REU 2019 Day 5

S. Hopkins

Minuscule doppelgängers

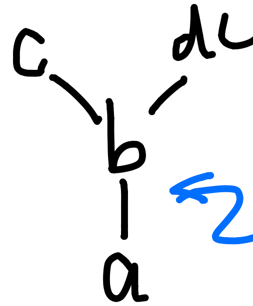
TA: Andy Hardt office hour @ 1:30pm

Poset basics:

$P = (P, \leq)$ will be a finite partially ordered set (poset).

Represent it by its Hasse diagram

$P =$



$a \leq b,\ b \leq d,\ b \leq c$
$(\Rightarrow a \leq c)$

edges = cover relations

e.g. $P = [a] := \{1,2,\dots,a\} = \mathbf{a}$

$$\begin{array}{c} a \\ | \\ \vdots \\ | \\ 3 \\ | \\ 2 \\ | \\ 1 \end{array}$$

$P \times Q :=$ Cartesian product

$= \{(p,q) : p \in P,\ q \in Q\}$

$(i,j) \le (i',j')$ if $i \le i',\ j \le j'$

e.g. 2×2

$$\begin{tikzcd}[column sep=small] & (2,2) \arrow[dl, dash] \arrow[dr, dash] & \\ (2,1) \arrow[dr, dash] & & (1,2) \arrow[dl, dash] \\ & (1,1) & \end{tikzcd}$$

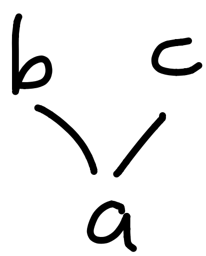
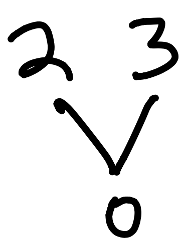
$a \times b =$ "rectangle poset"

P-partitions and order ideals

A P-partition of height m is a map $\pi: P \to \{0,1,2,\dots,m\}$ which is weakly order-preserving, i.e.

$$x \le y \text{ in } P \Rightarrow \pi(x) \le \pi(y)$$

Denote the set of these by $PP^m(P)$

e.g. $P=$   $\in PP^3(P)$

($\in PP^4(P)$ also)

An order ideal is a subset $I \subset P$ which is downward-closed, i.e. $y \in I$, $x \leq y \in P \Rightarrow x \in I$

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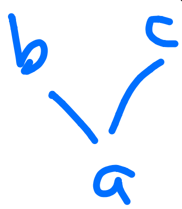
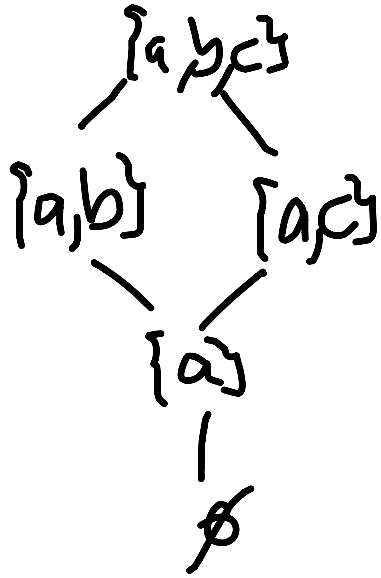
e.g. $P = \begin{tikzcd}[column sep=small, row sep=small] b & & c \\ & a \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$ $\{a,b\} = I$ is an order ideal, but $\{b,c\}$ is not.

We use $J(P)$ to denote the set of order ideals of P ordered by containment, viewed itself as a poset (even a "distributive lattice").

e.g. $J\Bigg($  $\Bigg) =$ 

$\exists$ a bijection

$$PP^m(P) \longleftrightarrow \{I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m \in J(P)\}$$

$$\pi \longmapsto \underset{\displaystyle I_1}{\underset{\|}{\pi^{-1}\{0\}}} \subseteq \underset{\displaystyle I_2}{\underset{\|}{\pi^{-1}\{0,1\}}} \subseteq \cdots \subseteq \underset{\displaystyle I_m}{\underset{\|}{\pi^{-1}\{0,1,\ldots,m-1\}}}$$

e.g.  $\in PP^3(P) \mapsto \{a\} \subseteq \{a\} \subseteq \{a,b\}$

$\big(\in PP^4(P) \mapsto \{a\} \subseteq \{a\} \subseteq \{a,b\} \subseteq \{a,b,c\}\big)$

Note: $PP^1(P) \cong J(P)$

Order polynomials & doppelgängers

The **order polynomial** $\Omega_P(m)$ of P is

$$\Omega_P(m) := \#\text{weakly order preserving maps } f: P \to [m] = \{1, \dots, m\}$$

$$= \# PP^{m-1}(P)$$

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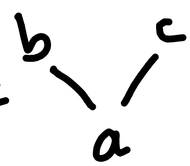
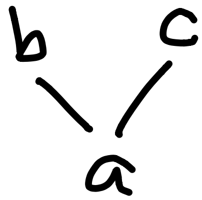
e.g. $P =$ 

How to compute $\Omega_P(m)$?

$P =$ 

What can $f : P \to [m]$ look like?

<u>Cases:</u>

1) $f(a) = f(b) = f(c)$ $\binom{m}{1}$ ways
2) $f(a) < f(b) = f(c)$ $\binom{m}{2}$ ways
3) $f(a) = f(b) < f(c)$ $\binom{m}{2}$ ways
4) $f(a) = f(c) < f(b)$ $\binom{m}{2}$ ways
5) $f(a) < f(b) < f(c)$ $\binom{m}{3}$ ways
6) $f(a) < f(c) < f(b)$ $\binom{m}{3}$ ways

$$\Rightarrow \Omega_P(m) = 2\binom{m}{3} + 3\binom{m}{2} + \binom{m}{1}.$$

PROPOSITION:

$\Omega_P(m)$ is a polynomial in $m \in \mathbb{N}$.

proof: Example shows it a sum of terms $\binom{m}{k} = \dfrac{m(m-1)\cdots(m-k+1)}{k!} \in \mathbb{Q}[m]$. $\square$

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e.g. $\Omega_{a}(m) = \left(\!\binom{m}{a}\!\right) =$ "m multichoose a"

a element chain

$=$ # of a-element multisubsets of $[m]$

$= \binom{m+a-1}{a}$

$a=3$ $\quad$ $m=4$

$$\begin{array}{ccc} x_3 & & 2 \\ | & & | \\ x_2 & & 2 \\ | & & | \\ x_1 & & 1 \end{array} \quad \longleftrightarrow \{1,2,2\} \subset [4]$$

e.g. $$\Omega_{1+1+\cdots+1}(m) = m^a$$

$$\bullet\ \bullet\ \cdots\ \bullet$$

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e.g.

THM (MacMahon 1915)

$$\Omega_{a\times b}(m) = \prod_{i=1}^{a}\prod_{j=1}^{b}\frac{i+j+m-2}{i+j-1}$$

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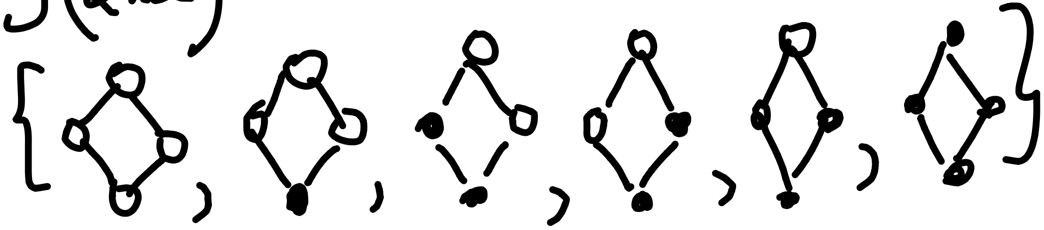
e.g. $$PP^1(2\times 2) = \Omega_{2\times 2}(2) = \frac{2\cdot 3\cdot 3\cdot 4}{1\cdot 2\cdot 2\cdot 3} = 6$$

$$= \# J(2\times 2)$$



DEF'N: We say P, Q are **doppelgängers** if $\Omega_P(m) = \Omega_Q(m)\ \forall m$

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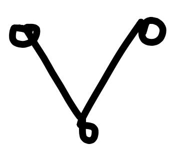
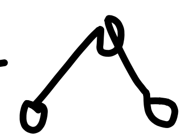
e.g. P and $P^* :=$ **dual** of P are doppelgängers

$P =$  $P^* =$ 

DEF'N: The **comparability** graph $\mathrm{com}(P)$ is the undirected graph with vertex set P and edges $\{x, y\}$ if $x < y$ or $y < x$ in P.

e.g. $\mathrm{com}\left(\begin{tikzcd}[column sep=small, row sep=small] & d & \\ & c \arrow[u, dash] & \\ a \arrow[ur, dash] & & b \arrow[ul, dash] \end{tikzcd}\right) = \begin{tikzcd}[column sep=small, row sep=small] & d \arrow[dl, dash] \arrow[dr, dash] \arrow[dd, dash] & \\ a \arrow[dr, dash] & & b \arrow[dl, dash] \\ & c & \end{tikzcd}$

THM (Stanley 1986) If $\mathrm{com}(P) \cong \mathrm{com}(Q)$ then P, Q are doppelgängers.

Note $\mathrm{com}(P) \cong \mathrm{com}(P^*)$.

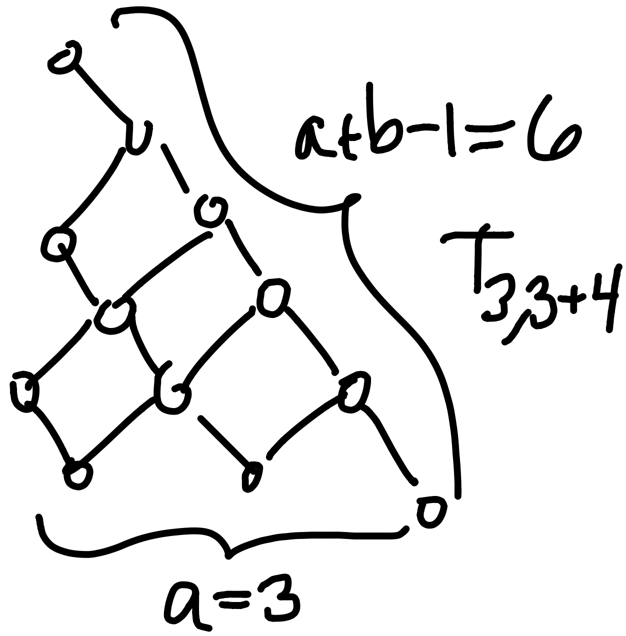
Main antagonist:

"trapezoid poset"

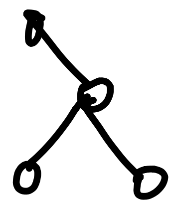
$T_{a,a+b}$

$(a \leq b)$

$a=3,\ b=4.$



$$\#T_{a,a+b} = a \cdot b = \# \, a \times b$$

THM (Proctor 1983) $P = a \times b$ and $Q = T_{a,a+b}$ are doppelgängers e.g. $T_{2,2+2} =$ 

$\#J(T_{2,2+2}) = 6$

REU Exercise 11

(a) Compute $\Omega_{2\times 2}(m)$. (by hand, not MacMahon!)

Compute$(\Omega_{T_{2,2+2}}(m))$

Conclude they are doppelgängers

(b) Say $n=\#P$. A *linear extension* of P is a list $p_1, p_2, \ldots, p_n$ of all the elements of P with $p_i < p_j \Rightarrow i<j$.

Show that

(i) $\Omega_P(m)$ has degree n as a polynomial in m

(ii) its coefficient of m^n is $\frac{1}{n!}$(# of linear extensions of P)

(c) Show that $2 \times b$ and $T_{2,2+b}$ have isomorphic comparability graphs but $\mathrm{com}(a \times b) \not\cong \mathrm{com}(T_{a,a+b})$ for all $3 \leq a \leq b$

Whence "minuscule"?

The minuscule posets are a certain family of posets coming from representation theory of Lie algebras with many remarkable properties.

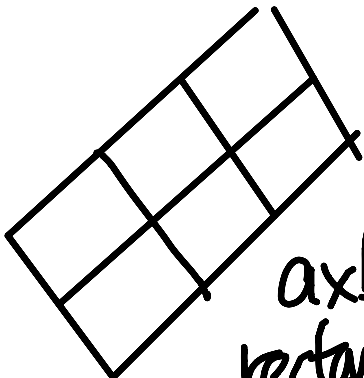
e.g. there is a product formula for

$$\sum_{\pi \in PP^m(P)} q^{|\pi|}$$

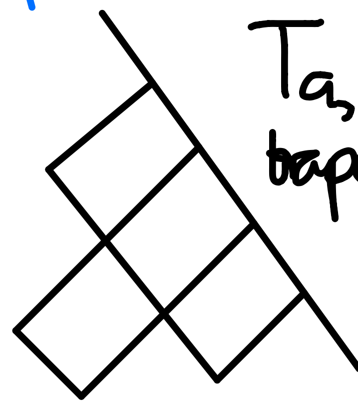
where $|\pi| = \sum_{p \in P} \pi(p)$.

Minuscule poset

Doppelgänger

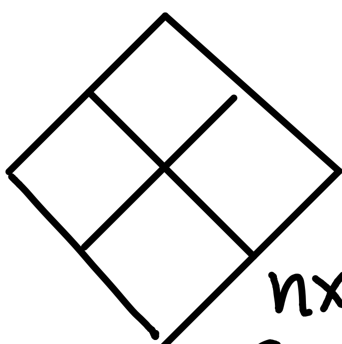


$a \times b$ rectangle

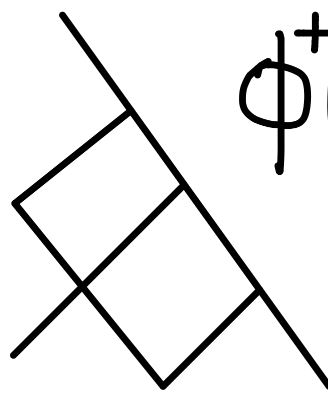


$T_{a,a+b}$ trapezoid

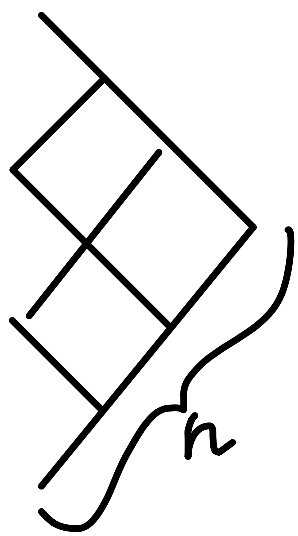
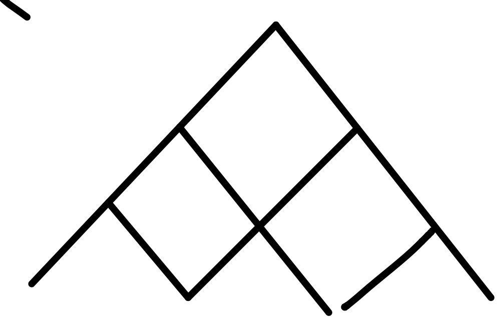
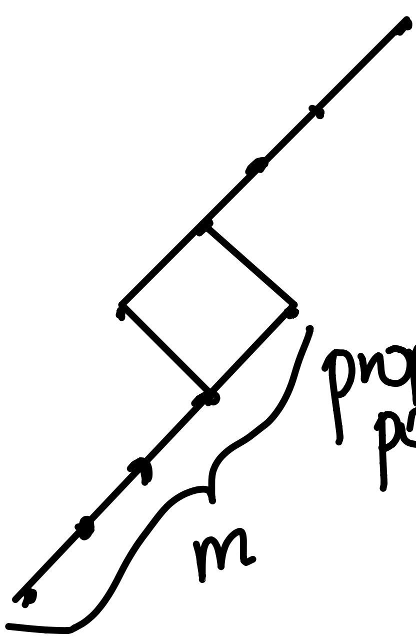
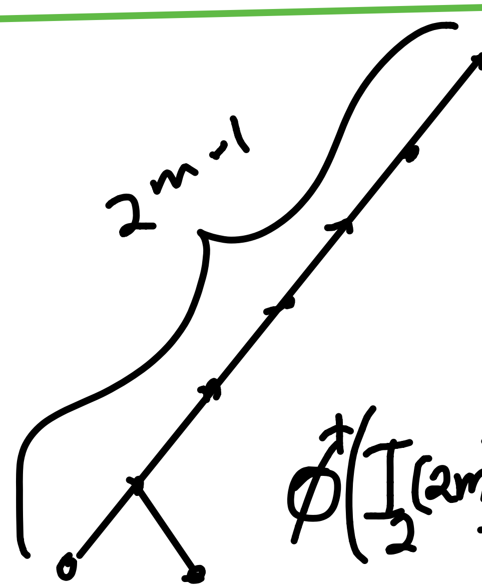
special case



$n \times n$ square



$\Phi^+(B_n)$

Minuscule poset	Doppelgänger
	(not a doppelgänger but a "lookalike") 
shifted staircase	$\Phi^+(A_n)$
 propeller poset	 $\Phi^+(I_2(2m))$

(special case $n=5$ of shifted staircase) $\longleftrightarrow \Phi^+(H_3)$

$\Phi^+(W)$ here = "root poset of coincidental type W"

E_6/E_7 minuscule posets have no doppelgängers.

THM (Hamaker-Patrias-Pechenik-Williams 2018)

For (P,Q) a minuscule doppelgänger pair, $\exists$ an explicit ("uniform") bijection

$$PP^m(P) \xrightarrow{\sim} PP^m(Q)$$

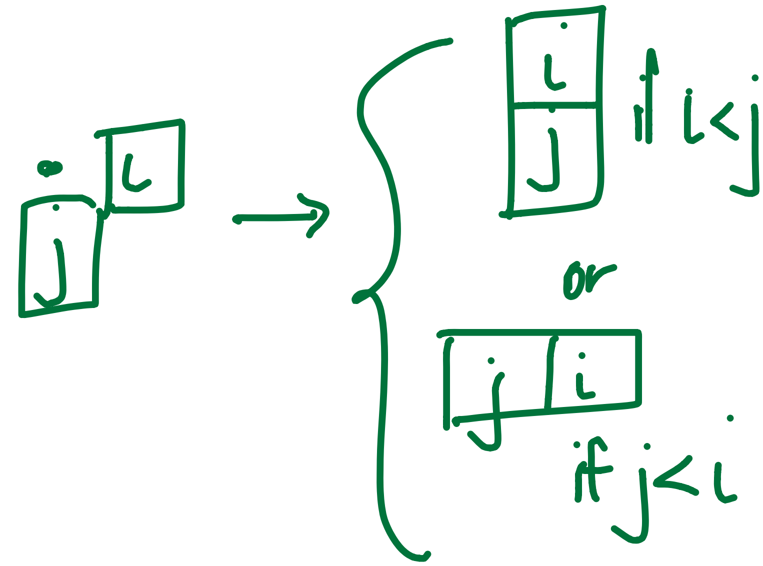
The bijection idea uses "jeu de taquin" (= French for "teasing game" or "Game of 15")

9	6	1	10
2	8	15	11
7		3	13
12	4	5	14

jdt slides a skew standard Young tableau to make it not skew

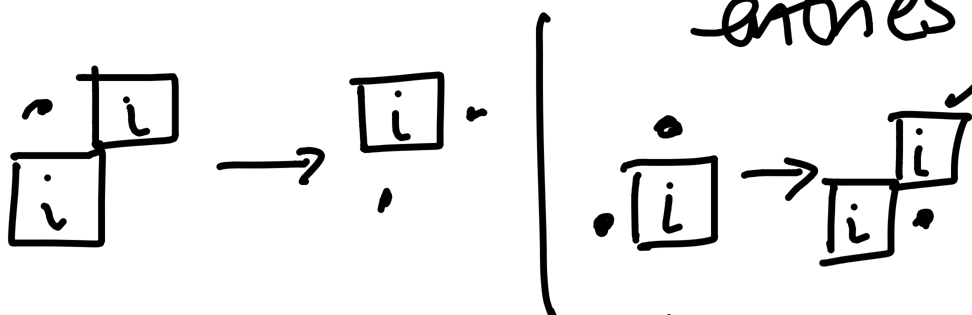
$$\begin{array}{ccc} \bullet & 1 & 4 \\ 2 & 3 & 5 \end{array}$$
$$\downarrow$$
$$\begin{array}{ccc} 1 & \bullet & 4 \\ 2 & 3 & 5 \end{array}$$
$$\downarrow$$
$$\begin{array}{ccc} 1 & 3 & 4 \\ 2 & \bullet & 5 \end{array}$$
$$\downarrow$$
$$\begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \bullet \end{array}$$

sliding rules:



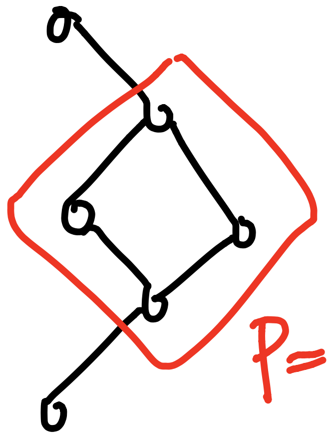
Two modifications to jdt

- K-theoretic jdt of increasing tableaux (can have repeated entries)

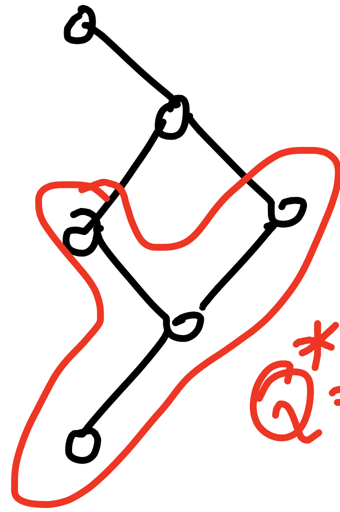


They convert P-partitions to increasing tableaux.

- Embed P and Q^* into a larger minuscule poset in which the K-jdt occurs.



$P = 2 \times 2$



$Q^* = \left(\emptyset^{+}(B_2)\right)^*$

Their proof that it works is based on computations in K-theory.

On to the REU problem…

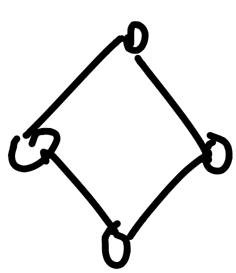
DEF'N: A set-valued P-partition of height m is a map

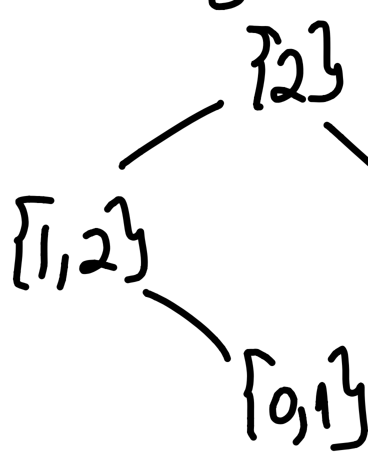
$$\pi : P \longrightarrow \underbrace{2^{\{0,1,2,\dots,m\}} - \emptyset}_{\text{nonempty subsets of } \{0,1,\dots,m\}}$$

such that

$\max \pi(x) \leq \min \pi(y)$ if $x < y \in P$

e.g.





The excess of π is $e = \sum_{p \in P} (\#\pi(p) - 1)$

Denote the set of such P-partitions by $PP^m_e(P)$ (so $PP^m_0(P) = PP^m(P)$).

DEF'N: Let $\pi \in PP^m(P)$.

View π as $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_m \in J(P)$

$I_i = \pi^{-1}\{0, 1, \ldots, i-1\}$

The down-degree of π is

$$\mathrm{ddeg}(\pi) := \sum_{i=1}^{m} \#\max(I_i)$$

e.g. $P = \begin{tikzcd}[column sep=small] b \arrow[dr, dash] & & c \arrow[dl, dash] \\ & a & \end{tikzcd} \quad \pi = \begin{tikzcd}[column sep=small] 2 \arrow[dr, dash] & & 3 \arrow[dl, dash] \\ & 0 & \end{tikzcd} \in PP^4(P)$

$$\longleftrightarrow \{a\} \subseteq \{a\} \subseteq \{a,b\} \subseteq \{a,b,c\}$$

$$\mathrm{ddeg}(\pi) = 1+1+1+2 = 5$$

Finally, **rowmotion**

$$\mathrm{row} : PP^m(P) \longrightarrow PP^m(P)$$

is a certain invertible (piecewise-linear) map on P-partitions.

REU Problem 5

(P,Q) a minuscule doppelgänger pair

PHILOSOPHY

If $\mathrm{com}(P) \cong \mathrm{com}(Q)$ then

1) Show $\#\mathcal{PP}^m_e(P) = \#\mathcal{PP}^m_e(Q)$

(idea: extend K-jdt to set valued tableaux)

2) Show

$$\sum_{\pi \in \mathcal{PP}^m(P)} q^{\mathrm{ddeg}(\pi)} = \sum_{\pi \in \mathcal{PP}^m(Q)} q^{\mathrm{ddeg}(\pi)}$$

(idea: ????)

3) on next page

MOTIVATION

Any of these imply $J(T_{a,a+b})$ is "CDE" (CONJ of Reiner-Tenner-Yong)

3) Show rowmotion "behaves the same" on $PP^m(P)$, $PP^m(Q)$

- same orbit structure
- same ddeg average along orbits

(idea for some part: maybe HPPW's bijection commutes with rowmotion)

REU Exercise 12

(a) Show that the HPPW bijection

$$PP(P) \xrightarrow{\varphi} PP(Q)$$

does not preserve ddeg(π).

In fact, it fails already for

$$P = 2\times 2, \quad Q = T_{2,2+2}$$

(b) Check (by hand/computer) that the HPPW bijection commutes with rowmotion in small cases, i.e.

$$\varphi(\mathrm{row}(\underset{\substack{\cap \\ PP^m(P)}}{\pi})) = \mathrm{row}(\varphi(\underset{\substack{\cap \\ PP^m(Q)}}{\pi}))$$

(c) Play with Sam's bad code or write your own better code to check 1), 2), 3) in the REU Problem.

Can you beat his code for $a \times b$, $T_{a,a+b}$, which can handle

1) $a, b \leq 4$, $m \leq 3$ (and all excesses e)

2) $a, b \leq 4$, $m \leq 4$

3) $a, b \leq 4$, $m \leq 4$?