Symmetric function theory - a very brief intro.

(TAs: Emily Tibor + Claire Frechette)

elements of $\underset{\text{integers}}{\mathbb{Z}}[\underbrace{x_1, \ldots, x_n}_{\text{sometimes } \underline{x}}]$ which remain

unchanged when we permute variables.

Examples: $n = 3$.

$$e_2(\underline{x}) := x_1x_2 + x_2x_3 + x_1x_3$$

$$h_2(\underline{x}) := x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 + x_1x_3$$

$$p_4(\underline{x}) := x_1^4 + x_2^4 + x_3^4$$

$$m_{(2,1,1)}(\underline{x}) := x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2$$

partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \overset{\geq 0}{\lambda_n})$

(often think of integer $k = \lambda_1 + \lambda_2 + \cdots + \lambda_n$)

write $|\lambda| = k$.

Also make partition versions of e's, h's, p's

$$p_\lambda := p_{\lambda_1} \cdots p_{\lambda_n} \qquad \lambda_i \in \mathbb{Z}_{\geq 0}$$

$S_n :=$ symm. gp. on n letters.

then $\Lambda_n := \mathbb{Z}[x_1, \dots, x_n]^{S_n}$ $S_n \curvearrowright$ super. : fixed pts under action of S_n.

<u>Fact</u>: $\Lambda_n \cong \mathbb{Z}[e_1, \dots, e_n]$

Or we can refine our set-up : $\Lambda_n = \bigoplus_{k \geq 0} \Lambda_n^k$ $\leftarrow$ deg. k homog. polys.

<u>Fact</u>: $\{m_\lambda\}_{\ell(\lambda) \leq n, |\lambda| = k}$ is $\mathbb{Z}$-basis for Λ_n^k.

"length" of λ
\# parts

Introduce scalar product $\langle\ ,\ \rangle$ defined by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu} = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{else.} \end{cases}$$

weird def'n out of nowhere. But has nice properties:
positive definite (inner product), symmetric.
"Hall inner product"

Does there exist an orthonormal basis for $\langle\ ,\ \rangle$?

Yes! Schur polynomials. Denote them by

$s_\lambda(x_1, \dots, x_n)$, so $\langle s_\lambda(\underline{x}), s_\mu(\underline{x}) \rangle = \delta_{\lambda\mu}$

Two definitions of Schur polys

① $$s_\lambda(x_1, \dots, x_n) = \frac{A_{\lambda+\rho}}{A_\rho} \qquad \rho = (n-1, n-2, \dots, 1, 0)$$

where
$$A_\mu = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \sigma(\underline{x}^\mu) \qquad \underline{x}^\mu := x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$$

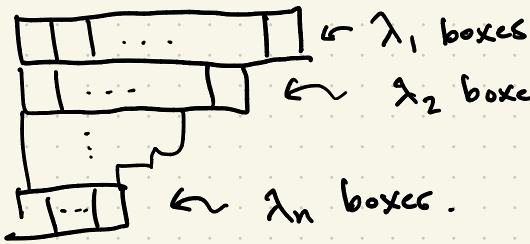
"alternator"

± 1 according to whether
σ is even, odd # of transp.

e.g. $\sigma = (12)$
then $\operatorname{sgn}(\sigma) = -1$, $\sigma(x_1^3 x_2 x_3) = x_1 x_2^3 x_3$.

note $A_\mu = 0$ if $\mu_i = \mu_{i+1}$ in μ.

② Given $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$, form a Young diagram:



Fill it with alphabet $\{1, \ldots, n\}$ so that it is

- weakly increasing in rows
- strictly increasing in columns.

"semistandard Young tableaux" (SSYT)

SSYT(λ) : set of SSYTs w/ shape λ

Example. $\lambda = (3,1,0)$

$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ ok

$\begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$ BAD!

$$s_\lambda(x_1, \cdots, x_n) := \sum_{T \in SSYT(\lambda)} \underline{x}^{wt(T)}$$

(Second defn. of Schur polys)

$wt(T) = (\#1\text{'s}, \#2\text{'s}, \ldots, \#n\text{'s in } T)$

So $wt\left(\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right) = (2,1,1)$

so $\underline{x}^{wt\left(\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right)} = x_1^2 x_2^1 x_3^1$

REU Exercise 1.1: (0) Compute $s_{5,4,1}(x_1, x_2, x_3)$ using two definitions.

(1) A_μ can be expressed as the determinant of a matrix. Show this.

(2) Evaluate A_ρ as an explicit product in x_i's.

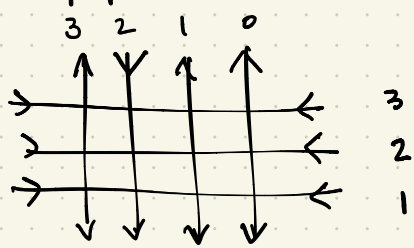
Part II: Statistical mechanical model.

Give 3rd def'n of Schur polynomials using stat. mech.

Rough form: $$s_\lambda(x_1, \dots, x_n) = \sum_{\substack{\text{admissible} \\ \text{states } S}} \text{weight}(S)$$

Make a grid with n rows, N columns.

e.g. $n = 3$, $N = 4$: $(N > \lambda_1)$



$\lambda = (1, 0, 0)$

$\lambda + \rho = (3, 1, 0)$ if $\rho = (2, 1, 0)$

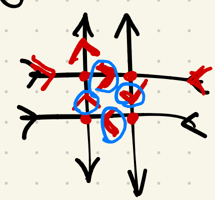
up at parts of $\lambda + \rho$.

Decorate boundary edges according to a partition λ. (with in/out arrows)

up arrows on top boundary at columns whose index matches parts of $\lambda + \rho$, down at rest.

To such a grid with boundary corresp. to λ, find all fillings with arrows on edges so that, at each vertex (i.e. crossing) there are 2 in arrows / 2 out arrows on edges adjacent to vertex.

Tiny example: $(\lambda = (0,0),\ \lambda+\rho = (1,0))$



How many fillings satisfy my rule?

$\leftarrow$ this is one such filling. In this example, there is one more.

"admissible state" of lattice model w/ λ boundary.

Make a function

$$P_\lambda(x_1, \ldots, x_n) = \sum_{\substack{\text{admissible} \\ \text{states } S \text{ of} \\ \lambda+\rho\text{-grid}}} wt(S).$$

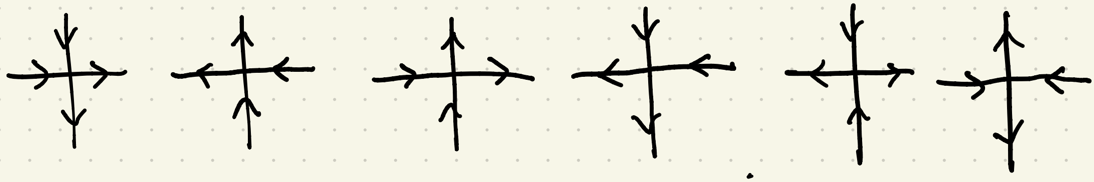
"good fillings" 2 ins / 2 outs.

$$wt(S) = \prod_{v \in S} wt(v)$$

all vertices in S.

and finally, $wt(v)$ will depend primarily on the adjacent edges.

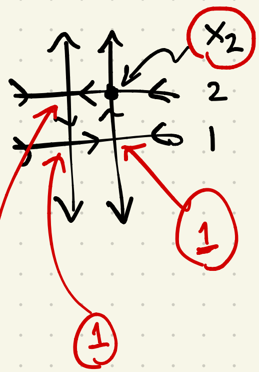
in arrows: "NW" "SE" "SW" "NE" "NS" "EW"



wt: 1 x_i in row i 0 x_i in row i x_i 1.

Example:

$\lambda+\rho = (1,0)$



$$\mathrm{wt}(S) = \prod_{v \in S} \mathrm{wt}(v) = x_2.$$

Claim: With these (Boltzmann) weights, $\mu = \lambda+\rho$.

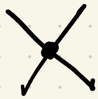
$$(N > \cancel{\lambda_1}\ \mu_1) \quad P_\lambda(x_1, \dots, x_n) = (*)\, S_\lambda(x_1, \dots, x_n)$$

$(N >> 0)$ n = # rows. $(*)$: up to very simple expression in x_i's.

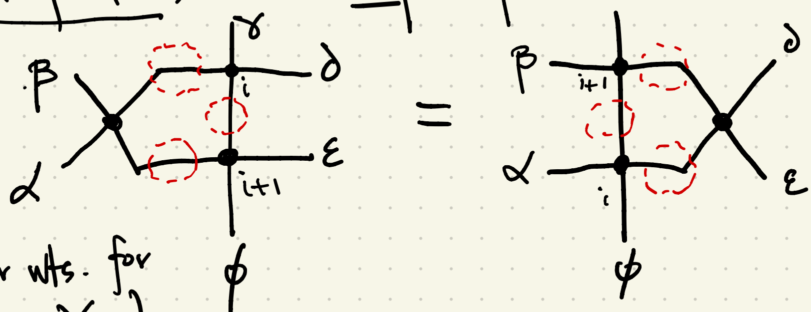
REU Exercise 1.2: Determine this simple expression for $(*)$ and prove the claim using def'n of s_λ as sum of SSYT.

(and determine effect, if any, on P_λ of the choice of N)

Introduce a tool which makes lattice model methods so powerful. - Yang-Baxter equation. (in picture form).

Goal: Find a new set of 6 weights for new family of vertices:  such that the following partition functions are always equal:

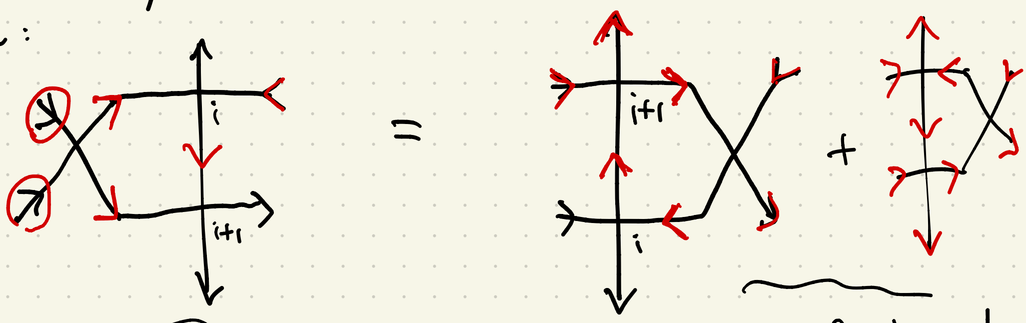
gen. funct. of wts



for arrow choices $\alpha, \beta, \gamma, \delta, \varepsilon, \phi$

(solve for wts. for )

Example:



gen. function with one state.

gen. function w/ two states.

$P($  $) = wt(S) = wt($  $)\, wt(EW_i)\, wt(NW_{i+1})$

unknown.

Cliff-hanger: Why is this solution in wts X to this "Yang-Baxter equation" so useful?

REU Exercise: 1.3: Find a solution to the YBE for the weights for P_λ described above.

(YBEs only exist for very special choices of weights)

REU Problem: (3 projects)

① (broad) Catalog identities satisfied by Schur functions. Determine which ones have lattice model proofs.

② (super specific) Evaluate a partition function of lattice models in earlier work of Brubaker-Schultz.

③ (shot in dark) Explore lattice models for k-Schur functions. Do they exist?