(1)

REU 2019 Day 7
Digraph associahedra

1. Associahedra, building sets, nested sets
2. Extended version
3. f, h, γ - vectors

REU Problem 7(a,b,c)

~~4. Polytope review~~

4~~5~~. REU Problem 7(d)

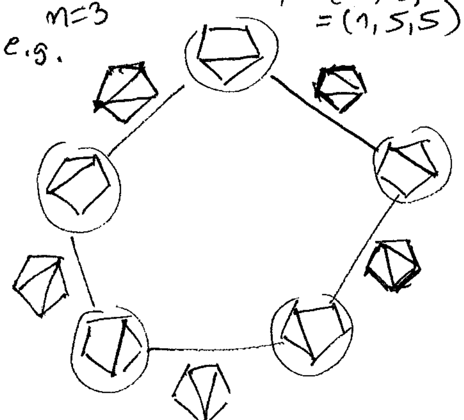
1. Associahedra

Recall from Day 4 that $\#\left\{\begin{array}{c}\text{dissections of } (n+2)\text{-gon}\\ \text{using } k \text{ diagonals}\end{array}\right\} \overset{\text{Kirkman Cayley}}{=} \frac{1}{n}\binom{n}{k+1}\binom{n+k+1}{k}$

This gives the f-vector $f := (\underset{\text{empty face}}{f_{-1}}, \underset{\text{vertices}}{f_0}, \underset{\text{edges}}{f_1}, \ldots, f_{n-2})$ where $f_i = \#$ i-dimensional faces in an interesting simplicial complex $Assoc_n$ (= boundary of the dual associahedron) having vertices = {diagonals of (n+2)-gon}

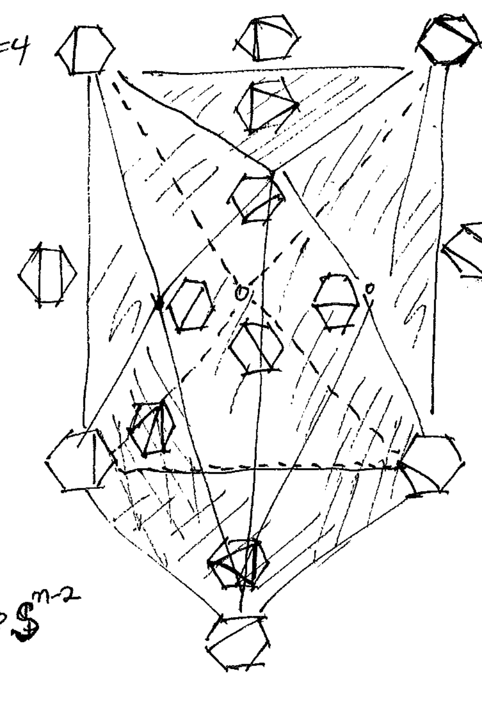
simplices = {pairwise noncrossing subsets of diagonals}

e.g. n=3 $f = (f_{-1}, f_0, f_1) = (1,5,5)$

$Assoc_3 =$ 

triangulates a 1-sphere $\mathbb{S}^1$

n=4 $f = (f_{-1}, f_0, f_1, f_2) = (1, 9, 21, 14)$



$= Assoc_4$ triangulates a 2-sphere $\mathbb{S}^2$

THM (Stasheff 1963) $Assoc_n$ triangulates an (n-2)-sphere $\mathbb{S}^{n-2}$

(2)

Let's generalize vastly.

DEF'N: A (connected) building set on $[n] = \{1,2,\dots,n\}$ is a collection $\mathcal{B} \subseteq 2^{[n]}$ = subsets of $[n]$
with $\mathcal{B}$ containing $[n]$ and all singletons $\{1\}, \{2\}, \dots, \{n\}$
and $I, J \in \mathcal{B}$ with $I \cap J \neq \emptyset \Rightarrow I \cup J \in \mathcal{B}$

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e.g. $\mathcal{B}_{min} = \{ \{1\}, \{2\}, \dots, \{n\}, [n] \}$

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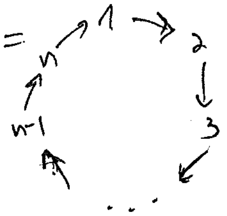
e.g. Γ a strongly connected digraph on vertex set $[n]$
$\rightsquigarrow$ $\mathcal{B}_\Gamma$ = digraph building set
$:= \{ I \subseteq [n] : \Gamma|_I \text{ is strongly connected} \}$

• If $\Gamma =$



then $\mathcal{B}_\Gamma = \mathcal{B}_{min}$

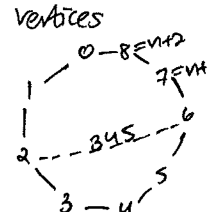
• If $\Gamma = 1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows \cdots \rightleftarrows n{-}1 \rightleftarrows n$ then $\mathcal{B}_\Gamma = \{$ contiguous subsets I
1, 2, ..., n
12, 23, ..., $n{-}1\,n$
123, 234, ...
1234, 2345, ... $\}$

$(= 1 - 2 - 3 - \cdots - n{-}1 - n)$
undirected path

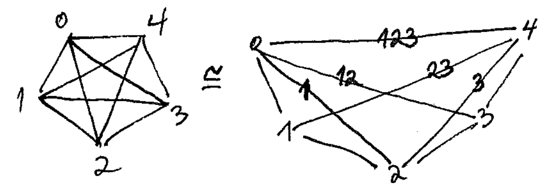
$\updownarrow$

$\{$diagonals of $(n+2)$-gon$\}$

number vertices



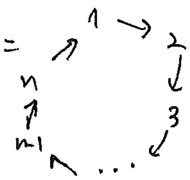
and send a diagonal to the vertices strictly below it (on the other side of the $0 - n{+}2$ edge):



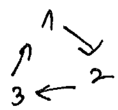
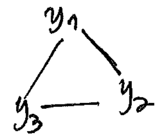
DEF'N: A nested set $\mathcal{F} = \{I_1, \dots, I_m\}$ with respect to $\mathcal{B}$ is a collection of subsets $I_j \in \mathcal{B}$
which are pairwise either nested ($I_i \subseteq I_j$ or $I_i \supseteq I_j$) or disjoint $I_i \cap I_j = \emptyset$
and whenever $I_{i_1}, I_{i_2}, \dots, I_{i_k}$ among them are pairwise disjoint,
their union $\bigcup_{j=1}^{k} I_{i_j} \notin \mathcal{B}$.

(3)

The complex of nested sets $\mathcal{N}(\mathcal{B})$ is the simplical complex on vertex set $\{y_I\}_{I \in \mathcal{B} - \{[n]\}}$ and simplices $\{y_{I_1}, \ldots, y_{I_m}\}$ for each nested set $\{I_1, \ldots, I_m\} \subseteq \mathcal{B} - \{[n]\}$

e.g. when $\mathcal{B} = \mathcal{B}_\Gamma$ for $\Gamma =$  , then $\mathcal{N}(\mathcal{B}_\Gamma)$ is the boundary of a simplex on vertices $y_1, y_2, \ldots, y_n$

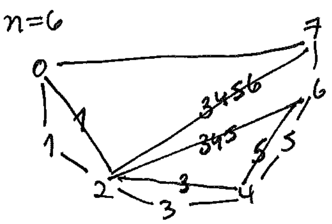
$= \mathcal{B}_{min}$

$n=3$ $\Gamma =$  $\mathcal{N}(\mathcal{B}_\Gamma) =$  $= \{\emptyset, y_1, y_2, y_3, y_1y_2, y_1y_3, y_2y_3\}$ (but not y_{123})

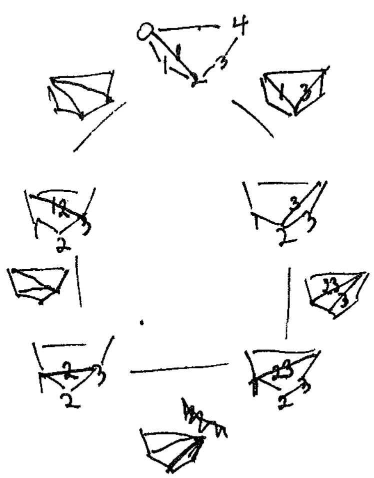
$\mathcal{B} = \{1, 2, 3, 123\}$

e.g. when $\mathcal{B} = \mathcal{B}_\Gamma$ for $\Gamma = 1 \rightleftarrows 2 \rightleftarrows \cdots \rightleftarrows n-1 \rightleftarrows n$, then $\mathcal{N}(\mathcal{B}_\Gamma) \cong Assoc_n$

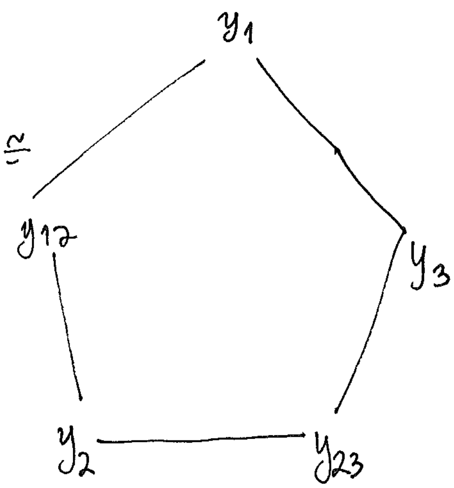
$n=6$



$\longleftrightarrow$ simplex $\{y_1, y_3, y_5, y_{345}, y_{3456}\} \in \mathcal{N}(\mathcal{B}_{1-2-3-4-5-6})$



$\cong$



THM (DeConcini-Procesi 1995, Feichtner-Yuzvinsky 2004, Postnikov 2005) For any building set $\mathcal{B}$ (on $[n]$), $\mathcal{N}(\mathcal{B})$ triangulates an S^{n-2}.

(4)

2. Extended version (Lam & Pylyavskyy 2012) (for digraphs)

DEF'N: $\tilde{N}(\mathcal{B})$:= extended nested set complex for the (connected) building set $\mathcal{B}$ on $[n]$

= simplicial complex having vertices $\{x_1, x_2, \ldots, x_n\} \cup \{y_I\}_{I \in \mathcal{B}}$

(so note $I=[n]$ has a vertex $y_{[n]}$ now!)

and simplices $\{x_{i_1}, \ldots, x_{i_r}, y_{I_1}, \ldots, y_{I_s}\}$

where $\{I_1, \ldots, I_s\}$ is a nested set for $\mathcal{B}$

and $x_{i_j} \notin I_k \ \forall j,k$ i.e. $\{x_{i_1}, \ldots, x_{i_r}\} \cap \bigcup_{j=1}^{s} I_j = \emptyset$

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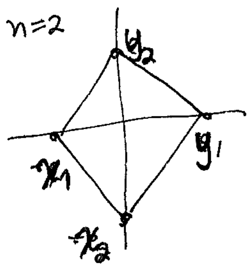
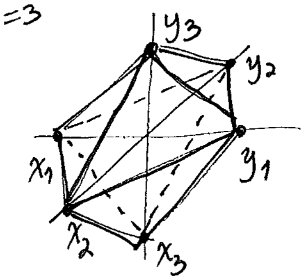
e.g. $\tilde{N}(\mathcal{B}_{1 \to 2 \to \cdots \to n \to 1})$ = boundary of n-dimensional cross-polytope/hyperoctahedron, with face $\{y_1, \ldots, y_n\}$ stellarly subdivided

cross-polytopes:

$n=2$



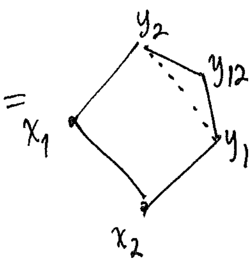
$n=3$

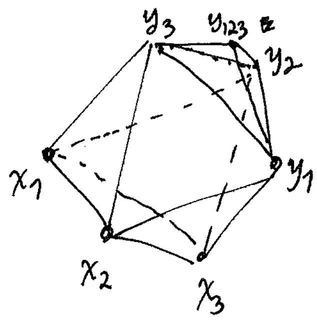


stellarly subdivide edge $y_1 - y_2$

stellarly subdivide triangle y_1, y_2, y_3

$\tilde{N}(\mathcal{B}_{1 \rightleftarrows 2}) =$





$= \tilde{N}(\mathcal{B}_{1 \to 2 \to 3 \to 1})$

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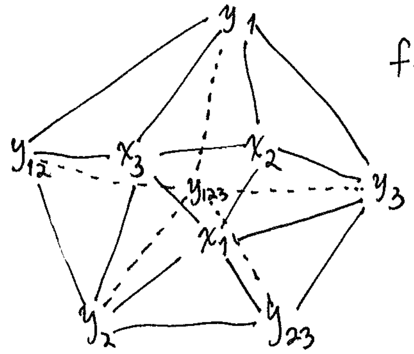
e.g. $\tilde{N}(\mathcal{B}_{1 \rightleftarrows 2 \rightleftarrows 3}) =$



$f = (f_{-1}, f_0, f_1, f_2)$
$= (1, 9, 21, 14)$ ← same as for $N(\mathcal{B}_{1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows 4})$!??
$\parallel$
$Assoc_4$

(5) REU Exercise 16

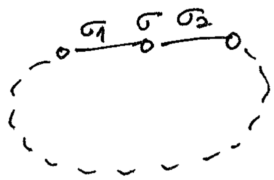
Show $\tilde{N}(\mathcal{B})$, $N(\mathcal{B})$ are both

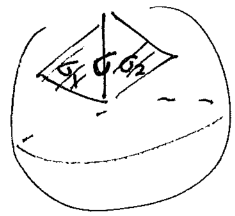
(a) pure of dimensions $n-1$, $n-2$

i.e. all facets of $\tilde{N}(\mathcal{B})$ have dim $n-1$

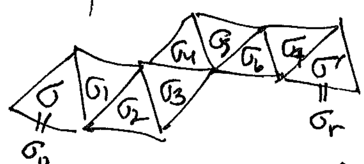
of $N(\mathcal{B})$ have dim $n-2$

(b) thin, meaning every codimension 1 face (= ridge) σ lies in exactly 2 facets σ_1, σ_2:





(c) pseudomanifolds, meaning thin and every pair σ, σ' of facets have a path $\sigma = \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_r = \sigma'$ where $\sigma_i \cap \sigma_{i+1}$ is a ridge $\forall i$



(d) ~~[illegible]~~ flag for any $\mathcal{B} = \mathcal{B}_\Gamma$ with undirected graph Γ, meaning their minimal nonfaces have size 2

$\Updownarrow$ equivalently

their Stanley-Reisner ideals $I_{\tilde{N}(\mathcal{B}_\Gamma)}$, $I_{N(\mathcal{B}_\Gamma)}$ are quadratic $\langle x_{i_1}x_{j_1}, \ldots, x_{i_\ell}x_{j_\ell} \rangle$

REU Problem 7 Parts (a,b,c)

(a) Prove CONJ (Lam-Pylyavskyy for $\mathcal{B} = \mathcal{B}_\Gamma$) $\tilde{N}(\mathcal{B})$ triangulates $\mathbb{S}^{n-2}$ and even the boundary of a (simplicial) convex polytope

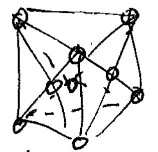
(b) Characterize the digraphs Γ for which $N(\mathcal{B}_\Gamma)$, $\tilde{N}(\mathcal{B}_\Gamma)$ are flag.

(c) Prove CONJ: $\tilde{N}(\overset{1}{\circ}\!-\!\overset{2}{\circ}\!-\!\cdots\!-\!\overset{n}{\circ})$ and $\mathrm{Assoc}_{n+1} = N(\overset{1}{\circ}\!-\!\overset{2}{\circ}\!-\!\cdots\!-\!\overset{n}{\circ}\!-\!\overset{n+1}{\circ})$ have same f-vector. Are they isomorphic?

3. f, h, γ-vectors

f-vectors of $(d-1)$-spheres are <u>too big</u> and <u>redundant</u>!

e.g. they satisfy Euler's relation $-f_{-1}+f_0-f_1+f_2-\cdots+(-1)^{d-1}f_d=\tilde{\chi}(\mathbb{S}^{d-1})=(-1)^{d-1}$

e.g.  $\cong \mathbb{S}^2$ $\quad f=(1,9,21,14)$

$$-1+9-21+14=+1=(-1)^2$$

Assoc_4

<u>THM</u> (Dehn 1905 – Sommerville 1927) All linear relations among f_i for $(d-1)$-spheres Δ are given by $h_i=h_{d-i}$ for $1\le i\le\lfloor\frac{d}{2}\rfloor$

where $h=(h_0,h_1,\ldots,h_d)$ is the <u>h-vector</u> of Δ defined by

$$\sum_{i=0}^{d} h_i(t+1)^{d-i}=\sum_{i=-1}^{d-1} f_i t^{d-1-i}$$

$$\Updownarrow$$

$$\sum_{i=0}^{d} h_i t^{d-i}=\sum_{i=-1}^{d-1} f_i (t-1)^{d-1-i}$$

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e.g. $f=(1,9,21,14)$ for $\Delta=\mathrm{Assoc}_4$ $\rightsquigarrow$ $1\cdot(t-1)^3+9(t-1)^2+21(t-1)^1+14(t-1)^0$

$$=t^3+t^2(-3+9)+t^1(3-18+21)+t^0(-1+9-21+14)$$

$$=t^3+6t^2+6t+1$$

$$\rightsquigarrow h=(1,6,6,1)$$

$$\underbrace{h_0\ \ h_1\ \ h_2\ \ h_3}_{\text{Dehn-Sommerville}}$$

<u>FACT:</u> In general, Assoc_n has $h_k=\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$ = <u>Narayana numbers</u>

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h-vector entries are also <u>smaller</u>, but still nonnegative:

<u>THM</u> (Stanley 1970) $\Delta\cong\mathbb{S}^{d-1}$ (or even Δ which are <u>Cohen-Macaulay</u>) have $h=(h_0,h_1,\ldots,h_d)$ <u>nonnegative</u>.

:= MFR of $\Bbbk[\Delta]$ is as <u>short as possible</u>, namely #vertices(Δ) − dim(Δ)

$\Leftrightarrow$ topological definitions...

(7) REU Exercise 17

Show that the $\mathbb{Z}$-graded Hilbert series of $\Bbbk[\Delta]$ for a d-dimensional complex Δ

$$\mathrm{Hilb}(\Bbbk[\Delta],t) := \sum_{i\geq 0} \dim_{\Bbbk}\big(\underbrace{\Bbbk[\Delta]_i}_{i^{\text{th}}\text{ graded component}}\big)\cdot t^i$$

has two expressions:

$$\mathrm{Hilb}(\Bbbk[\Delta],t) \overset{(a)}{=} \sum_{i=-1}^{d-1} f_i \frac{t^{i+1}}{(1-t)^{i+1}}$$

$$\overset{(b)}{=} \frac{\sum_{i=0}^{d} h_i t^i}{(1-t)^d}$$

For flag simplicial spheres $\Delta \cong \mathbb{S}^{d-1}$, even the h-vector is (conjecturally) too big:

write the h-polynomial $\overset{1}{\overset{\shortparallel}{h_0}}t^d + h_1 t^{d-1} + \ldots + h_{d-1}t + \overset{1}{\overset{\shortparallel}{h_d}}$

uniquely as a sum of centered binomials:

$$= 1\,(t+1)^d + \gamma_1 t^1 (t+1)^{d-2} + \gamma_2 t^2 (t+1)^{d-4} + \ldots$$

to get the γ-vector $(\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_{\lfloor d}$

e.g. $h(\mathrm{Assoc}_3,t) = t^2+3t+1 = 1\,(t+1)^2 + 1\cdot t(t+1)^0 \Rightarrow \gamma = (1,1)$

1	3	1
1	2	1
	1	

$h(\mathrm{Assoc}_4,t) = t^3+6t^2+6t+1 = 1\cdot(t+1)^3 + 3\cdot t(t+1)^1 \Rightarrow \gamma = (1,3)$

1	6	6	1
1	3	3	1
	3	3	

$h(\mathrm{Assoc}_5,t) = t^4+10t^3+20t^2+10t+1 = 1\cdot(t+1)^4 + 6t(t+1)^2 + 2t^2(t+1)^0 \Rightarrow \gamma = (1,6,2)$

1	10	20	10	1
1	4	6	4	1
	6	12	6	
		2		

(8) CONJ (S. Gal 2005) Flag simplicial spheres have γ nonnegative,
(proven for flag $N(\mathcal{B})$ by N. Aisbett 2012, V. Volodin 2009)

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4. <u>REU Problem 7(d)</u>

- Study h-vectors of $\tilde{N}(\mathcal{B})$,
  and γ-vectors of $\tilde{N}(\mathcal{B})$ when it is a <u>flag</u> complex
- How do they relate to h-vectors or γ-vectors of $N(\mathcal{B})$?
- Do they have combinatorial interpretations
  similar to those for $N(\mathcal{B})$ and $N(\mathcal{B}_\Gamma)$, Γ undirected
  as in Postnikov-Williams-R.?