Extended Nestohedra and their Face Numbers

Quang Dao, Christina Meng, Julian Wellman, Zixuan Xu,
Calvin Yost-Wolff, Teresa Yu

UMN REU

July 24, 2019

Introduction

- **Nestohedra** are a well-understood class of convex polytopes
- Generalized by Lam–Pylyavskyy '15 and Devadoss–Heath–Vipismakul '11 independently
  - LP-algebras
  - Moduli space of a Riemann surface

What is known so far

	Non-extended	Extended ($\square$)
When flag	Y	
Link decomposition	Y	
Polytopality	Y	
Gal's conjecture	Y	
Combinatorial interpretation for γ-vector	chordal $\mathcal{B}$	
Shellings	$\mathcal{B}_{K_n}$	
Cluster/LP algebras	Y	
How are they related?		

Goal: fill in the column!

Building Sets

Definition

A **(connected) building set** $\mathcal{B}$ on $[n] := \{1, \ldots, n\}$ is a collection of subsets of $[n]$ such that

1. $\mathcal{B}$ contains all singletons $\{i\}$ and the whole set $[n]$
2. if $I, J \in \mathcal{B}$ with $I \cap J \neq \varnothing$, then $I \cup J \in \mathcal{B}$.

Definition

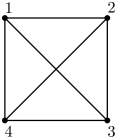
For an undirected graph G, its corresponding **graphical building set** $\mathcal{B}_G$ is

$$\mathcal{B}_G = \{I \subseteq V(G) \mid G[I] \text{ is connected}\}.$$

Examples of Building Sets

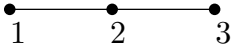
Complete graph K_n

- all subsets of $[n]$
- $\mathcal{B}_{K_4} = \{1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 234, 124, 134, 1234\}$



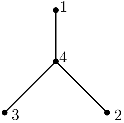
Path graph P_n

- all interval subsets of $[n]$
- $\mathcal{B}_{P_3} = \{1, 2, 3, 12, 23, 123\}$



Star graph $K_{1,n}$

- All singletons and all subsets of $[n+1]$ that contain $n+1$
- $\mathcal{B}_{K_{1,3}} = \{1, 2, 3, 4, 14, 24, 34, 124, 134, 234, 1234\}$



Nested Collections

Definition

For a building set $\mathcal{B}$, a **nested collection** N of $\mathcal{B}$ is a collection of elements $\{I_1, \ldots, I_m\}$ of $\mathcal{B} \setminus \{[n]\}$ such that

1. for any $i \neq j$, I_i and I_j are either nested or disjoint
2. for any $I_{i_1}, \ldots, I_{i_k}$ pairwise disjoint, their union is not an element of $\mathcal{B}$

Consider $\mathcal{B} = \mathcal{B}_{P_4} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$.

- $\{1, 3, 34\}$ is a nested collection
- $\{1, 2, 23\}$ **is not** a nested collection since $\{1\} \cup \{2\} \in \mathcal{B}$.

Nested Complexes

Definition

For a connected building set $\mathcal{B}$ on $[n]$, the **nested set complex** $\mathcal{N}(\mathcal{B})$ is the simplicial complex with

- vertices $\{I \mid I \in \mathcal{B} \setminus [n]\}$
- faces $\{I_1, \ldots, I_m\}$ that are nested collections of $\mathcal{B}$

Definition

The **nestohedron** $\mathcal{P}(\mathcal{B})$ is the polytope dual to the nested set complex $\mathcal{N}(\mathcal{B})$.

In the literature, $\mathcal{P}(\mathcal{B}_{P_n})$ is known as the **associahedron**, and $\mathcal{P}(\mathcal{B}_{K_n})$ is known as the **permutohedron**.

Extended Nested Collections

Definition

For a building set $\mathcal{B}$ on $[n]$, an **extended nested collection** $N^\square$ of $\mathcal{B}$ is a collection of elements $\{I_1, \ldots, I_m, x_{i_1}, \ldots, x_{i_r}\}$ such that

1. $I_k \in \mathcal{B}$ for all k, and $\{I_1, \ldots, I_m\}$ form a nested collection of $\mathcal{B}$
2. $i_j \in [n]$ for all j, and $i_j \notin I_k$ for all $1 \leq k \leq m$

$\mathcal{B} = \mathcal{B}_{P_4}$

- $\{1, 3, 34, x_2\}$ is an extended nested collection
- $\{1, 3, 34, x_4\}$ **is not** an extended nested collection

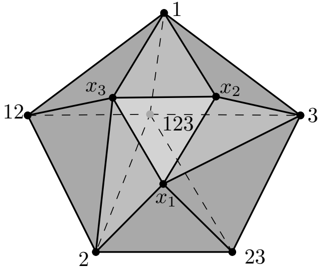
Extended Nested Complexes and Nestohedra

Definition

For a building set $\mathcal{B}$ on $[n]$, the **extended nested set complex** $\mathcal{N}^{\square}(\mathcal{B})$ is the simplicial complex with

- vertices $\{I \mid I \in \mathcal{B}\} \cup \{x_i \mid i \in [n]\}$
- faces $\{I_1, \ldots, I_m, x_{i_1}, \ldots, x_{i_r}\}$ that are extended nested collections of $\mathcal{B}$

$$\mathcal{B} = \{1, 2, 3, 12, 23, 123\}$$



Extended Nested Complexes and Nestohedra

Definition

For a building set $\mathcal{B}$ on $[n]$, the **extended nested set complex** $\mathcal{N}^{\square}(\mathcal{B})$ is the simplicial complex with

- vertices $\{I \mid I \in \mathcal{B}\} \cup \{x_i \mid i \in [n]\}$
- faces $\{I_1, \ldots, I_m, x_{i_1}, \ldots, x_{i_r}\}$ that are extended nested collections of $\mathcal{B}$

Definition

The **extended nestohedron** $\mathcal{P}^{\square}(\mathcal{B})$ is the polytope dual to the extended nested set complex

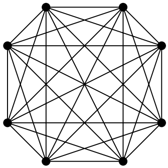
What is known so far

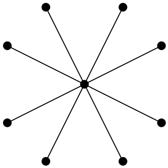
	Non-extended	Extended ($\square$)
When flag	Y	
Link decomposition	Y	
Polytopality	Y	
Gal's conjecture	Y	
Combinatorial interpretation for γ-vector	chordal $\mathcal{B}$	
Shellings	$\mathcal{B}_{K_n}$	
Cluster/LP algebras	Y	
How are they related?	$\mathcal{N}^{\square}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$ sometimes	

When is $\mathcal{N}^{\square}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$?

Theorem (Manneville – Pilaud '17)

Let G, G' be undirected graphs such that $\mathcal{N}^{\square}(\mathcal{B}_G) \simeq \mathcal{N}(\mathcal{B}_{G'})$. Then G is a spider and G' is the corresponding octopus.

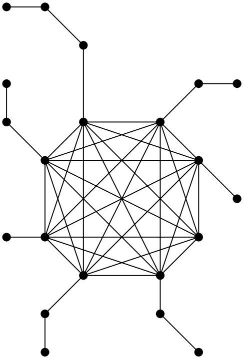




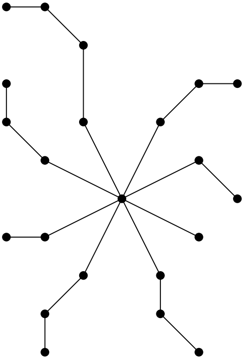
When is $\mathcal{N}^\square(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$?

Theorem (Manneville–Pilaud '17)

Let G, G' be undirected graphs such that $\mathcal{N}^\square(\mathcal{B}_G) \simeq \mathcal{N}(\mathcal{B}_{G'})$. Then G is a spider and G' is the octopus.



spider



octopus

When is $\mathcal{N}^{\square}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$?

Corollary (Manneville–Pilaud '17)

- $\mathcal{N}^{\square}(\mathcal{B}_{K_n}) \simeq \mathcal{N}(\mathcal{B}_{K_{1,n}})$ is the dual of the **stellohedron**.
- $\mathcal{N}^{\square}(\mathcal{B}_{P_n}) \simeq \mathcal{N}(\mathcal{B}_{P_{n+1}})$ is the dual of the $(n-2)$-**associahedron**.

Remark (REU '19)

When $G = C_4$, we do not have $\mathcal{N}^{\square}(\mathcal{B}_G) \simeq \mathcal{N}(\mathcal{B}')$ for any other building set $\mathcal{B}'$.

Theorem (REU '19)

If $\mathcal{B}$ is a building set on $[n]$ such that all elements $I \in \mathcal{B}$ are intervals, then there exists $\mathcal{B}'$ such that $\mathcal{N}^{\square}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$.

What is known so far

	Non-extended	Extended ($\square$)
When flag	Y	Y
Link decomposition	Y	
Polytopality	Y	
Gal's conjecture	Y	
Combinatorial interpretation for γ-vector	chordal $\mathcal{B}$	
Shellings	$\mathcal{B}_{K_n}$	
Cluster/LP algebras	Y	
How are they related?	$\mathcal{N}^{\square}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$ sometimes	

When is $\mathcal{N}^{\square}(\mathcal{B})$ flag?

Definition

A simplicial complex Δ is **flag** if Δ has no minimal non-faces of degree greater than 2. In other words, Δ is determined by its 1-skeleton.

Proposition (REU '19)

$\mathcal{N}(\mathcal{B})$ is flag if and only if $\mathcal{N}^{\square}(\mathcal{B})$ is flag.

For a graphical building set $\mathcal{B} = \mathcal{B}_G$, it was shown in (PRW '08) that $\mathcal{N}(\mathcal{B})$ is a flag simplicial complex.

Corollary (REU '19)

If G is an undirected graph, then $\mathcal{N}^{\square}(\mathcal{B}_G)$ is flag.

What is known so far

	Non-extended	Extended ($\square$)
When flag	Y	Y
Link decomposition	Y	Y
Polytopality	Y	
Gal's conjecture	Y	
Combinatorial interpretation for γ-vector	chordal $\mathcal{B}$	
Shellings	$\mathcal{B}_{K_n}$	
Cluster/LP algebras	Y	
How are they related?	$\mathcal{N}^{\square}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$ sometimes	

Link Decompositions of $\mathcal{N}(\mathcal{B})$ and $\mathcal{N}^\square(\mathcal{B})$

Theorem (Zelevinsky '06)

Let $\mathcal{B}$ be a building set on S. Then the link of $C \in \mathcal{B}$ in $\mathcal{N}(\mathcal{B})$

$$\mathcal{N}(\mathcal{B})_C \simeq \mathcal{N}(\mathcal{B}|_C) * \mathcal{N}(\mathcal{B}/C).$$

Theorem (REU '19)

For the extended nested complex $\mathcal{N}^\square(\mathcal{B})$, we have:

$$\mathcal{N}^\square(\mathcal{B})_{x_i} \simeq \mathcal{N}^\square(\mathcal{B}_1) * \cdots * \mathcal{N}^\square(\mathcal{B}_k)$$

where $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are the connected components of $\mathcal{B}|_{[n]\setminus\{i\}}$, and

$$\mathcal{N}^\square(\mathcal{B})_C \simeq \mathcal{N}(\mathcal{B}|_C) * \mathcal{N}^\square(\mathcal{B}/C)$$

for $C \in \mathcal{B}$.

What is known so far

	Non-extended	Extended ($\square$)
When flag	Y	Y
Link decomposition	Y	Y
Polytopality	Y	Y
Gal's conjecture	Y	
Combinatorial interpretation for γ-vector	chordal $\mathcal{B}$	
Shellings	$\mathcal{B}_{K_n}$	
Cluster/LP algebras	Y	
How are they related?	$\mathcal{N}^{\square}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$ sometimes	

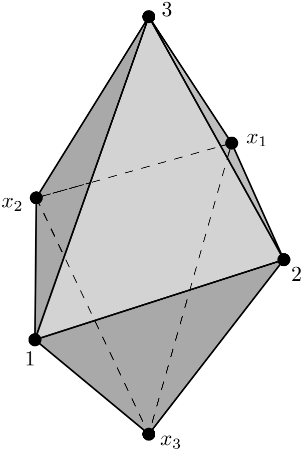
Polytopality

Theorem (REU '19)

For any building set B, $\mathcal{N}^{\square}(\mathcal{B})$ can be realized as the boundary of a polytope $\mathcal{N}_{\mathcal{B}}$.

Polytopality

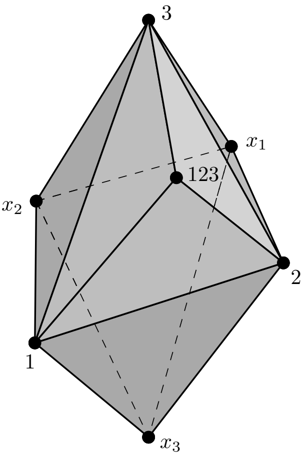
- Consider $\mathbb{R}^n$ with standard basis vectors $e_1, \ldots, e_n$. Start with cross polytope in $\mathbb{R}^n$ with vertices e_i labeled $\{i\} \in \mathcal{B}$ and vertices $-e_i$ labeled x_i for all $i \in [n]$.



$$\mathcal{B} = \{1, 2, 3, 12, 123\}$$

Polytopality

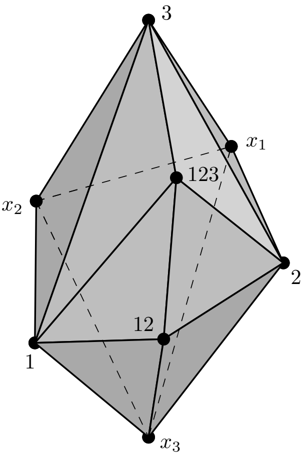
- Consider $\mathbb{R}^n$ with standard basis vectors $e_1, \ldots, e_n$. Start with cross polytope in $\mathbb{R}^n$ with vertices e_i labeled $\{i\} \in \mathcal{B}$ and vertices $-e_i$ labeled x_i for all $i \in [n]$.
- Order the non-singletons of $\mathcal{B}$ by decreasing cardinality, then for each $I \in \mathcal{B}$ a non-singleton, perform stellar subdivision on the face $\mathcal{I} = \{\{i\} \mid i \in I\}$, with the new added vertex labeled I.



$$\mathcal{B} = \{1, 2, 3, 12, 123\}$$

Polytopality

- Consider $\mathbb{R}^n$ with standard basis vectors $e_1, \ldots, e_n$. Start with cross polytope in $\mathbb{R}^n$ with vertices e_i labeled $\{i\} \in \mathcal{B}$ and vertices $-e_i$ labeled x_i for all $i \in [n]$.
- Order the non-singletons of $\mathcal{B}$ by decreasing cardinality, then for each $I \in \mathcal{B}$ a non-singleton, perform stellar subdivision on the face $\mathcal{I} = \{\{i\} \mid i \in I\}$, with the new added vertex labeled I.
- The boundary of the resulting polytope $\mathcal{N}_\mathcal{B}$ will be isomorphic to $\mathcal{N}^\square(\mathcal{B})$.



$$\mathcal{B} = \{1, 2, 3, 12, 123\}$$

Polytopality

We also obtain a polytopal realization of $\mathcal{P}^\square(\mathcal{B})$ as a Minkowski sum.

Theorem (REU '19)

Let B a building set on $[n]$, and consider $\mathbb{R}^n$ with standard basis vectors $e_1, \dots, e_n$. Then $\mathcal{P}^\square(\mathcal{B})$ is isomorphic to the boundary of the polytope:

$$\mathcal{P} := \sum_{i \in [n]} \mathrm{Conv}(0, e_i) + \sum_{I \in B} \mathrm{Conv}(\{e_S | S \subsetneq I\}),$$

where the coordinates of e_S are given by the indicator function on S i.e. $(e_S)_i = 1$ if and only if $i \in S$.

Intuitively, the first sum is the n-dimensional cube $\mathcal{C}^n$, while each term of the next sum corresponds to shaving a face $I \in \mathcal{B}$ from the cube.

What is known so far

	Non-extended	Extended ($\square$)
When flag	Y	Y
Link decomposition	Y	Y
Polytopality	Y	Y
Gal's conjecture	Y	
Combinatorial interpretation for γ-vector	chordal $\mathcal{B}$	
Shellings	$\mathcal{B}_{K_n}$	
Cluster/LP algebras	Y	
How are they related?	$\mathcal{N}^{\square}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$ sometimes, through f- and h-vectors	

f, h, γ-vectors for $\mathcal{P}^{\square}(\mathcal{B})$

Definition

For a polytope $\mathcal{P}$, let f_k be the number of k-dimensional faces of $\mathcal{P}$. The f-vector of $\mathcal{P}$ is defined to be $f = (f_{-1}, \dots, f_{d-1})$.

Definition

The h-vector $h = (h_0, \dots, h_d)$ of $\mathcal{P}$ is defined by

$$\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} f_{i-1}(t-1)^{i-1}$$

If $\mathcal{P}$ is a simple polytope, then we have $h_i = h_{d-i}$ for all $i = 0, \dots, \lfloor \frac{d}{2} \rfloor$.

f, h, γ-vectors for $\mathcal{P}^{\square}(\mathcal{B})$

Proposition (REU '19)

$$f_{\mathcal{P}^{\square}(\mathcal{B})}(t) = \sum_{S \subseteq [n]} (t+1)^{n-|S|} f_{\mathcal{P}(\mathcal{B}|_S)}(t)$$

What is known so far

	Non-extended	Extended ($\square$)
When flag	Y	Y
Link decomposition	Y	Y
Polytopality	Y	Y
Gal's conjecture	Y	Y
Combinatorial interpretation for γ-vector	chordal $\mathcal{B}$	
Shellings	$\mathcal{B}_{K_n}$	
Cluster/LP algebras	Y	
How are they related?	$\mathcal{N}^{\square}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$ sometimes, through f- and h-vectors	

Gal's Conjecture for Flag $\mathcal{P}^{\Box}(\mathcal{B})$

Definition

The γ-**vector** for a simple polytope $\mathcal{P}$ is given by

$$\sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (t+1)^{d-2i} = \sum_{j=0}^{d} h_j t^j.$$

Gal's Conjecture (2005)

The γ-vector of any flag simple polytope is nonnegative.

- Shown true for $\mathcal{P}(\mathcal{B})$ by Volodin '10

Gal's Conjecture for Flag $\mathcal{P}^{\Box}(\mathcal{B})$

Theorem (REU '19)

Gal's conjecture is true for flag extended nestohedra $\mathcal{P}^{\Box}(\mathcal{B})$.

- Start with flag building set $\mathcal{B}$
- There exists minimal flag building set $\mathcal{B}_{\text{min}} \subseteq \mathcal{B}$, and $\mathcal{P}^{\Box}(\mathcal{B}_{\text{min}})$ has nonnegative γ-vector
- Add back in elements $\mathcal{B} \setminus \mathcal{B}_{\text{min}}$
  - Corresponds to shaving a codimension 2 face
- Use link decomposition to show that γ-vector remains nonnegative

What is known so far

	Non-extended	Extended ($\square$)
When flag	Y	Y
Link decomposition	Y	Y
Polytopality	Y	Y
Gal's conjecture	Y	Y
Combinatorial interpretation for γ-vector	chordal $\mathcal{B}$	chordal $\mathcal{B}$
Shellings	$\mathcal{B}_{K_n}$	
Cluster/LP algebras	Y	
How are they related?	$\mathcal{N}^{\square}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$ sometimes, through f- and h-vectors	

Gal's Conjecture for Flag $\mathcal{P}^\square(\mathcal{B})$

- **Chordal**: nice class of building sets, includes $\mathcal{B}_{K_n}, \mathcal{B}_{P_n}, \mathcal{P}_{K_{1,n}}$
- $\widehat{\mathfrak{S}}_n(\mathcal{B}) = \{\mathcal{B}\text{-permutations with no double or final descents}\}$

Theorem (Postnikov–Reiner–Williams '08)

For chordal $\mathcal{B}$ on $[n]$, $$\gamma_{\mathcal{P}(\mathcal{B})}(t) = \sum_{w \in \widehat{\mathfrak{S}}_n(\mathcal{B})} t^{\mathrm{des}(w)}.$$

- $\widehat{\mathfrak{S}}^\square_{n+1} = \{\text{extended } \mathcal{B}\text{-permutations with no double or final descents}\}$

Theorem (REU '19)

For chordal $\mathcal{B}$ on $[n]$, $$\gamma_{\mathcal{P}^\square(\mathcal{B})}(t) = \sum_{w \in \widehat{\mathfrak{S}}^\square_{n+1}(\mathcal{B})} t^{\mathrm{des}(w)}.$$

What is known so far

	Non-extended	Extended ($\square$)
When flag	Y	Y
Link decomposition	Y	Y
Polytopality	Y	Y
Gal's conjecture	Y	Y
Combinatorial interpretation for γ-vector	chordal $\mathcal{B}$	chordal $\mathcal{B}$
Shellings	$\mathcal{B}_{K_n}$	$\mathcal{B}_{K_n}$
Cluster/LP algebras	Y	
How are they related?	$\mathcal{N}^{\square}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$ sometimes, through f- and h-vectors	

Weak Bruhat Order

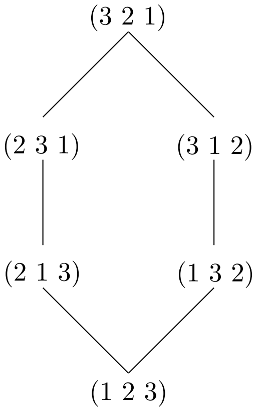
- $w = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \in \mathfrak{S}_n$
- Transpositions $s_i = \begin{pmatrix} i & i+1 \end{pmatrix}$
- $\ell(w) := |\{1 \leq i < j \leq n \mid a_i > a_j\}|$, i.e. the minimum number of transpositions

Definition

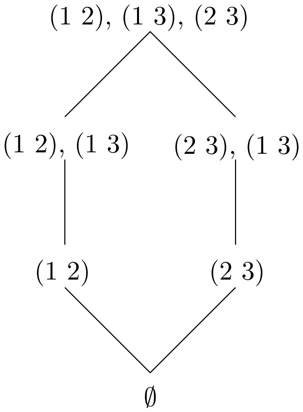
The **weak Bruhat order** on $\mathfrak{S}_n$ is defined by the following:

$$\pi \lessdot \sigma \text{ if and only if } \ell(\sigma) = \ell(\pi) + 1 \text{ and } \sigma = \pi \cdot s_i$$

Weak Bruhat Order



weak Bruhat order on $\mathfrak{S}_3$



inversion sets

Partial Permutations

Definition

Define the set of **partial permutations** on $[n]$, denoted $\mathfrak{P}_n$, to be set of permutations $w \in \mathfrak{S}_S$ for some $S \subseteq [n]$.

$$\mathfrak{P}_2 : \quad \underbrace{\begin{pmatrix}1 & 2\end{pmatrix}, \begin{pmatrix}2 & 1\end{pmatrix}}_{S=\{1,2\}}, \underbrace{\begin{pmatrix}1\end{pmatrix}}_{S=\{1\}}, \underbrace{\begin{pmatrix}2\end{pmatrix}}_{S=\{2\}}, \underbrace{\begin{pmatrix}\end{pmatrix}}_{S=\varnothing}$$

Remark

- $\mathfrak{S}_n$ is in bijection with facets of $\mathcal{N}(\mathcal{B}_{K_n})$
- $\mathfrak{P}_n$ is in bijection with the facets of $\mathcal{N}^{\square}(\mathcal{B}_{K_n})$

Partial Order on $\mathfrak{P}_n$

Definition (REU '19)

Define map $\varphi : \mathfrak{P}_n \to \mathfrak{S}_{n+1}$ as follows.

- Consider partial permutation $w \in \mathfrak{S}_S$, $S \subseteq [n]$
- Append numbers in $[n+1] \setminus S$ to end of w in descending order
- Resulting permutation $\varphi(w) \in \mathfrak{S}_{n+1}$

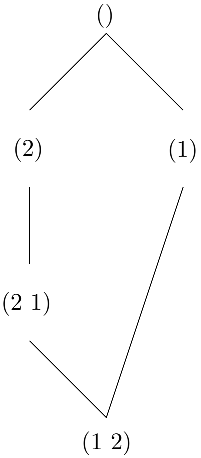
$$w = \begin{pmatrix} 2 & 4 & 1 \end{pmatrix} \in \mathfrak{P}_5 \implies \varphi(w) = \begin{pmatrix} 2 & 4 & 1 & 6 & 5 & 3 \end{pmatrix}$$

Definition (REU '19)

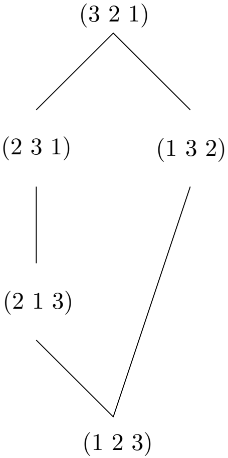
The **partial order** on $\mathfrak{P}_n$ defined by the following:

$$\pi < \sigma \text{ if and only if } \varphi(\pi) < \varphi(\sigma) \text{ in the weak Bruhat order on } \mathfrak{S}_{n+1}$$

Partial Order on $\mathfrak{P}_n$



$\mathfrak{P}_2$



weak Bruhat order on $\varphi(\mathfrak{P}_2)$

Definition

A **congruence** on a lattice L is an equivalence relation Θ on elements of L which respects joins and meets, i.e. if $a_1 \equiv a_2$ and $b_1 \equiv b_2$, then

$$a_1 \wedge b_1 \equiv a_2 \wedge b_2, \qquad a_1 \vee b_1 \equiv a_2 \vee b_2.$$

A **lattice quotient** L/Θ is a partial order on the equivalence classes under Θ:

$$[a]_\Theta \leq [b]_\Theta \Leftrightarrow x \leq_L y \text{ for some } x \in [a], y \in [b].$$

Proposition (REU '19)

The defined partial order on $\mathfrak{P}_n$ is a lattice quotient of the weak Bruhat order on $\mathfrak{S}_{n+1}$.

Partial Order on $\mathfrak{P}_n$

Corollary (McConville '16, Reading '02)

- Every interval of $\mathfrak{P}_n$ is contractible or homotopy equivalent to a sphere
- If $x = \bigvee^{\mathfrak{P}_n} Y$ for some $Y \subseteq \mathfrak{P}_n$, then $x = \bigvee^{\mathfrak{S}_{n+1}} Y$
- Möbius function $\mu(u, v)$ only takes on values $0, \pm 1$

Shellings of $\mathcal{N}(\mathcal{B}_{K_n}), \mathcal{N}^{\square}(\mathcal{B}_{K_n})$

- **Shellings**: nice way to build up a simplicial complex facet by facet

Theorem (Björner '84)

Label facets of $\mathcal{N}(\mathcal{B}_{K_n})$ by permutations $w \in \mathfrak{S}_n$. If $\pi_1 < \cdots < \pi_k$ is a linear extension of the weak Bruhat order on $\mathfrak{S}_n$, then $F_{\pi_1}, \ldots, F_{\pi_k}$ gives a shelling of $\mathcal{N}(\mathcal{B}_{K_n})$.

Theorem (REU '19)

Label facets of $\mathcal{N}^{\square}(\mathcal{B}_{K_n})$ by partial permutations $w \in \mathfrak{P}_n$. If $\pi_1 < \cdots < \pi_k$ is a linear extension of the partial order on $\mathfrak{P}_n$, then $F_{\pi_1}, \ldots, F_{\pi_k}$ gives a shelling of $\mathcal{N}^{\square}(\mathcal{B}_{K_n})$.

What is known so far

	Non-extended	Extended ($\square$)
When flag	Y	Y
Link decomposition	Y	Y
Polytopality	Y	Y
Gal's conjecture	Y	Y
Combinatorial interpretation for γ-vector	chordal $\mathcal{B}$	chordal $\mathcal{B}$
Shellings	$\mathcal{B}_{K_n}$	$\mathcal{B}_{K_n}$
Cluster/LP algebras	Y	?
How are they related?	$\mathcal{N}^{\square}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$ sometimes, through f- and h-vectors, ...?	

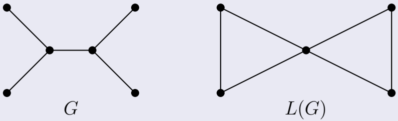
Future Work

- Is there a combinatorial interpretation for the γ-vector of $\mathcal{P}(\mathcal{B}), \mathcal{P}^{\square}(\mathcal{B})$ of arbitrary flag building sets?
- When does a total ordering on (extended) $\mathcal{B}$-permutations give a shelling of the (extended) nested complexes?
- Can $\mathcal{N}^{\square}(\mathcal{B})$ provide a combinatorial interpretation of the exchange polynomials of LP-algebras? (Lam–Pylyavskyy)

Conjecture

Let G be a forest and $L(G)$ be the line graph of G. Then

$$f_{\mathcal{P}(\mathcal{B}_G)}(t) = f_{\mathcal{P}^{\square}(\mathcal{B}_{L(G)})}(t).$$



Acknowledgements and References

- Thank you to Vic Reiner and Sarah Brauner for all of their support and guidance!

- See our REU report for a complete set of references

Questions?