{w_0} - \sum_{i=1}^{k} q_{w_0(S - \tau(F_i))}$.

In the sequel only the case where the deleted faces F_i are vertices will be of interest. For simplicity, if Δ is a finite complex on vertex set V and $U=\{u_1, u_2,..., u_r\}\subseteq V$ let us write just $\Delta - U$ for the subcomplex $\Delta(u_1, u_2,..., u_r)=\{F\in\Delta \mid F\not\ni u_i \text{ for } i=1,2,..., r\}$. Given an integer $k\geqslant 1$ we shall say, following Baclawski, that Δ is *k-Cohen–Macaulay* (over the field $\mathbf{k}$) if $\dim(\Delta - U)=\dim\Delta$ and the Stanley–Reisner ring $\mathbf{k}[\Delta - U]$ is Cohen–Macaulay for all $U\subseteq V$ such that $|U|\leqslant k-1$. Baclawski observes in [1] that the k-CM condition for $k\geqslant 2$ makes it possible to drastically simplify Hochster's expression for the last $k-1$ Betti numbers of $\mathbf{k}[\Delta]$, and he proves in [2] that it implies a simple description of the canonical module of $\mathbf{k}[\Delta]$ as an embedded ideal. We will discuss what these results say about the face ring of a finite building after quoting the general results in more detail.

Suppose Δ is a finite $(d-1)$-dimensional complex on a vertex set V of cardinality n. Let $\mathbf{k}$ be a field, and $\mathbf{k}[\Delta]$ the face ring (cf. Section 1, part (C)). For $i\geqslant 0$ the *Betti number* $\beta_i(\mathbf{k}[\Delta])$ is the rank of the ith resolvant in a minimal free graded resolution of $\mathbf{k}[\Delta]$ over $\mathbf{A}=\mathbf{k}[x_i \mid x_i\in V]$. Equivalently, $\beta_i(\mathbf{k}[\Delta])=\dim_{\mathbf{k}}\operatorname{Tor}_i^{\mathbf{A}}(\mathbf{k}[\Delta],\mathbf{k})$. Hochster has given the formula [18, p. 194]

$$\beta_{n-d-i}(\mathbf{k}[\Delta])=\sum_{U\subseteq V}\dim_{\mathbf{k}}\tilde{H}_{d-1+i-|U|}(\Delta - U,\mathbf{k}).$$

Let us write $\tilde{h}_j(\Delta)=\dim_{\mathbf{k}}\tilde{H}_j(\Delta,\mathbf{k})$. For dimensional reasons, if $i\geqslant 0$ the formula reduces to

$$\beta_{n-d-i}=\sum_{\substack{U\subseteq V\\ |U|=i}}\tilde{h}_{d-1}(\Delta-U)+\sum_{\substack{U\subseteq V\\ |U|>i}}\tilde{h}_{d-1+i-|U|}(\Delta-U). \tag{4.10}$$

Now suppose that $\mathbf{k}[\Delta]$ is Cohen–Macaulay. This is equivalent on the one hand to $\beta_{n-d+1}(\mathbf{k}[\Delta])=0$, and on the other (by a theorem of Reisner [24]) to $\tilde{H}_j(\operatorname{lk}_\Delta F,\mathbf{k})=0$ for all $F\in\Delta$ and all $j<\dim(\operatorname{lk}_\Delta F)$. For any vertex v let $\Delta_v=\{F\in\Delta \mid F\cup v\in\Delta\}$. Then $\Delta=(\Delta - v)\cup\Delta_v$ and $(\Delta - v)\cap\Delta_v=\operatorname{lk}_\Delta v$, so a Mayer–Vietoris exact sequence shows that $\tilde{H}_i(\Delta - v,\mathbf{k})=0$ for $i\leqslant d-3$. By repeated application of the same argument one finds that $\tilde{H}_{d-1-|U|-j}(\Delta - U,\mathbf{k})=0$ for all $U\subseteq V$ and all $j\geqslant 1$.

Finally, suppose that Δ is k-CM, $k\geqslant 2$. Then if $U'\subseteq V$, $|U'|\leqslant k-1$, it follows from the last paragraph that $U'\subseteq U\subseteq V$ and $j\geqslant 1$ imply $\tilde{h}_{d-1-|U-U'|-j}(\Delta - U)=0$. In particular, $0\leqslant i\leqslant k-2$ and $|U|>i$ imply $\tilde{h}_{d-1+i-|U|}(\Delta - U)=0$. Consequently, formula (4.10) simplifies to

$$\beta_{n-d-i}=\sum_{\substack{U\subseteq V\\ |U|=i}}\tilde{h}_{d-1}(\Delta - U) \qquad \text{for} \quad i=0,1,..., k-2. \tag{4.11}$$

This expression was obtained by Baclawski as part of the proof of Theorem 4.5 of [1].

Now, let G be a finite group with BN-pair and with building $\Delta = \Delta(G; B, N)$. Define $q_G = \min_{s \in S} q_s$. For instance, if G is a finite Chevalley group over $GF(q)$ then $q_G = q$. The following is a consequence of Theorem 4.8.

4.12. COROLLARY. *Δ is $(q_G + 1)$-Cohen–Macaulay.*

This result is sharp in the sense that Δ is not $(q_G + 2)$-CM. To see this, choose $s \in S$ so that $q_s = q_G$ and let F be a wall of type (s). Then $\mathrm{lk}_\Delta F$ consists of $q_s + 1$ vertices, and if these are removed from Δ the remaining complex is no longer pure, hence not CM.

In view of Corollary 4.9 and formula (4.11) it is in principle possible to express the q_G last Betti numbers of the face ring of a building in terms of indices $q_{w_0(J)}$ related to the top elements of parabolic subgroups. Letting $n = \sum_{s \in S} [G: G_{(s)}]$, one obtains

$$\begin{aligned} \beta_{n-|S|}(\mathbf{k}[\Delta]) &= q_{w_0}, \\ \beta_{n-|S|-1}(\mathbf{k}[\Delta]) &= \sum_{s \in S} [G: G_{(s)}](q_{w_0} - q_{w_0((s))}). \end{aligned} \tag{4.13}$$

If $q_G > 2$ the remaining Betti numbers $\beta_{n-|S|-i}$, $i = 2, 3, \ldots, q_G - 1$, have increasingly awkward expressions, e.g.,

$$\begin{aligned} \beta_{n-|S|-2}(\mathbf{k}[\Delta]) = \sum_{s \in S} & \binom{|G: G_{(s)}|}{2} (q_{w_0} - 2q_{w_0((s))}) \\ & + \sum_{\substack{s,t \in S \\ s \neq t}} (|G: G_{(s)}||G: G_{(t)}|(q_{w_0} - q_{w_0((s))} - q_{w_0((t))}) \\ & + |G: G_{S-\{s,t\}}| \, q_{w_0(S-\{s,t\})}). \end{aligned} \tag{4.14}$$

Let once more Δ be a finite $(d-1)$-dimensional CM complex on vertex set V, $|V| = n$. Then $\mathrm{Ext}^i_{\mathbf{A}}(\mathbf{k}[\Delta], \mathbf{A}) = 0$ for $i \neq n - d$, and $\Omega(\mathbf{k}[\Delta]) = \mathrm{Ext}^{n-d}_{\mathbf{A}}(\mathbf{k}[\Delta], \mathbf{A})$ is known as the *canonical module* of $\mathbf{k}[\Delta]$. For a general discussion of canonical modules of CM face rings see Stanley [37]. If Δ is balanced with a corresponding system of parameters $\theta = \{\theta_s \mid s \in S\}$ (as in Section 1, part (C)) then $\Omega(\mathbf{k}[\Delta]) \cong \mathrm{Hom}_{\mathbf{k}[\theta]}(\mathbf{k}[\Delta], \mathbf{k}[\theta])$. Baclawski uses this in [2] to prove that if Δ is 2-CM and balanced then $\Omega(\mathbf{k}[\Delta]) \cong I \subseteq \mathbf{k}[\Delta]$, where I is the homogeneous ideal generated by $\tilde{H}_{d-1}(\Delta, \mathbf{k})$. Notice that the elements of $\tilde{H}_{d-1}(\Delta, \mathbf{k})$, being $\mathbf{k}$-linear combinations of chambers C, can be considered as elements in $\mathbf{k}[\Delta]$ by passing to the corresponding monomials $x(C)$.

Now let $\Delta = \Delta(G; B, N)$ be a finite building. By combining Proposition 4.5, Theorem 4.8, and Baclawski's result, one obtains the following concrete characterization of the canonical module of a building.

4.15. PROPOSITION. *Let* $C \in \mathscr{C}\!h(\Delta)$ and $A_C = \{\Sigma \in \mathscr{A}p(\Delta) \mid C \in \Sigma\}$.

Then $\Omega(\mathbf{k}[\Delta])$ is isomorphic with the ideal I in $\mathbf{k}[\Delta]$ which is generated by the elementary cycles ρ_Σ, $\Sigma \in A_C$. Furthermore, this is a minimal generating set for I.

4.16. *Remark.* Let Δ be a finite building. The map $\mathscr{A}\mathit{p}(\Delta) \to \tilde{H}_{|S|-1}(\Delta, \mathbb{Z})$ given by $\Sigma \to \rho_\Sigma$ induces a structure of *simple matroid* $\mathbf{Ap}(\Delta)$ on the set $\mathscr{A}\mathit{p}(\Delta)$ by linear independence of elementary cycles ρ_Σ. (For matroid theory see, e.g., Welsh [43].) The situation is analogous to that of the basis geometry of a geometric lattice [5, Sect. 6]. Proposition 4.5 shows that A_C, i.e., the collection of apartments through an arbitrary fixed chamber C, is a matroid base, and that the standard matrix representation of $\mathbf{Ap}(\Delta)$ with respect to such a base is a matrix with entries 0, $+1$, and -1. It is natural to ask whether $\mathbf{Ap}(\Delta)$ is a unimodular geometry, i.e., whether all minors of such a matrix equal 0, $+1$, -1? By a method similar to [5, Proposition 6] one can show that $\mathbf{Ap}(\Delta)$ is 2-partitionable, a somewhat weaker property.

4.17. *Remark.* Since the paper [25] by Rota there has been a continuing interest in the Möbius function of a poset (partially ordered set) and its role in combinatorial mathematics. For some classes of sufficiently structured posets one can find expressions for the Möbius function of all rank-selected subposets, see, e.g., Stanley [33, 35] and also [4]. The results of this paper imply that this is the case for all face-lattices of regular convex polytopes and for certain q-analogues which are subspace lattices of projective, polar, and metasymplectic spaces. Let us sketch the connection. The reader is referred to the cited sources for further background.

Given a poset P let the *order complex* $\Delta(P)$ be the simplicial complex of all finite chains $x_0 < x_1 < \cdots < x_k$ in P. The following result was obtained jointly with Wachs (and the "if" part also independently by Surowski [38]).

4.18. Proposition. *Let Δ be a Coxeter complex or building of finite rank. Then $\Delta \cong \Delta(P)$ for some poset P if and only if the corresponding Coxeter diagram is linear.*

The finite irreducible Coxeter groups whose diagrams are linear (i.e., whose nodes can be numbered $1, 2, \ldots, d$ in such a way that i is adjacent to j only if $|i-j| = 1$) are the symmetry groups of regular polytopes. If Δ is the Coxeter complex of such a group and $\bar{L} = L - \{\hat{0}, \hat{1}\}$ is the proper part of the face-lattice L of the polytope, then $\Delta \cong \Delta(\bar{L})$. Furthermore, the balancing of Δ by the nodes of the linear Coxeter diagram taken in natural linear order corresponds to the balancing of $\Delta(\bar{L})$ by dimension. Since the Möbius function μ of a poset is the reduced Euler characteristic of its order complex one deduces the following formulas (4.19) and (4.20) from Theorems 2.1, 4.1, and their Corollaries.

Let L be the face-lattice of a d-dimensional regular convex polytope $\mathscr{P}$, and for $J \subseteq [d] = \{1, 2, \ldots, d\}$ consider the rank-selected subposet $L_J = \{x \in L \mid (\dim x + 1) \in \{0, d+1\} \cup J\}$. Let W be the symmetry group of $\mathscr{P}$ and S a set of Coxeter generators. By naturally ordering the nodes of the linear Coxeter diagram one may identify S with $[d]$. Then the Möbius function of L_J, $J \subseteq [d]$, has the expression:

$$\mu_{L_J}(\hat{0}, \hat{1}) = (-1)^{|J|+1} \operatorname{card} \mathscr{D}_J = \sum_{I \cup J = [d]} (-1)^{d+1+|I|} [W: W_I]. \quad (4.19)$$

The finite buildings of irreducible linear diagram type are those of type A_d, C_d, F_4, or $G_2^{(m)}$, $m = 6$ or 8. They are all isomorphic with the order complexes of proper parts of certain lattices. In particular, if Δ is a finite building of type A_d, C_d, or F_4 then $\Delta \cong \Delta(\bar{L})$, where L is the subspace lattice of a d-dimensional projective or polar space or a metasymplectic space, respectively. Such spaces are discussed in Tits [39].

For simplicity we assume from now on that G is a finite Chevalley group over $GF(q)$ of type A_d, C_d, F_4, or G_2, and that Δ and L are the corresponding building and lattice, such that $\Delta \cong \Delta(\bar{L})$. Let (W, S) be the Weyl group of G. As before, one may in a natural way identify S with $[d] = \{1, 2, \ldots, d\}$. Defining as before the rank-selected subposet L_J, $J \subseteq [d]$, one obtains the formula:

$$\mu_{L_J}(\hat{0}, \hat{1}) = (-1)^{|J|+1} \sum_{w \in \mathscr{D}_J} q^{l(w)} = \sum_{I \cup J = [d]} (-1)^{d+1+|I|} [G: G_I]. \quad (4.20)$$

Stanley has pointed out that for lattices of type A_d, i.e., for the subset lattice of a $(d+1)$-set and the subspace lattice of a $(d+1)$-dimensional vector space over $GF(q)$, the expressions (4.19) and (4.20) in terms of descent classes $\mathscr{D}_J$ are obtainable by the method of supersolvable lattices [33], (cf. [35, Eq. (13)]).

4.21. *Remark.* It is straightforward to extend Definition 1.1 and the relevant parts of the theory of shellable complexes to arbitrary cardinality as follows. Let Δ be a pure d-dimensional complex. A *shelling* is a well-ordering of $\mathscr{C}\!h(\Delta) = \{C_\alpha \mid \alpha \in A\}$ such that $(\bigcup_{\beta<\alpha} \bar{C}_\beta) \cap \bar{C}_\alpha$ is pure $(d-1)$-dimensional for all elements α in A except the first. Let $\Delta_{<\alpha} = \bigcup_{\beta<\alpha} \bar{C}_\beta$ and define $\mathscr{R}(C_\alpha) = \{x \in C_\alpha \mid C_\alpha - \{x\} \in \Delta_{<\alpha}\}$. Now Proposition 1.2 goes through after only notational adjustment, and Theorems 1.3 and 1.6 remain valid. In the proof of Theorem 1.3 one may need to use transfinite induction and the fact that an increasing union of contractible complexes is contractible to show that Δ^* is contractible. The proof of Theorem 1.6 is unchanged.

The extended notion of shellability in terms of well-orderings applies very naturally to buildings and also to some other complexes of combinatorial interest, e.g., the order complexes of infinite geometric lattices and the independence and broken circuit complexes of infinite matroids (cf. [4, 5]).

For infinite buildings the proofs of Theorems 4.1 and 4.8 depend on the possibility of finding linear extensions of the weak ordering of G/B. A sufficient supply of such extensions can be obtained by variations of the following principle: well-order each family BwB/B, $w \in W$, and then arrange these families in sequence according to some linear extension of the weak ordering of W.

There is another generalized notion of shellability which is satisfactory for infinite buildings, and which avoids transfinite induction. Suppose Δ is a balanced complex on vertex set $V = \dot{\bigcup}_{s \in S} V_s$. A *multishelling* is a sequence $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3, \ldots,$ of families of chambers together with a map $\mathscr{R}: \mathscr{C}\!h\,(\Delta) \to \Delta$ such that (letting $\Delta_k = \bigcup_{i=1}^{k} \bigcup_{C \in \mathscr{F}_i} \bar{C}$, $\Delta_0 = \varnothing$):

(i) $\mathscr{C}\!h\,(\Delta) = \dot{\bigcup}_{k \geqslant 1} \mathscr{F}_k$, and $|\mathscr{F}_1| = 1$,

(ii) if $F \subseteq C \in \mathscr{F}_k$, $k \geqslant 1$, then $F \in \Delta_{k-1} \Leftrightarrow F \not\supseteq \mathscr{R}(C)$,

(iii) if $C, C' \in \mathscr{F}_k$, $C \neq C'$, then $\mathscr{R}(C) \neq \mathscr{R}(C')$ and $\tau(\mathscr{R}(C)) = \tau(\mathscr{R}(C'))$.

One recognizes the kind of shelling discussed in Section 1 as the special case when $|\mathscr{F}_k| = 1$ for all k. The added generality here is that we now allow arbitrarily many new chambers (topologically: cells) to be attached at each step. Define Δ_k^* to be Δ_k minus all chambers C such that $\mathscr{R}(C) = C$. Then $\|\Delta_{k-1}^*\|$ is a strong deformation retract of $\|\Delta_k^*\|$, since the cells $\|\bar{C}_\alpha\|$, $C_\alpha \in \mathscr{F}_k \cap \Delta_k^*$, can be retracted simultaneously and independently (because of (iii)). Hence, the proof of Theorem 1.3 goes through, and under these conditions it makes sense to say that Δ is *multishellable of characteristic* card$\{C \in \mathscr{C}\!h\,(\Delta) \mid \mathscr{R}(C) = C\}$. One then proves the following form of Proposition 1.2:

An ordered sequence $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3, \ldots$ of families of chambers and a map $\mathscr{R}: \mathscr{C}\!h\,(\Delta) \to \Delta$ such that conditions (i) and (iii) above are valid is a multishelling if and only if (α) $\Delta = \dot{\bigcup}_{i \geqslant 1} \dot{\bigcup}_{C \in \mathscr{F}_i} [\mathscr{R}(C), C]$ and (β) $\mathscr{R}(C) \subseteq C'$, $C \in \mathscr{F}_i$, $C' \in \mathscr{F}_j \Rightarrow i \leqslant j$, for all $i, j \geqslant 1$. Using this, one proves Theorem 1.6 as before.

Now, suppose $\Delta = \Delta(G; B, N)$ is a building of finite rank $|S|$. For $w \in W$ let $\mathscr{F}_w = \{C_g \mid g \in BwB\}$, and for $C_g \in \mathscr{F}_w$ let $\mathscr{R}(C_g)$ be its face of type $\mathscr{D}(w)$. Then give the countable collection $\{\mathscr{F}_w \mid w \in W\}$ a simple linear ordering such that $w < w'$ implies $\mathscr{F}_w < \mathscr{F}_{w'}$. Condition (i) and the $\tau(\mathscr{R}(C)) = \tau(\mathscr{R}(C'))$ part of condition (iii) in the definition of a multishelling are immediate. The details of the proof of Theorem 4.1 settle condition (ii). We are left to verify only that if $C_{bw}, C_{b'w} \in \mathscr{F}_w$, $b, b' \in B$, and $\mathscr{R}(C_{bw}) = \mathscr{R}(C_{b'w})$, then $C_{bw} = C_{b'w}$. If $bwG_{(s)} = b'wG_{(s)}$ for all $s \in \mathscr{D}(w)$, then $bwG_{S - \mathscr{D}(w)} = b'wG_{S - \mathscr{D}(w)}$, and hence by Proposition 3.3,

$bwB = b'wB$, which was to be shown. Finally, since $|\mathscr{F}_w| = q_w$ we get $\operatorname{card}\{C \in \mathscr{Ch}(\Delta) \mid \tau(\mathscr{R}(C)) = J\} = \sum_{w \in \mathscr{D}_J} q_w$.

5. Type-Selected Subcomplexes

In the preceding sections certain properties of type-selected subcomplexes of Coxeter complexes and buildings have been obtained as byproducts of results for the full complexes. Taking a closer look one finds that these subcomplexes have an interesting combinatorial structure of their own, which in fact makes them reminiscent of weak buildings. In particular, they possess a natural class of apartments. For simplicity we will formulate the following discussion mostly in terms of Coxeter complexes.

Let $\Delta = \Delta(W, S)$ be a Coxeter complex. For $J \subseteq S$, write $\hat{J} = S - J$ and $(C_w)_J = \{wW_{(s)} \mid s \in J\}$. The group W acts transitively on $\mathscr{Ch}(\Delta_J)$ and the fundamental chamber $(C_e)_J$ is stabilized by $W_{\hat{J}}$. Hence, $(C_w)_J = (C_{w'})_J$ if and only if $wW_{\hat{J}} = w'W_{\hat{J}}$, and the chambers of Δ_J are naturally indexed by $W/W_{\hat{J}}$ or $W^{\hat{J}}$. In particular, if $C \in \mathscr{Ch}(\Delta_J)$ then $C = (C_w)_J$ for a unique $w \in W^{\hat{J}}$, for which we write $w = w(C)$.

The induced shellings of Δ_J have a simple direct description which we want to mention. Given a shelling of Δ, the proof of Theorem 1.6 shows how to obtain one of Δ_J: the map $\mathscr{E}: \mathscr{Ch}(\Delta_J) \to \mathscr{Ch}(\Delta)$ which sends $C \in \mathscr{Ch}(\Delta_J)$ to the first element of $\{C' \in \mathscr{Ch}(\Delta) \mid C' \supseteq C\}$ is injective and via $\mathscr{E}$ the shelling order of $\mathscr{Ch}(\Delta)$ induces one of $\mathscr{Ch}(\Delta_J)$ (for a simple direct proof of this see [4, p. 170]). One can then from Theorem 2.1 deduce the following.

5.1. Theorem. *Any linear extension of the weak ordering of $W^{\hat{J}}$ assigns a shelling order to $\mathscr{Ch}(\Delta_J)$ with restriction map $\mathscr{R}(C) = (C)_{\mathscr{D}(w(C))}$.*

Let Δ be a Coxeter complex as before. For $F \in \mathscr{Ch}(\Delta_{\hat{J}})$ define $\Sigma_F = \operatorname{lk}_\Delta F$. Then on the one hand $\Sigma_F \cong \Delta(W_J, J)$ (cf. [39, p. 19]) and on the other Σ_F is a subcomplex of Δ_J. We will call such subcomplexes *apartments* of Δ_J and write $\mathscr{Ap}(\Delta_J) = \{\Sigma_F \mid F \in \mathscr{Ch}(\Delta_{\hat{J}})\}$. The map $F \to \Sigma_F$ is, in general, a many-to-one correspondence $\mathscr{Ch}(\Delta_{\hat{J}}) \to \mathscr{Ap}(\Delta_J)$. However, it is in many cases one-to-one, and at least for the complexes Δ which come from posets (cf. Proposition 4.18) it is easy to characterize when this occurs. In any case, there is a kind of "duality" between chambers and apartments in the two "dual" complexes Δ_J and $\Delta_{\hat{J}}$. Let us formally state some basic combinatorial properties of the pair $(\Delta_J, \mathscr{Ap}(\Delta_J))$.

5.2. Proposition. (i) *Δ_J is a shellable and balanced complex.*

(ii) *Every $\Sigma \in \mathscr{Ap}(\Delta_J)$ is a Coxeter complex of fixed type.*

(iii) *Given* $C, C' \in \mathscr{Ch}(\Delta_J)$ *there exist* $C = C_0, C_1, \ldots, C_k = C'$ *in* $\mathscr{Ch}(\Delta_J)$ *and* $\Sigma_1, \Sigma_2, \ldots, \Sigma_k$ *in* $\mathscr{A}p(\Delta_J)$ *such that* $C_{i-1}, C_i \in \Sigma_i$ *for* $i = 1, 2, \ldots, k$.

(iv) *There exists a group of type-preserving automorphisms of* Δ_J *which is transitive on pairs* $(C, \Sigma) \in \mathscr{Ch}(\Delta_J) \times \mathscr{A}p(\Delta_J)$ *such that* $C \in \Sigma$.

Proof. Parts (i) and (ii) are clear, and it is easy to see that W acting on Δ_J has property (iv).

Recall that given $I \subseteq S$ every $w \in W$ can be uniquely represented on the form $w = uv$, $u \in W^I$, $v \in W_I$. Define $\pi_I(w) = u$. For (iii) we may assume that $C' = (C_e)_J$. Let $w_1 = w(C)$. Define $w_2 = \pi_J(w_1)$, $w_3 = \pi_{\hat{J}}(w_2)$, $w_4 = \pi_J(w_3)$, $w_5 = \pi_{\hat{J}}(w_4)$, and so on. Then $w_1 \geqslant w_2 \geqslant w_3 \geqslant \cdots$, and if $w_j = w_{j+1}$ then $w_j = e$. This is because $w_j \in W^J$ and $w_{j+1} \in W^{\hat{J}}$ or conversely, and $W^J \cap W^{\hat{J}} = \{e\}$. Since $(C_{w_{2i}})_J$ and $(C_{w_{2i-1}})_J$ both lie in Σ_{F_i}, where $F_i = (C_{w_{2i}})_{\hat{J}}$, and $(C_{w_{2i}})_J = (C_{w_{2i+1}})_J$ for all $i \geqslant 1$, property (iii) follows. ∎

It is a consequence of (iv) that (v) if $\Sigma, \Sigma' \in \mathscr{A}p(\Delta_J)$ and $\Gamma = \Sigma \cap \Sigma' \cap \mathscr{Ch}(\Delta_J)$ then given any $C \in \Gamma$ there exists a type-preserving isomorphism $\Sigma \to \Sigma'$ which leaves C invariant. We do not know whether, like in a building, the same is true with respect to any pair $C, C' \in \Gamma$. It can be shown that if $\Sigma \neq \Sigma'$ and $0 < \operatorname{card} \Gamma < \infty$ then Γ is the set of maximal faces of a shellable triangulation of the $(|J| - 1)$-ball (the same is true for apartments in a building). Tits has pointed out [39, p. 38] that a system for which (i), (ii), (iii) with $k = 1$, and a certain strengthening of (v), hold, is a weak building having many of the properties of a building. The ways in which the pair $(\Delta_J, \mathscr{A}p(\Delta_J))$ of a type-selected Coxeter subcomplex fails to be a building is in a sense analogous to the ways in which the pair $(W_{\hat{J}}, W_J)$ fails to be a BN-pair in W.

5.3. Lemma. *Suppose* $\Delta = \Delta(W, S)$, $J \subseteq S$, *and* $\Sigma \in \mathscr{A}p(\Delta_J)$.

(i) *There exists a unique chamber* $C_\Sigma \in \mathscr{Ch}(\Sigma)$ *such that* $w(C_\Sigma) < w(C)$ *under Bruhat order, for all other* $C \in \mathscr{Ch}(\Sigma)$.

(ii) *If* W_J *is finite there exists a unique chamber* $C^\Sigma \in \mathscr{Ch}(\Sigma)$ *such that* $w(C^\Sigma) > w(C)$ *under Bruhat order, for all other* $C \in \mathscr{Ch}(\Sigma)$. *Furthermore,* $w(C^\Sigma) \in \mathscr{D}_J$.

Proof. Recall that the projection map $\pi_I\colon W \to W^I$ is order-preserving with respect to Bruhat order (Deodhar [16, Lemma 3.5]). Suppose $\Sigma = \Sigma_F$ for $F \in \mathscr{Ch}(\Delta_{\hat{J}})$. Then $\{w \in W \mid C_w \supseteq F\} = w' W_J$ for a unique $w' \in W^J$, and so $\mathscr{Ch}(\Sigma) = \{(C_w)_J \mid w \in w' W_J\}$. Since $(C_w)_J = (C_{\pi_{\hat{J}}(w)})_J$ and $w' \leqslant w$ for all $w \in w' W_J$, it follows that $C_\Sigma = (C_{\pi_{\hat{J}}(w')})_J$ satisfies (i). Similarly, for the first part of (ii) the choice $C^{\Sigma} = (C_{\pi_{\hat{J}}(w'')})_J$ will do, where $w'' = w'w_0(J) \in {}^JW$. It remains to show that $\pi_{\hat{J}}(w'') \in \mathscr{D}_J$.

Let $u'' = \pi_{\hat{J}}(w'')$. Since $u'' \in W^{\hat{J}}$ we have $\mathscr{D}(u'') \subseteq J$. Suppose $\mathscr{D}(u'') \neq J$ and let $s \in J - \mathscr{D}(u'')$. Since $w'' \in {}^JW$ we have $s \in \mathscr{D}(w'')$. Now, using well-known properties of Bruhat order (cf. Deodhar [16, Theorem 1.1]), first the hypotheses $u'' \leqslant w''$, $u''s > u''$, and $w''s < w''$ imply that $u''s \leqslant w''$, and then if $w'' = u''v''$ and $u'' = s_1 s_2 \cdots s_k$ and $v'' = s'_1 s'_2 \cdots s'_q \in W_{\hat{J}}$ are reduced expressions then $u''s$ can be obtained as a reduced subword $u''s = (s_{i_1} s_{i_2} \cdots s_{i_d})(s'_{j_1} s'_{j_2} \cdots s'_{j_e})$, $1 \leqslant i_1 < i_2 < \cdots < i_d \leqslant k$, $1 \leqslant j_1 < j_2 < \cdots < j_e \leqslant q$. Furthermore, since $u'' < u''s$ it is possible to delete some letter in this reduced expression for $u''s$ and get one for u''. If $d < k$, such a reduced expression for u'' would end in an $s'_j \in \hat{J}$, contradicting the fact that $\mathscr{D}(u'') \cap \hat{J} = \varnothing$. If $d = k$, then $u''s = u''s'_j$ for some $s'_j \in \hat{J}$ contradicting the fact that $s \notin \hat{J}$. Hence, $\mathscr{D}(u'') = J$. ∎

Let us now assume that $\Delta = \Delta(W, S)$, $J \subseteq S$, and that W_J is finite. We know from Theorem 2.1 and Corollary 1.4 that $\tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z})$ is a free Abelian group of rank $|\mathscr{D}_J|$, and it is natural to expect to find a basis for homology in terms of the spherical apartments.

For $\Sigma \in \mathscr{A}\!\mathscr{h}(\Delta_J)$ let ρ_Σ be the fundamental cycle of the spherical complex $\Sigma \cong \Delta(W_J, J)$. Thus, ρ_Σ is a nonzero element of $\tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z})$ uniquely determined up to sign. Specifically, if $\mathscr{C}\!\mathscr{h}(\Sigma) = \{(C_w)_J \mid w \in uW_J\}$, $u \in W^J$, then $\rho_\Sigma = \sum_{v \in W_J} (-1)^{l(v)} (C_{uv})_J$. In analogy with the case of buildings, cycles of the form ρ_Σ will be called *elementary*.

The map $\psi: \mathscr{A}\!\mathscr{h}(\Delta_J) \to \mathscr{D}_J$ defined by $\psi(\Sigma) = w(C^{\Sigma})$ is well defined (by Lemma 5.3) and clearly surjective.

5.4 THEOREM. *Let A be a system of representatives of the sets $\psi^{-1}(w)$, $w \in \mathscr{D}_J$. Then the elementary cycles $\{\rho_\Sigma \mid \Sigma \in A\}$ form a basis for $\tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z})$.*

Proof. Consider a linear combination $\sigma = \sum_{i=1}^k n_i \rho_{\Sigma_i}$ for which $n_i \in \mathbb{Z} - \{0\}$, $\Sigma_i \in A$, $i = 1, 2, \ldots, k$. If $w(C^{\Sigma_j})$ is a maximal element in the Bruhat ordering of the set $\{w(C^{\Sigma_i}) \mid i = 1, 2, \ldots, k\}$, then from Lemma 5.3(ii) one sees that ρ_{Σ_i} has a nonzero C^{Σ_j}-term if and only if $i = j$. Hence, $\sigma \neq 0$.

Let $\rho \in \tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z})$, and define $D(\rho) = \{w \in \mathscr{D}_J \mid \rho$ has a nonzero $(C_w)_J$-term$\}$. If $D(\rho) \neq \varnothing$ take a maximal element $w_1 \in D(\rho)$ and let Σ_1 be A's representative from the set $\psi^{-1}(w_1)$. Then choose $n_1 \in \mathbb{Z}$ so that $w_1 \notin D(\rho - n_1 \rho_{\Sigma_1})$. Lemma 5.3(ii) shows that the process can be continued until for a sequence $\Sigma_1, \Sigma_2, \ldots, \Sigma_k \in A$ and $n_1, n_2, \ldots, n_k \in \mathbb{Z}$ one gets $D(\rho - \sum_{i=1}^k n_i \rho_{\Sigma_i}) = \varnothing$. Then $\rho - \sum_{i=1}^k n_i \rho_{\Sigma_i}$ is a cycle in the contractible subcomplex $\Delta_J^* = \Delta_J - \{(C_w)_J \mid w \in \mathscr{D}_J\}$ and hence equals zero. ∎

The development of an investigation into the structure of type-selected subcomplexes of buildings runs parallel to the preceding discussion, so to avoid tedious repetition we shall merely mention a few key points and state the results. For the rest of this section let $\Delta = \Delta(G; B, N)$ be the building of a BN-pair G with Weyl group (W, S), and let $J \subseteq S$. The chambers of Δ_J are naturally indexed by $G^{\hat{J}}/B$: if $C \in \mathscr{C}h\,(\Delta_J)$ then $C = (C_{gB})_J = \{gG_{(s)} \mid s \in J\}$ for a unique $gB \in G^{\hat{J}}/B$, and we will write $gB = \theta(C)$ and $w = w(C)$, where $w \in W$ is uniquely determined by $gB = bwB$, $b \in B$, $w \in W$. The weak partial ordering is inherited by $G^{\hat{J}}/B$ as a subset of G/B.

5.5. THEOREM. *Any linear extension of the weak ordering of* $G^{\hat{J}}/B$ *assigns a shelling order to* $\mathscr{C}h\,(\Delta_J)$ *with restriction map* $\mathscr{R}(C) = (C)_{\mathscr{D}(w(C))}$.

For any $F \in \mathscr{C}h\,(\Delta_{\hat{J}})$ we have that $\mathrm{lk}_\Delta F \cong \Delta(G_J; B, N_J)$, so $\mathrm{lk}_\Delta F$ is a building whose apartments are Coxeter complexes of type (W_J, J). Let $\mathscr{A}p(\Delta_J) = \{\Sigma \mid \Sigma \in \mathscr{A}p(\mathrm{lk}_\Delta F),\ F \in \mathscr{C}h\,(\Delta_{\hat{J}})\}$, and consider this the class of *apartments* in Δ_J. The pair $(\Delta_J, \mathscr{A}p(\Delta_J))$ of a type-selected building satisfies all the properties of Proposition 5.2. Part (iv) follows from the corresponding property for G acting on $(\Delta, \mathscr{A}p(\Delta))$. To prove (iii) one uses the projection operator $\pi_I: G/B \to G^I/B$, $I \subseteq S$, which sends gB to the minimal representative of the coset gG_I (cf. Proposition 3.3), and argues essentially as before.

If $F \in \mathscr{C}h\,(\Delta_{\hat{J}})$ then $F = (C_{gB})_{\hat{J}}$ for a unique $gB \in G^J/B$ for which we write $gB = \hat{\theta}(F)$. Define the class of *special apartments* as follows:

$$\mathscr{S}\mathscr{A}p(\Delta_J) = \{\Sigma \mid \Sigma \in \mathscr{A}p(\mathrm{lk}_\Delta F),\ (C_{\hat{\theta}(F)})_J \in \Sigma,\ F \in \mathscr{C}h\,(\Delta_{\hat{J}})\}.$$

For example, $\mathscr{S}\mathscr{A}p(\Delta) = \{\Sigma \in \mathscr{A}p(\Delta) \mid C_e \in \Sigma\}$. Then Lemma 5.3 has the following counterpart. Suppose that $\Sigma \in \mathscr{S}\mathscr{A}p(\Delta_J)$.

(i) Then there exists a unique chamber $C_\Sigma \in \mathscr{C}h\,(\Sigma)$ such that $\theta(C_\Sigma) < \theta(C)$ in the strong ordering of G/B for all chambers $C \neq C_\Sigma$ in $\mathscr{C}h\,(\Sigma)$.

(ii) If W_J is finite there exists a unique chamber $C^\Sigma \in \mathscr{C}h\,(\Sigma)$ such that $\theta(C^\Sigma) > \theta(C)$ in the strong ordering for all $C \neq C^\Sigma$ in $\mathscr{C}h\,(\Sigma)$, and furthermore $\theta(C^\Sigma) \in B\mathscr{D}_J B/B$.

The proof is based on Lemma 5.3 and uses the fact that the projection map $\pi_I: G/B \to G^I/B$ is order-preserving for the strong ordering.

Assume from now on that W_J is finite. Then each $\Sigma \in \mathscr{A}p(\Delta_J)$ triangulates the $(|J|-1)$-sphere and hence provides an *elementary cycle* $\rho_\Sigma \in \tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z})$, uniquely determined up to sign. The map $\psi: \mathscr{S}\mathscr{A}p(\Delta_J) \to B\mathscr{D}_J B/B$ given by $\psi(\Sigma) = \theta(C^\Sigma)$ is well defined and surjective.

5.6. THEOREM. *Let A be a system of representatives of the sets $\psi^{-1}(gB)$, $gB \in B\mathscr{D}_J B/B$. Then the elementary cycles $\{\rho_\Sigma \mid \Sigma \in A\}$ form a basis for $\tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z})$.*

This is proved in analogy with Theorem 5.4. Notice that $B\mathscr{D}_J B/B = \dot{\bigcup}_{w\in\mathscr{D}_J}(BwB/B)$ so $\operatorname{card}(B\mathscr{D}_J B/B) = \sum_{w\in\mathscr{D}_J} q_w$, in agreement with Theorem 4.1. Notice also that if $J = S$ then ψ is the natural bijection $\{\Sigma \in \mathscr{A}\!p(\Delta) \mid C_e \in \Sigma\} \leftrightarrow Bw_0B/B$, and the construction yields the basis of Proposition 4.5. A very intriguing open question is whether the bases constructed in Theorems 5.4 and 5.6, or some derivatives of them, have the $\{0, +1, -1\}$-property mentioned in Proposition 4.5. The significance of this question and some partial results will be discussed in the next section.

The remarks made in Section 4 about the Stanley–Reisner ring of a building can be routinely extended to type-selected subcomplexes. Let Δ be a finite building (resp. Coxeter complex), and $\varnothing \neq J \subseteq S$. Since Δ is (q_G+1)-CM (resp. 2-CM) it follows that Δ_J is also (q_G+1)-CM (resp. 2-CM). Consequently, as in Section 4, $\mathbf{k}[\Delta_J]$ is a Cohen–Macaulay ring of type $\sum_{w\in\mathscr{D}_J} q_w$ (resp. card $\mathscr{D}_J$) whose canonical module is isomorphic to the ideal generated by elementary cycles chosen as in Theorem 5.6 (resp. Theorem 5.4).

In [9] Bromwich constructed bases for the homology of type-selected subcomplexes of finite buildings and Coxeter complexes. Her bases can be shown to correspond to a certain choice for A in Theorems 5.4 and 5.6.

6. HOMOLOGY REPRESENTATIONS

The action of a finite Coxeter group or group with a BN-pair on a type-selected subcomplex Δ_J of its associated complex induces a representation by linear transformations of $\tilde{H}_{|J|-1}(\Delta_J, \mathbb{C})$. In this section we will investigate some properties of such homology representations. We begin with a quick review of the general construction. No notational distinction is made between a representation and its character.

Let Δ be a finite, shellable, and balanced complex on vertex set $V = \dot{\bigcup}_{s\in S} V_s$, and suppose that G is a group of type-preserving automorphisms of Δ. For each $J \subseteq S$, G permutes the set $\{F \in \Delta \mid \tau(F) = J\}$. Call this permutation representation α_J. Also, G's action on Δ_J induces an action on $\tilde{H}_{|J|-1}(\Delta_J, \mathbb{C})$. Call this complex representation β_J. The characters of α_J and β_J are related in the following simple way. If $g \in G$ then by the Hopf trace formula

$$\sum_{i=-1}^{|J|-1} (-1)^i \operatorname{Tr} g_i^* = \sum_{i=-1}^{|J|-1} (-1)^i \operatorname{Tr} g_i,$$

where $g_i^*: \tilde{H}_i(\Delta_J, \mathbb{C}) \to \tilde{H}_i(\Delta_J, \mathbb{C})$ and $g_i: C_i(\Delta_J, \mathbb{C}) \to C_i(\Delta_J, \mathbb{C})$, $C_i(\Delta_J, \mathbb{C}) = \{\sum c_k F_k \mid c_k \in \mathbb{C},\ F_k \in \Delta_J,\ \dim F_k = i\}$. We have $\tilde{H}_i(\Delta_J, \mathbb{C}) = 0$ if $i < |J| - 1$, $\operatorname{Tr} g^*_{|J|-1} = \beta_J$ and $\operatorname{Tr} g_i = \sum \alpha_I$, the sum extending over all $I \subseteq J$ such that $|I| = i + 1$. Hence, the Hopf formula simplifies to

$$\beta_J = \sum_{I \subseteq J} (-1)^{|J-I|} \alpha_I. \tag{6.1}$$

A Möbius inversion then gives

$$\alpha_J = \sum_{I \subseteq J} \beta_I. \tag{6.2}$$

Notice that α_J is a character of degree card $\mathscr{C}\!\ell\,(\Delta_J)$, while β_J is of degree h_J, a number combinatorially determined by any shelling of Δ (cf. Theorem 1.6). Furthermore, β_J can be realized by $\mathbb{Z}$-matrices. Under type-preserving action, if $g(F) = F$ and $E \subseteq F$ then $g(E) = E$. Thus a fixed subcomplex $\Delta_J^g = \{F \in \Delta_J \mid g(F) = F\}$ exists. The formula (6.1) is clearly equivalent to

$$\beta_J(g) = (-1)^{|J|-1} \tilde{\chi}(\Delta_J^g), \tag{6.3}$$

where $\tilde{\chi}$ denotes reduced Euler characteristic.

The idea to use the Hopf trace formula in character theory in this way first appeared in the work of Solomon [29, 31]. It has later been used by Bromwich [9], Curtis [13], Curtis, Lehrer, and Tits [14], Stanley [36], Surowski [38], and others. Our formulation in terms of type-selected subcomplexes is close to that of Bromwich and Stanley.

From now on, let (W, S) be a finite Coxeter group (resp. let $(G; B, N)$ be a finite group with BN-pair having Weyl group (W, S)), and denote by Δ its complex $\Delta(W, S)$ (resp. building $\Delta(G; B, N)$). What can one say about the homology characters β_J, $J \subseteq S$? Let us make a quick summary of some known facts:

(I) β_J is a character of degree $|\mathscr{D}_J|$ (resp. $\sum_{w \in \mathscr{D}_J} q_w$) satisfying

$$\beta_J = \sum_{I \subseteq J} (-1)^{|J-I|} \operatorname{Ind}_{W_{\bar{I}}}^{W}(1) \qquad (\text{resp. } \ldots \operatorname{Ind}_{G_{\bar{I}}}^{G}(1)).$$

In fact, since W (resp. G) acts transitively on $\{F \in \Delta \mid \tau(F) = I\}$ and $\operatorname{Stab}((C_e)_I) = W_{\bar{I}}$ (resp. $G_{\bar{I}}$), this action is equivalent with left coset action, and consequently its character is induced from the principal character of $W_{\bar{I}}$ (resp. $G_{\bar{I}}$), i.e., $\alpha_I = \operatorname{Ind}_{W_{\bar{I}}}^{W}(1)$ (resp. $\alpha_I = \operatorname{Ind}_{G_{\bar{I}}}^{G}(1)$). Now use (6.1), and for the degree Theorems 2.1 and 4.1. Notice that in the Coxeter case α_S, and hence $\sum_{J \subseteq S} \beta_J$, is the left regular representation.

Alternating sums of induced characters as above for Coxeter groups were first shown to be genuine characters (and not merely virtual) by Solomon

[30]. The identification with β_J is due to Solomon [29, 31] for $J = S$ and to Bromwich [9] for $J \neq S$.

(II) In the Coxeter case β_S equals ε, the *alternating character* $\varepsilon(w) = (-1)^{l(w)}$. For a BN-pair β_S is known as the *Steinberg character* (cf. Curtis [12] and Solomon [31]). The character values of the Steinberg representation have been computed using (6.3) (cf. Curtis [13] and Curtis, Lehrer, and Tits [14]).

(III) For Coxeter groups there is a duality $\beta_J = \varepsilon\beta_{\bar{J}}$, due to Solomon [30, Theorem 2]. (Remark: Stanley has shown that this remains true whenever Δ is a homology $(|S|-1)$-sphere [36, Sect. 2].)

(IV) Let G be a finite BN-pair with Weyl group (W, S). Denote by β_J^G and β_J^W the respective homology characters. Then for $I, J \subseteq S$:

$$(\beta_I^G, \beta_J^G) = (\beta_I^W, \beta_J^W) = \operatorname{card}\{w \in W \mid \mathscr{D}(w^{-1}) = I,\ \mathscr{D}(w) = J\}.$$

The first equality is due to Curtis [12], and the second was pointed out by Stanley (personal communication). It is equivalent to $(\alpha_I^W, \alpha_J^W) = \operatorname{card}\{w \in W \mid \mathscr{D}(w^{-1}) \subseteq I,\ \mathscr{D}(w) \subseteq J\}$, which can be obtained via Frobenius reciprocity. In particular, it follows that β_J^G is irreducible if and only if β_J^W is. Also, if all the irreducible characters of W lie in the linear span of $\{\beta_J^W \mid J \subseteq S\}$ (which is the case, e.g., for the symmetric group) then a complete decomposition of β_J^W into irreducibles automatically produces one of β_J^G.

(V) For the symmetric groups the complete decomposition of β_J into irreducibles has been accomplished by Solomon [30, Sect. 6]. The following formulation is due to Stanley [36, Sect. 4], to where we refer for explanation of undefined terminology. For S_{n+1}, the Coxeter group of type A_n, one identifies the set of Coxeter generators with $[n] = \{1, 2, \ldots, n\}$ in the natural way. Let $J \subseteq [n]$, and let χ^λ be the irreducible character corresponding to the partition $\lambda \vdash n+1$. Then the multiplicity of χ^λ in β_J equals the number of standard Young tableaux having shape λ and descent set J. Stanley has given a similar complete decomposition of β_J also for the hyperoctahedral groups [36, Sect. 6].

(VI) One sees from the Solomon–Stanley decomposition for the symmetric group S_{n+1} that the character β_J is irreducible if and only if $J = \varnothing$, $J = \{1, 2, \ldots, k\}$, or $J = \{k, k+1, \ldots, n\}$, $k = 1, 2, \ldots, n$. By (IV) the same is true for BN-pairs of type A_n. The irreducible characters of S_{n+1} which arise in this way are the "hook" characters, i.e., characters χ^λ where λ is a hook partition $\lambda = (k, 1, 1, \ldots, 1) \vdash n+1$.

After this summary of facts let us again consider the general situation.

6.4. PROPOSITION. *The homology character β_J can be realized by matrices having all entries equal to* 0, +1, *or* −1 *in the following cases*:

(i) *for a Coxeter group or* BN-*pair if* $|J| \leqslant 2$,

(ii) *for a Coxeter group if* $|S-J| \leqslant 2$,

(iii) *for a* BN-*pair if* $J=S$ *(the Steinberg character)*,

(iv) *for a Coxeter group or* BN-*pair of type* A_n *if* $J=\{1,2,..., k\}$ *or* $J=\{k, k+1,..., n\}$, $k=1,2,..., n$.

Proof. It suffices to find elementary cycles $\{\rho_\Sigma \mid \Sigma \in A\}$ forming a basis for $\tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z})$ such that for any $\Sigma' \in \mathscr{Ch}(\Delta_J)$ if $\rho_{\Sigma'} = \sum n_\Sigma \rho_\Sigma$ then $n_\Sigma \in \{0, +1, -1\}$ for all $\Sigma \in A$. The case (iii) is therefore settled by Proposition 4.5. In the case (iv) the complex Δ_J is a truncated Boolean algebra or truncated projective geometry and hence comes from a geometric lattice. Bases with the required property were constructed in [5, Sect. 4] for the homology of any geometric lattice. Once case (i) is established case (ii) will automatically follow by duality $\beta_J = \varepsilon\beta_{\bar{J}}$. It remains to prove (i), which will be done following a general remark.

Consider first the Coxeter group case. Suppose that for each $w \in \mathscr{D}_J$ there is a cycle ρ_w in $\tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z})$ such that

(a) $\rho_w = \sum n_C C$, with $n_C \in \{0, +1, -1\}$ for all $C \in \mathscr{Ch}(\Delta_J)$, and

(b) $\pm(C_{w'})_J$ is a nonzero term in $\rho_w \Leftrightarrow w' = w$, for all $w' \in \mathscr{D}_J$.

Then $\{\rho_w \mid w \in \mathscr{D}_J\}$ is a basis for $\tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z})$ (by a proof similar to that of Theorem 5.4), and if $\rho = \sum n_w \rho_w$, where $\rho = \sum m_C C$ and $m_C \in \{0, +1, -1\}$ for all $C \in \mathscr{Ch}(\Delta_J)$, then $n_w \in \{0, +1, -1\}$ for all $w \in \mathscr{D}_J$. For convenience, let us call such a basis *unitary*. Clearly, the β_J-representation is realized by $\{0, +1, -1\}$-matrices in any unitary basis. In the BN-pair case we similarly define a unitary basis $\{\rho_{gB} \mid gB \in B\mathscr{D}_J B/B\}$ for $\tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z})$ by properties (a′) and (b′) corresponding to (a) and (b).

Now, suppose Δ_J is a type-selected subcomplex of a finite Coxeter complex and $|J|=2$. Select a basis $\{\bar{\rho}_w \mid w \in \mathscr{D}_J\}$ for $\tilde{H}_1(\Delta_J, \mathbb{Z})$ as in Theorem 5.4. Such a basis satisfies condition (a) but not necessarily (b). One can change it into a basis satisfying also (b) in the following way. As a partially ordered set under Bruhat order $\mathscr{D}_J$ has least element $w_0(J)$. Let $\rho_{w_0(J)} = \bar{\rho}_{w_0(J)}$. Conditions (a) and (b) are then satisfied for $w = w_0(J)$. Suppose that $\rho_{w'}$ has been defined for all w' strictly less than some $w \in \mathscr{D}_J$, and that (a) and (b) hold for all such w'. Δ_J is a graph and the edges which occur in $\bar{\rho}_w$ form a circuit $\bar{K}_w$. If $(C_w)_J$ is the only edge from $Q = \{(C_{w'})_J \mid w' \in \mathscr{D}_J\}$ in $\bar{K}_w$, let $\rho_w = \bar{\rho}_w$. If not, and $(C_{w'})_J \in Q$ is another such edge then by Lemma 5.3(ii) $w' < w$. The edges which occur in $\rho_{w'}$ form a circuit $K_{w'}$ and the symmetric difference $\bar{K}_w \Delta K_{w'}$ is a disjoint union of circuits. Hence, we may find in $\bar{K}_w \Delta K_{w'}$ a circuit which contains $(C_w)_J$, and by the construction such a circuit cannot contain $(C_{w'})_J$ nor any edge from Q which was not already an edge of $\bar{K}_{w'}$. Repeating the process if necessary to get rid of more edges from Q, we are able to construct in a finite number of steps a circuit K_w which is free of all edges from Q except $(C_w)_J$. Let ρ_w be the fundamental cycle of the 1-sphere K_w. Then conditions (a) and (b) hold also for w. By induction, then, $\{\rho_w \mid w \in \mathscr{D}_J\}$ is a unitary basis.

For the case of a type-selected subcomplex Δ_J of a building, $|J| = 2$, the proof is completely analogous, a key point being the analogue of Lemma 5.3 for special apartments. Since the cases $|J| \leqslant 1$ are trivial, statement (i) is proved. ∎

6.5. *Remark.* Schur observed in [27, p. 678] that the hook irreducible representations of S_{n+1} can be realized by $\{0, +1, -1\}$-matrices. Thus the Coxeter group case of part (iv) is due to him. Also, the $J = [n]$ case of (iv) for $GL_n(q)$ is due to Stanley [36, Sect. 5].

Schur (loc. cit.) also raised the question whether all irreducible representations of S_{n+1} have the $\{0, +1, -1\}$-property. This was later answered in the negative by Young [42, p. 261], at least for the Schur–Young integral realizations. We are grateful to Garsia for providing this reference. The available evidence suggests that instead the homology characters β_J might be a natural class to which Schur's discovery for the hook characters could be generalized.

We conjecture that all β_J-characters for all finite Coxeter groups and BN-pairs can be realized by $\{0, +1, -1\}$-matrices. A likely method of proof would be to find unitary bases for homology, either by an explicit construction starting from elementary cycles as in the preceding proof or by indirect methods. For instance, it can be shown that if the complexes $\Delta_J^* \cup (C_w)_J$ are 2-Cohen–Macaulay for all $w \in \mathscr{D}_J$, then $\tilde{H}_{|J|-1}(\Delta_J, \mathbb{Z})$ has a unitary basis (here $\Delta_J^* = \Delta_J - \{(C_w)_J \mid w \in \mathscr{D}_J\}$), and similarly for buildings.

6.6. *Remark.* The proof for case (i) was purposely phrased in the particular terminology of Coxeter complexes in order to suggest a method which might work also when $|J| \geqslant 3$. The $|J| = 2$ case is actually a special case of the following more general fact. Let Γ be any connected graph (1-dimensional complex) and let G be a group of automorphisms. Then G's induced action on $\tilde{H}_1(\Gamma, \mathbb{C})$ yields a representation which can be realized by $\{0, +1, -1\}$-matrices.

6.7. *Remark.* In [20] Kazhdan and Lusztig define a remarkable class of representations of Coxeter groups. To describe how these are related to the homology representations considered here we must recall some facts.

Let (W, S) be a finite Coxeter group. Kazhdan and Lusztig define a certain preorder relation $\leqslant_{\mathrm{L}}$ on W, and write $w \sim_{\mathrm{L}} w'$ for the equivalence relation $w \leqslant_{\mathrm{L}} w' \leqslant_{\mathrm{L}} w$. The equivalence classes under $\sim_{\mathrm{L}}$ are called (*left*) *cells*. The cells are partially ordered by $\leqslant_L$. If $w \leqslant_L w'$, then $\mathscr{D}(w') \subseteq \mathscr{D}(w)$ [20, p. 172]. Hence, if $w \sim_L w'$ then $\mathscr{D}(w) = \mathscr{D}(w')$, and it follows that every cell $\mathscr{C}$ has a well-defined descent set $\mathscr{D}(\mathscr{C}) = \mathscr{D}(w)$, any $w \in \mathscr{C}$. For every cell $\mathscr{C}$ Kazhdan and Lusztig construct a representation $KL^{\mathscr{C}}$ of W of degree $|\mathscr{C}|$ and a distinguished basis with respect to which W acts by $\mathbb{Z}$-matrices.

The Kazhdan–Lusztig characters $KL^{\mathscr{C}}$ are related to the homology characters β_J, $J \subseteq S$, in the following way:

$$\beta_J = \sum_{\mathscr{D}(\mathscr{C}) = J} KL^{\mathscr{C}}. \tag{6.8}$$

By (6.2) and the duality $\beta_J = \varepsilon\beta_{\bar{J}}$ we have $\mathrm{Ind}_{W_J}^W(1) = \sum_{I \subseteq \bar{J}} \beta_I = \sum_{I \supseteq J} \varepsilon\beta_I$. Hence, (6.8) is equivalent to

$$\mathrm{Ind}_{W_J}^W(1) = \sum_{\mathscr{C} \subseteq {}^JW} \varepsilon \cdot KL^{\mathscr{C}}. \tag{6.9}$$

Lusztig has kindly provided a proof for (6.9) (personal communication in response to a conjecture by the author), which we now sketch. Let $\mathbb{C}[W]$ be the group algebra of W or, equivalently, the Hecke algebra $\mathscr{H}$ of [20] specialized to $q^{1/2} = 1$. There are certain elements $C_w = \sum_{y \leqslant w} n_y y$, $n_y \in \mathbb{Z}$, which form a distinguished basis $\{C_w \mid w \in W\}$ for $\mathbb{C}[W]$ [20, Theorem 1.1]. If $\mathscr{C}$ is a cell and $y \in \mathscr{C}$, let $E_{\mathscr{C}} = \bigoplus_{w \leqslant_L y} \mathbb{C} \cdot C_w$. The subspace $E_{\mathscr{C}}$ is well defined and stable under W (cf. [20, (2.3a) and (2.3b)]). Furthermore, $\bar{E}_{\mathscr{C}} = E_{\mathscr{C}} / \sum_{\mathscr{C}' <_L \mathscr{C}} E_{\mathscr{C}'}$ affords the character $KL^{\mathscr{C}}$ and ${}^JE = \bigoplus_{w \in {}^JW} \mathbb{C} \cdot C_w = \sum_{\mathscr{C} \subseteq {}^JW} E_{\mathscr{C}}$ affords the character $\varepsilon \cdot \mathrm{Ind}_{W_J}^W(1)$. Finally, the submodules $\{E_{\mathscr{C}} \mid \mathscr{C} \subseteq {}^JW\}$ form a filtration of ${}^JE$, isomorphic as a partially ordered set with the $\leqslant_L$ ordering of the corresponding cells, so ${}^JE = \bigoplus_{\mathscr{C} \subseteq {}^JW} \bar{E}_{\mathscr{C}}$.

In a sense formula (6.8) generalizes the Solomon–Stanley decomposition of β_J for symmetric groups to general Coxeter groups. However, the Kazhdan–Lusztig representations are not necessarily irreducible in general. To see the connection in the case of symmetric groups, recall the *Robinson–Schensted correspondence*, which is a fundamental bijection $w \to (P(w), Q(w))$ between S_{n+1} and the set of pairs of standard Young tableaux of size $n+1$ and equal shape (cf. Knuth [22, Sect. 5.1.4]). Kazhdan and Lusztig [20, Sect. 5] show that if $w, w' \in S_{n+1}$, then $w \sim_L w'$ if and only if $Q(w) = Q(w')$. Furthermore, if $\mathscr{C}$ is a cell in S_{n+1}, $w \in \mathscr{C}$, and shape $Q(w) = \lambda$, then $KL^{\mathscr{C}} = \chi^\lambda$. Thus by (6.8) a given irreducible character χ^λ occurs in β_J as many times as there are standard Young tableaux of shape λ having descent set J.

It is unknown to us whether there is any natural class of representations of finite Chevalley groups which provides nontrivial decompositions of their homology characters analogous to (6.8). It appears that the existence of such a class would be related to the existence of a combinatorially significant cell structure on G/B.

Appendix: Weak Buildings and Tits Geometries

In this paper we have considered only "concrete" buildings, i.e., buildings which are constructed on an underlying group. Tits has axiomatically defined the notion of an "abstract" building [39, p. 38], for which the buildings of BN-pairs are the motivating examples. His classification theorem [39] shows that all abstract buildings of irreducible spherical type and rank $\geqslant 3$ actually arise from BN-pairs. Tits has kindly pointed out to us that the shellability proof can be adapted to the wider setting of "weak" buildings, which includes all Coxeter complexes and abstract buildings and also some other similar geometries. The weak buildings have recently gained importance because of their central role in the emerging theory of Tits–Buekenhout diagram geometries [41, 45], and particularly in view of the recent work of Tits [42]. In this Appendix we will comment on the role of shellability in this wider context.

Let Δ be a pure d-dimensional complex. The distance $d(C, C')$ between two chambers $C, C' \in \mathscr{Ch}(\Delta)$ is as usual defined as the minimal length of a gallery beginning with C and ending with C' (we assume that finite connecting galleries always exist). Let $C^* \in \mathscr{Ch}(\Delta)$. Then the *radial ordering* of $\mathscr{Ch}(\Delta)$ away from the fundamental chamber C^* is defined by: $C \overset{*}{\leqslant} C' \Leftrightarrow d(C^*, C) + d(C, C') = d(C^*, C')$. This defines a partial ordering of $\mathscr{Ch}(\Delta)$ which is ranked and with the least element C^*. It is easy to see that if $\Delta = \Delta(W, S)$ or $\Delta = \Delta(G; B, N)$ and $C^* = C_e$ as in Sections 2 and 4 then the radial ordering of $\mathscr{Ch}(\Delta)$ is equivalent to the weak ordering of W and of G/B, respectively.

A pair $(\Delta, \mathscr{A})$ consisting of a complex Δ and a family $\mathscr{A}$ of subcomplexes thereof is called a *weak building* if

(B1) the members of $\mathscr{A}$ (called *apartments*) are isomorphic to Coxeter complexes,

(B2) any two faces of Δ belong to a common apartment,

(B3) if $F, F' \in \Sigma \cap \Sigma'$, $\Sigma, \Sigma' \in \mathscr{A}$, there is an isomorphism $\Sigma \to \Sigma'$ which leaves the set $F \cup F'$ pointwise fixed.

The terminology used for this concept varies somewhat: weak buildings are called "weak buildings whose apartments are Coxeter complexes" in [39] and just "buildings" in [42]. As pointed out by Tits [39, pp. 38, 234] many of the formal properties of actual buildings (and certainly the ones used below) remain true for weak buildings.

A1. Theorem. *Let $\Delta = (\Delta, \mathscr{A})$ be a weak building and $C^* \in \mathscr{Ch}(\Delta)$. Then any linear extension of the radial ordering away from C^* assigns a shelling order to $\mathscr{Ch}(\Delta)$.*

Proof. Suppose that $<$ is a well-ordering of $\mathscr{Ch}(\Delta)$ such that $C \overset{*}{<} C'$ implies $C < C'$. For $C \neq C^*$ let $\Delta_{<C} = \bigcup_{C' < C} \bar{C}'$. Suppose that $F \in \bar{C} \cap \Delta_{<C}$. The projection $D = \mathrm{proj}_F C^*$ is a chamber containing F [39, p. 50] and by Lemma 3.19.6 of [39] we have $d(C^*, C') = d(C^*, D) + d(D, C')$, i.e., $D \overset{*}{\leqslant} C'$ for all chambers C' containing F. Since by assumption C is not the first chamber with this property it follows that $D \overset{*}{<} C$. Let $D = C_0 \overset{*}{<} C_1 \overset{*}{<} \cdots \overset{*}{<} C_n = C$ be a maximal chain (equivalently, $C_0, C_1, \ldots, C_n$ is a minimal length gallery from D to C). Then the wall $G = C_{n-1} \cap C$ lies in $\Delta_{<C}$ since $C_{n-1} < C$, and $F \subseteq G$ since in fact $F \subseteq C_i$ for all $0 \leqslant i \leqslant n$ by [39, Proposition 3.13]. Hence, $\bar{C} \cap \Delta_{<C}$ is pure $(\dim \Delta - 1)$-dimensional.

The preceding result provides a common generalization to the qualitative part of Theorems 2.1 and 4.1. It can be shown that there are similar extensions to weak buildings of many of the properties derived in Sections 4 and 5. However, we will not pursue this further.

A2. *Remark.* Let I be a finite set and let $\mathbf{M}$ be a Coxeter diagram over I. This means that $\mathbf{M}: I \times I \to \mathbb{N} \cup \{\infty\}$ and $\mathbf{M}(i, i) = 1$, $\mathbf{M}(i, j) = \mathbf{M}(j, i) \geqslant 2$ for all $i, j \in I$, $i \neq j$. By a *Tits geometry* (*of type* $\mathbf{M}$) is meant a pure $(|I| - 1)$-dimensional complex Δ on vertex set V such that

(G1) Δ is a flag complex (i.e., any set of pairwise incident vertices is a face),

(G2) for every face F of cardinality $\leqslant |I| - 2$ the link (or residue) $\mathrm{lk}_\Delta F$ is connected,

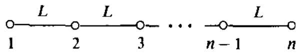
(G3) Δ is balanced $V = \bigcup_{i \in I} V_i$ so that if F is a face of type $\tau(F) = I - \{i, j\}$, $i \neq j$, then $\mathrm{lk}_\Delta F$ is a generalized $\mathbf{M}(i, j)$-gon.

For a complete discussion of this concept see Tits [41, 42]. If $(\Delta, \mathscr{A})$ is a weak building and $\mathbf{M}$ is the Coxeter diagram associated to the isomorphism class of its apartments then Δ is a Tits geometry of type $\mathbf{M}$. Theorem A.1 suggests that one might ask whether other Tits geometries are shellable. The work of Tits [42] shows that this is hardly ever possible. If $F \in \Delta$, $I' = I - \tau(F)$ and $\mathbf{M}'$ is the Coxeter diagram on I' obtained by restricting $\mathbf{M}$, then we will say that $\mathrm{lk}_\Delta F$ is a residue of type $\mathbf{M}'$.

A3. Proposition. *Suppose that Δ is a Tits geometry for which all residues of type $\mathbf{C}_3$ and $\mathbf{H}_3$ are weak buildings. Then Δ is shellable if and only if Δ is a weak building.*

Proof. One direction is provided by Theorem A1. Suppose that Δ is shellable. Then every link $\mathrm{lk}_\Delta F$, $F \in \Delta$, is shellable, as can easily be seen, e.g., from Proposition 1.2. It follows from Theorem 1.3 that all links $\mathrm{lk}_\Delta F$ of dimension $\geqslant 2$ are simply connected (in the topological sense). Hence, Δ is residually simply connected (in the combinatorial sense of [42]), and Theorem 1 of Tits [42] shows that Δ is a weak building.

A4. *Remark.* Buekenhout [45] has defined a class of geometries of type **M**, where **M** is a diagram of a more general kind than a Coxeter diagram. The construction is essentially the same as in the definition of Tits geometries above except that axiom (G3) now also allows rank 2 residues of types other than generalized n-gons, see [41, 45] for details. In this general setting there are shellable geometries which are not weak buildings. For instance, the Buekenhout geometries of type



are the proper parts of geometric lattices of rank $n+1$ [45, Theorem 7], and these are known to be shellable [4]. These geometries have a rich supply of Coxeter apartments of type $\mathbf{A}_n$ (corresponding to Boolean sublattices), and their homology is completely determined by the corresponding fundamental cycles [5, Theorem 4.2].

Acknowledgments

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