

Schur Function Identities and The Yang-Baxter Equation

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Abstract

We construct a eight-vertex lattice model which includes two different types of free-fermionic Boltzmann weights. We show our model admits a solution to the Yang-Baxter equation. Moreover we prove that the partition function of our model is equal to a sum of products of Schur functions which appears in the statement of the Cauchy identity by a partitioning the states of the model a special way. We conjecture that once it is understood how to use the Yang-Baxter equation to evaluate the partition function of our model this will yield a lattice model proof of the Cauchy identity for Schur polynomials.

1 Introduction

In statistical mechanics one attempts to recover global properties of a system by considering local interactions. In order to achieve this goal one defines the partition function of a system to be a sum of locally defined Boltzmann weights over all states of the system. Baxter realized that the existence of a star-triangle identity, now known as the Yang-Baxter equation, allowed one to deduce certain symmetry properties of the partition function which in many situations allowed for the explicit evaluation of the partition functions [1, 2]. As it turns out these tools from statistical mechanics are intimately connected to many areas of mathematics. Most importantly for us, since many symmetric polynomials may be represented as partition functions of solvable lattice models, these methods are useful for studying symmetric function theory and its generalizations

For example Brubaker, Bump, and Friedberg [3] showed that the six-vertex model in the free-fermionic regime satisfies a parametrized Yang-Baxter equation with non-abelian parameter group $GL(2, \mathbb{C}) \times \mathbb{C}^\times$ and as consequence they were able to construct a six-vertex lattice model with partition function given by a Schur polynomial times a deformation of the Weyl character formula. A whole host of other symmetric functions have been studied fruitfully using lattice models [4-9].

There are three classical multiplicative identities for Schur functions: The Littlewood-Richardson rule, the dual Cauchy identity, and the Cauchy identity [10]. A lattice model proof of the Littlewood-Richardson rule was given by Zinn-Justin in [11] and later more refined identities were proven by Zinn-Justin and Wheeler [6] again using lattice models. Also a lattice model proof of the dual Cauchy identity for factorial Schur functions, a specialization gives Schur functions, was given in [8].

For us a lattice model proof consists of evaluating the partition function of a lattice model in two different ways. Usually one way is by partitioning the states of the model under consideration in a clever way allowing for evaluation of the partition function and the other is by using the existence of a Yang-Baxter equation to evaluate the partition function. The power of this technique is that any lattice model proof should be generalizable to any polynomial which may be represented by a lattice model. So producing a lattice model proof of any Schur function identity should give proofs of analogous identities for the wide range of polynomials which may

be represented using lattice models.

The aim of the present report is to show how lattice models and the Yang-Baxter equation may be used to give proofs of Schur function identities. We focus on the Cauchy identity for Schur polynomials:

Theorem 1.1 (Cauchy Identity). *For two finite alphabets of variables $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_m)$, we have*

$$\sum_{\lambda} s_{\lambda}(\mathbf{x})s_{\lambda}(\mathbf{y}) = \prod_{i=1}^n \prod_{j=1}^m \frac{1}{1 - x_i y_j}. \quad (1.1)$$

Where s_{λ} is a Schur polynomial [10] and the sum is over all partitions λ with at most $\min(n, m)$ parts.

Our main result is the construction of a new lattice model, which lies in the eight-vertex free-fermionic regime, and a proof that its partition function realizes the left hand side of the Cauchy identity up to a simple monomial factor. We were not quite able to achieve a lattice model proof of the Cauchy identity as we have been unable to use the Yang-Baxter equation to evaluate the partition function in a second way and thus realize the Cauchy identity. We fully expect that it should be possible to use the Yang-Baxter equation evaluate the partition function. The main difficulty has been finding a way to make the computation tractable.

The structure of this report is as follows: First in Section 2 we review the Yang-Baxter equation in the context of the free-fermionic eight vertex model. Then after reviewing the Gamma ice model of [3] we make a review of Bump, McNamara, and Nakasuji's [8] lattice model proof of the dual Cauchy identity as it served as the main inspiration for our work on the Cauchy identity. We then proceed to define new Boltzmann weights in Section 5 and derive additional Yang-Baxter equations in Section 6 we use later. We then define a new half-infinite lattice model in Section 7. In Section 8 we prove that our half infinite partition function realizes one side of the Cauchy identity up to a simple monomial factor. We then explain the techniques we tried to evaluate the partition function of our model in a second way, which uses the Yang-Baxter equation, and the difficulties it presents.

2 The Yang-Baxter Equation

We review the eight-vertex model from statistical mechanics. We will consider two dimensional lattices, i.e. a planar graph where it is assumed that each vertex has four adjacent edges. Edges that join two vertices are referred to as *interior edges* while edges adjoined to only a single vertex are called *boundary edges*. Each edge is assigned a spin, \pm signs. Depending on the configuration of spins at each vertex, and possibly the vertices' location in the overall lattice each vertex will be assigned a *Boltzmann weight*. Write $\beta(v)$ for the Boltzmann weight of the vertex v . The Boltzmann weight at a vertex will be zero unless the number of adjacent edges labeled $-$ is even. So each vertex v has eight numbers associated to it. Let us denote these possibly non-zero Boltzmann weights at v as follows:

$a_1(v)$	$a_2(v)$	$b_1(v)$	$b_2(v)$	$c_1(v)$	$c_2(v)$	$d_1(v)$	$d_2(v)$

A lattice model, or statistical ensemble, \mathfrak{S} is the data of a graph, a choice of the possible Boltzmann weights at each vertex, and the boundary spins. A *state* \mathfrak{s} of the system \mathfrak{S} is defined to be an assignment of spins to the interior edges such that the Boltzmann weight of every vertex is non-zero. The *Boltzmann weight of a state* $\beta(\mathfrak{s})$ is the product $\prod_{v \in \mathfrak{S}} \beta(v)$. The *partition function* of a lattice model $Z(\mathfrak{S})$ is the sum $\sum_{\mathfrak{s} \in \mathfrak{S}} \beta(\mathfrak{s})$.

If at a vertex v the Boltzmann weights obey $a_1 a_2 + b_1 b_2 \neq 0$, $d_1 = d_2 = 0$ and

$$a_1 a_2 + b_1 b_2 - c_1 c_2 = 0$$

then v is said to be free-fermionic of *type C*. If the Boltzmann weights at v obey $a_1 a_2 + b_1 b_2$, $c_1 = c_2 = 0$ and

$$a_1 a_2 + b_1 b_2 - d_1 d_2 = 0$$

then the vertex is said to be free-fermionic of *type D*. In this report we are only concerned with lattice models where each vertex is free-fermionic of type C or type D. Brubaker, Bump and Friedberg [3] previously described a non-abelian parametrized Yang-Baxter equation for the eight-vertex model, when all vertices are type C or type D, which has $GL(2, \mathbb{C}) \times \mathbb{C}^\times$ as a subgroup of index two. Explicitly we have:

Theorem 2.1. [3, Theorem 8] *If u and v are vertices of type C then define a new rotated vertex w by*

$$\begin{aligned} a_1(w) &= a_1(u)a_2(v) + b_2(u)b_1(v), \\ a_2(w) &= b_1(u)b_2(v) + a_2(u)a_1(v), \\ b_1(w) &= b_1(u)a_2(v) - a_2(u)b_1(v), \\ b_2(w) &= -a_1(u)b_2(v) + b_2(u)a_1(v), \\ c_1(w) &= c_1(u)c_2(v), \\ c_2(w) &= c_2(u)c_1(v) \end{aligned}$$

If u is a vertex of type C and v is a vertex of type D then define a new rotated vertex w by

$$\begin{aligned} a_1(w) &= a_2(u)a_1(v) + b_1(u)b_1(v), \\ a_2(w) &= a_1(u)a_2(v) + b_2(u)b_2(v), \\ b_1(w) &= -b_2(u)a_1(v) + a_1(u)b_1(v), \\ b_2(w) &= -b_1(u)a_2(v) + a_2(u)b_2(v), \\ d_1(w) &= c_1(u)d_1(v), \\ d_2(w) &= c_2(u)d_1(v) \end{aligned}$$

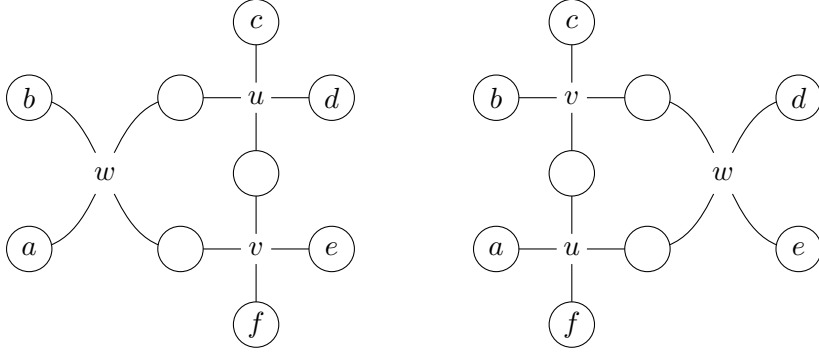
If u is of type D and v is of type C then define a new rotated vertex w by

$$\begin{aligned} a_1(w) &= a_1(u)a_2(v) + b_2(u)b_2(v), \\ a_2(w) &= a_2(u)a_1(v) + b_1(u)b_1(v), \\ b_1(w) &= b_1(u)a_1(v) + a_2(u)b_2(v), \\ b_2(w) &= b_2(u)a_1(v) + a_1(u)b_1(v), \\ d_1(w) &= d_1(u)c_2(v), \\ d_2(w) &= d_2(u)c_1(v) \end{aligned}$$

Finally, if u and v are both of type D then define a new rotated vertex w by

$$\begin{aligned} a_1(w) &= -a_2(u)a_2(v) + b_1(u)b_2(v), \\ a_2(w) &= -a_1(u)a_1(v) + b_2(u)b_1(v), \\ b_1(w) &= b_2(u)a_2(v) + a_1(u)b_2(v), \\ b_2(w) &= b_1(u)a_1(v) + a_2(u)b_1(v), \\ c_1(w) &= d_1(u)d_2(v), \\ c_2(w) &= d_2(u)d_1(v) \end{aligned}$$

For any assignment of edge spins $a, b, c, d, e, f \in \{\pm\}$ the following two configurations have the same partition function:



Meaning when we fix a, b, c, d, e, f and sum the product of the Boltzmann weights of each vertices over all possible spins appearing in the blank circles on either side we have equality.

3 Gamma ice

	a_1	a_2	b_1	b_2	c_1	c_2
Gamma Ice						
Boltzmann weight	1	z_i	t_i	z_i	$z_i(t_i + 1)$	1
Gamma-Gamma R-ice						
Boltzmann weight	$t_j z_i + z_j$	$t_i z_j + z_i$	$t_i z_j - t_j z_i$	$z_i - z_j$	$(t_i + 1)z_i$	$(t_j + 1)z_j$

Figure 1: The Boltzmann weights for Gamma ice and Gamma-Gamma R-ice.

Here we review the six-vertex Gamma ice model and how it may be used to represent Schur polynomials [3]. Let $\mathbf{z} = (z_1, \dots, z_n)$ be a finite alphabet of variables. We will refer to the z_i as *spectral parameters*. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition. Let $\rho = (n-1, n-2, \dots, 0)$. The model is then a rectangular grid with n rows and $\lambda_1 + n$ columns. Number the columns starting at 0 and increasing to the left. Number the rows starting with z_1 at the top and increasing the index as you move down. The boundary conditions are $+$ spins along the left and bottom edges, $-$ spins along the right edges. On the top edges we put $-$ at every column labeled $\lambda_i + n - i$ ($1 \leq i \leq n$), i.e. in the columns labeled with values in $\lambda + \rho$. The Boltzmann weights at each vertex depend only on the row the vertex lies in and are given by the Gamma weights appearing in Figure 1. Denote the Gamma ice model indexed by λ by $\mathfrak{S}_\lambda^\Gamma$.

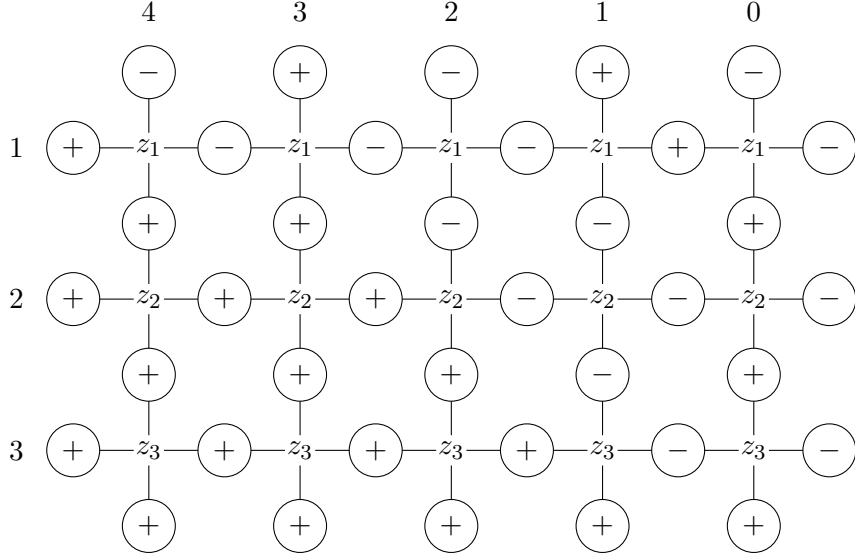
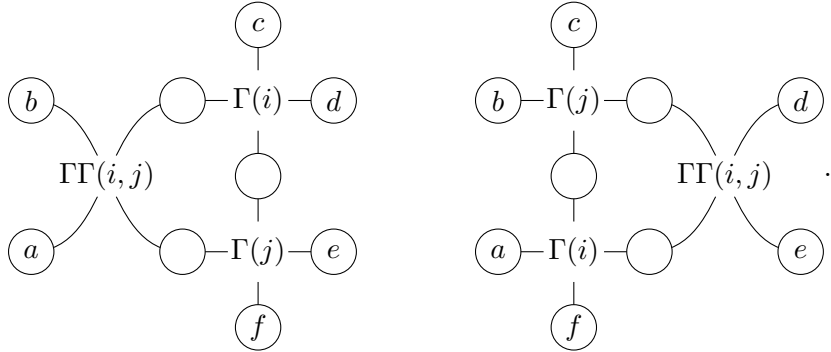


Figure 2: A state for the $\mathfrak{S}_\lambda^\Gamma$ system with $\lambda = (2, 1, 0)$.

Theorem 3.1. [3, Theorem 4] For any assignment of edge spins $a, b, c, d, e, f \in \{\pm\}$ the following two configurations have the same partition function:



Where $\Gamma(i)$ represents the Gamma weights in the i th row and $\Gamma\Gamma(i, j)$ represents the Gamma-Gamma weights in the i, j rows from Figure 1.

Proof. Since the Gamma weights are free-fermionic of type C the conclusion is a consequence of Theorem 2.1. \square

Theorem 3.2. [3, Theorem 5] Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition. Then

$$Z(\mathfrak{S}_\lambda^\Gamma) = \prod_{i < j} (t_i z_j + z_i) s_\lambda(\mathbf{z}).$$

Note that in the special case of five-vertex Gamma ice, i.e. when we take $t = 0$ in Figure 1, then we have

$$Z(\mathfrak{S}_\lambda^{\Gamma, t=0}) = \mathbf{z}^\rho s_\lambda(\mathbf{z}).$$

4 The Dual Cauchy Identity

A lattice model proof of the dual Cauchy identity for factorial Schur functions was given in [8]. As this provided the inspiration for our work on the Cauchy identity we review their argument, but in the simpler case of Schur polynomials, in this section before moving on to discuss the Cauchy identity.

Definition 4.1. Let m and n be positive integers. For a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_1 \leq m$ define the complement partition by

$$\hat{\lambda}_i = |\{\lambda_j : \lambda_j \leq m - i\}|.$$

Theorem 4.2 (Dual Cauchy Identity). [8, Theorem 7] For two finite alphabets of variables $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_m)$, we have

$$\sum_{\lambda} s_{\lambda}(\mathbf{x}) s_{\hat{\lambda}}(\mathbf{y}) = \prod_{i=1}^n \prod_{j=1}^m (x_i + y_j). \quad (4.1)$$

Proof. The proof consists of evaluating a partition function of a certain lattice model in two different ways. The model we consider is defined as follows: The model is a $(n + m) \times (n + m)$ lattice with $+$ spins along the left and bottom boundary, and $-$ spins along the right and top boundary. The spectral parameters are the given by the sequence $(y_m, \dots, y_1, x_1, \dots, x_n)$ starting from the top row. We use the Gamma weights with $t = 1$, defined in Figure 1, at each vertex with $t = 1$. Then by Theorem 3.2 the partition function of this model is given by

$$\prod_{i < j} (y_i + y_j) \prod_{i, j} (y_i + x_j) \prod_{i < j} (x_i + y_j).$$

Now we evaluate the partition function in a different way by partitioning the states according to the set of spins that appear between the y_1 and x_1 rows. These spins correspond to a partition in the usual way, i.e. the negative spins are in columns with index $\lambda_i + n - i$. Note that since we use Gamma ice the minus spins on vertical edges of our model interleave [3, Lemma 2]. Hence, in any state of our model there are exactly n negative spins appearing in on the vertical edges between the y_1 and x_1 rows. Since any configuration of spins which fit within our $(n + m) \times (n + m)$ model may appear between the y_1 and x_1 rows the possible configurations of spins between the y_1 and x_1 row are in bijection with the set of all partitions with at most n non-zero parts with $\lambda_1 \leq m$. So we may write the partition function as

$$\sum_{\lambda} Z_{\lambda}^{\text{top}} Z_{\lambda}^{\text{bottom}}.$$

For a pictorial description of of this partitioning see Figure 3

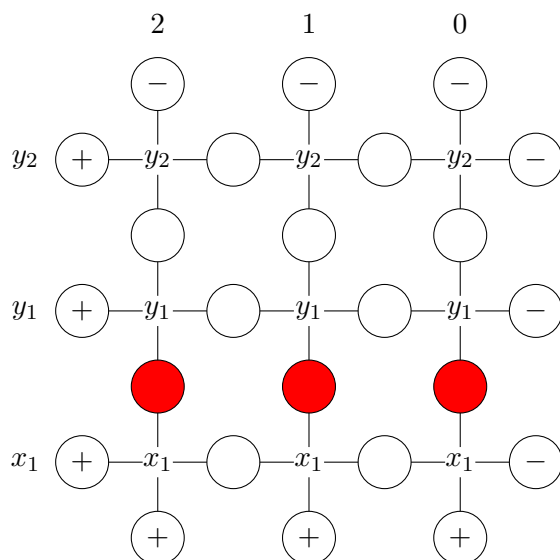


Figure 3: The model with $n = 1, m = 2$. We partition the states of the model by the spins appearing in the red circles.

where the sum is over all partitions λ with at most n non-zero parts and with $\lambda_1 \leq m$. Here $Z_\lambda^{\text{bottom}}$ is the partition function of the model with n rows and $n+m$ columns, parameters x_1, \dots, x_n and boundary conditions of $+$ spins along the left and bottom, $-$ spins along the right, and λ boundary conditions along the top. Furthermore, Z_λ^{top} is the partition function of the model with $+$ spins along the left, $-$ spins along the top and right, and λ boundary conditions along the bottom edge. Since all vertices carry Gamma weights by [Theorem 3.2](#) we have

$$Z_\lambda^{\text{bottom}} = \prod_{i < j} (x_i + x_j) s_\lambda(\mathbf{x}).$$

To evaluate Z_λ^{top} we will transform the top part of our system into a more familiar lattice model. First flip all spins along each vertical strand, then reflect the model over a horizontal axis. It is not hard to check that this defines a weight preserving bijection (in fact the weight of each type of vertex remains unchanged) between the top part of our model and the m by $n+m$ model with boundary conditions of $+$ spins along the left and bottom, $-$ spins along the right and spins determined by $\hat{\lambda}$ along the top boundary, i.e. negative spins in the columns labeled by $\hat{\lambda}_i + m - i$. Hence, we may again apply [Theorem 3.2](#) to conclude

$$Z_\lambda^{\text{top}} = \prod_{i,j} (y_i + y_j) s_{\hat{\lambda}}(\mathbf{y}).$$

The dual Cauchy identity then follows from comparing the two evaluations of the partition function. \square

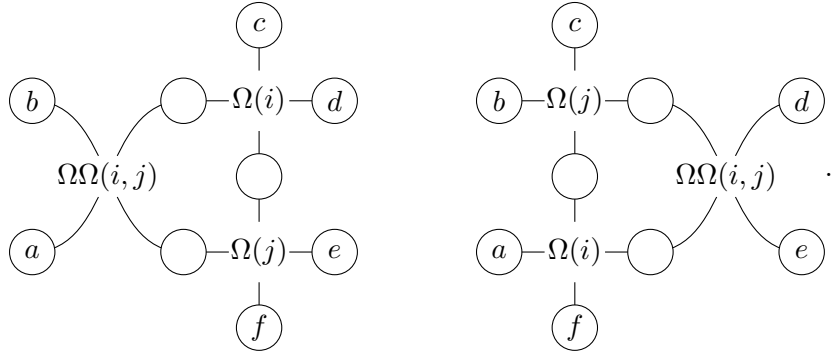
5 Omega ice

In this section we define Omega ice which is a five-vertex lattice model closely related to the five-vertex ($t = 0$) Gamma ice of [\[3\]](#). Let $\mathbf{z} = (z_1, \dots, z_n)$ be a finite alphabet of variables.

	a_1	a_2	b_1	b_2	d_1	d_2
Omega Ice						
Boltzmann weight	1	z_i	0	z_i	z_i	1
Omega-Omega R-ice						
Boltzmann weight	z_i	z_j	$z_j - z_i$	0	z_i	z_j

Figure 4: The Boltzmann weights for Omega ice and Omega-Omega ice.

Theorem 5.1. *For any assignment of edge spins $a, b, c, d, e, f \in \{\pm\}$ the following two configurations have the same partition function:*



Where $\Omega(i)$ represents the Omega weights in the i th row and $\Omega\Omega(i, j)$ denoted the Omega-Omega weights in the i, j rows from [Figure 4](#).

Proof. Since the Omega weights are free-fermionic of type D the conclusion is a consequence of [Theorem 2.1](#). \square

We will use Omega ice to represent Schur polynomials. Let us now define a lattice model denoted $\mathfrak{S}_\lambda^\Omega$. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a fixed partition. We consider a rectangular grid with n rows and $\lambda_1 + n$ columns. We will label the columns of the lattice from left to right in descending order from $\lambda_1 + n - 1$ to 0. The boundary conditions are as follows: On the left and top boundary we put +; on the right we put -. On the bottom, we put - at every column labeled $\lambda_i + n - i$ ($1 \leq i \leq n$), i.e. in the columns labeled with values in $\lambda + \rho$. Bottom edges not labeled by $\lambda_i + n - i$ for any i are given + spin.

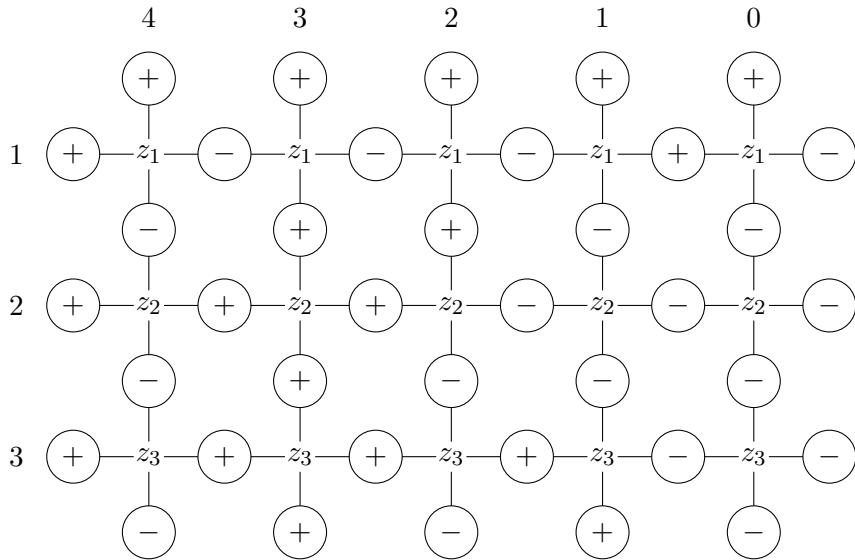


Figure 5: A state for the $\mathfrak{S}_\lambda^\Omega$ system with $\lambda = (2, 1, 0)$.

Proposition 5.2. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition. Then*

$$Z(\mathfrak{S}_\lambda^\Omega) = z^\rho s_\lambda(z).$$

Proof. Consider the following operation on a state of the Omega ice model: reflect the state over a horizontal axis. It is not hard to see that this operation defines a weight preserving bijection, in fact it preserves the weight of each vertex, between the states of our Omega ice model and the Gamma ice model with $t = 0$ appearing in [\[3, Section 3\]](#). The conclusion is then the result of [Theorem 3.2](#). \square

Lemma 5.3. *If $\mathfrak{s} \in \mathfrak{S}_\lambda^\Omega$ then $-$ spins appearing on vertical edges of \mathfrak{s} anti-interleave. More specifically if β_1, \dots, β_l are the column indices of the $-$ spins on the vertical edges above the z_i row and $\alpha_1, \dots, \alpha_{l'}$ are the $-$ spins on the vertical edges below the z_i row then we have $l' = l + 1$ and $\beta_1 \geq \alpha_1 \geq \beta_2 \geq \alpha_2 \geq \dots$.*

Proof. In light of the bijection between Omega ice and five-vertex Gamma ice, discussed in Proposition 5.2, the result is equivalent to [3, Lemma 2]. \square

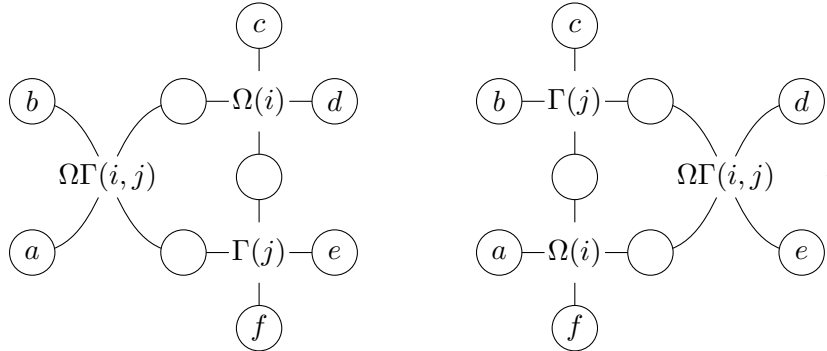
6 The Mixed Yang-Baxter Equation for Omega-Gamma and Gamma-Omega R-ice

Since the model we are ultimately interested uses vertices with a mixture of Omega and five-vertex Gamma weights (i.e. $t = 0$) we will need further mixed Yang-Baxter equations.

$\Omega\Gamma(i, j)$						
Boltzmann weights	$1 - z_i z_j$	$z_i z_j$	$z_i z_j$	$z_i z_j$	$z_i z_j$	1
$\Gamma\Omega(i, j)$						
Boltzmann weights	$-z_i z_j$	$z_i z_j - 1$	$z_i z_j$	$z_i z_j$	$z_i z_j$	1

Figure 6: The Boltzmann weights for Omega-Gamma and Gamma-Omega R-ice.

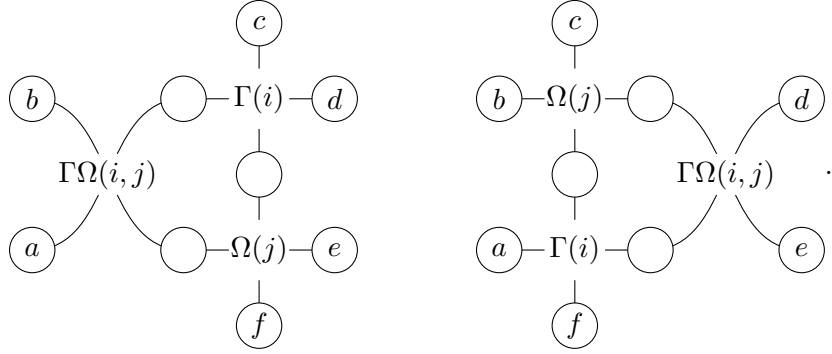
Theorem 6.1. *For any assignment of edge spins $a, b, c, d, e, f \in \{\pm\}$ the following two configurations have the same partition function:*



The Boltzmann weights are specified in Figures 1, 4 and 6.

Proof. Since the Omega weights are free-fermionic of type D and the Gamma weights are free-fermionic of type C the conclusion is a consequence of Theorem 2.1. \square

Theorem 6.2. *For any assignment of edge spins $a, b, c, d, e, f \in \{\pm\}$ the following two configurations have the same partition function:*



The Boltzmann weights are specified in [Figures 1, 4 and 6](#).

Proof. Since the Omega weights are free-fermionic of type D and the Gamma weights are free-fermionic of type C the conclusion is a consequence of [Theorem 2.1](#). \square

7 A Half Infinite Lattice Model

We will now define a new half infinite lattice model which has partition function equal to one side of the Cauchy identity (cf. [Section 8](#)). Let x_1, \dots, x_n and y_1, \dots, y_m be two sets of variables and assume further without loss that $m > n$. We now define the model denoted $\mathfrak{S}_\infty^{\Omega\Gamma}$. For the size of the model we take infinitely many columns and $n + m$ rows. The columns are labeled starting from 0 and increasing to the left. The spectral parameters are taken to be $(x_n, \dots, x_1, y_1, \dots, y_m)$ starting from the top of the model. For the weights the vertices with row label equal to x_i for any i are given the Ω weights while the remaining vertices (those with row label y_j for some j) are given the Γ weights. The boundary conditions are set as follows: let $k = m - n$ then the top boundary conditions are set as $+$ spins for each column with index greater than or equal to k and with $-$ spins the remaining edges. The spins along the right boundary are all set as $-$ spins. The spins along the bottom boundary are set to be all $+$ spins.

A state of our model $\mathfrak{S}_\infty^{\Omega\Gamma}$ is, as usual, an assignment of spins to the internal edges of the boundary value problem described such that each vertex has non-zero Boltzmann weight. Furthermore, since we are dealing with an infinite model, we also require that a state has a finite number of $-$ spins appearing in each row on horizontal edges. In other words for each state there exists a $r \in \mathbb{N}$ such that in each row all horizontal edges which lie to the left of the r th column carry a $+$ spin. The last condition is a finiteness condition which guarantees that each state has weight equal to a polynomial in the spectral parameters (cf. [Proposition 7.2](#)).

Definition 7.1. Let $\mathfrak{S}_r^{\Omega\Gamma}$ denote the truncation of the half-infinite model $\mathfrak{S}_\infty^{\Omega\Gamma}$ to a model with r columns with boundary edges on the new left boundary set as all $+$ spins.

Proposition 7.2. *In any state $\mathfrak{s} \in \mathfrak{S}_\infty^{\Omega\Gamma}$ there exists some $r \in \mathbb{N}$ such that for all columns with index greater than r all vertices appearing in the column are of type a_1 hence the weight of any $\mathfrak{s} \in \mathfrak{S}_\infty^{\Omega\Gamma}$ is a polynomial in the spectral parameters.*

Proof. The number of $-$ spins on horizontal edges in each row is finite by assumption, so if you go far enough (some finite number of steps) to the left in any state eventually all spins appearing on horizontal edges are $+$ spins. Starting from the top row since the top boundary is all $+$ considering that each vertex must have an even number of $+$ spins surrounding it this implies that to the left of some finite column index all spins appearing on vertical edges are $+$ spins. Continuing down inductively gives the result. \square

Proposition 7.3. *Let $k = m - n$ then in every state $\mathfrak{s} \in \mathfrak{S}_\infty^{\Omega\Gamma}$ the top right $k \times n$ block consists of only type a_2 vertices.*

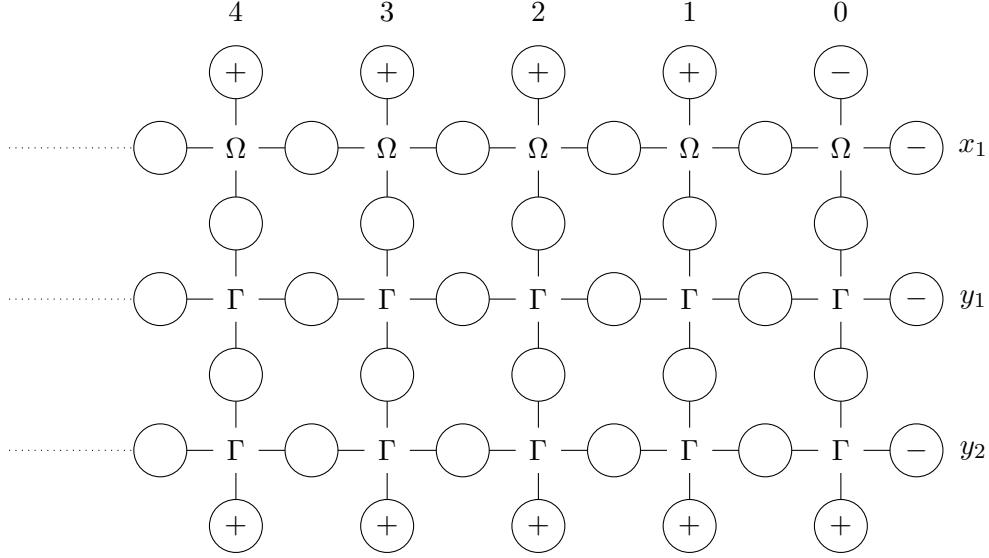


Figure 7: A diagram of $\mathfrak{S}_{\infty}^{\Omega\Gamma}$ with $n = 1, m = 2$. The dotted lines indicate that the picture continues infinitely to the left.

Proof. The only non-zero Ω vertex with $-$ spin on the north and east edge is the type a_1 vertex. It is easy to see that each vertex in the top right $k \times n$ block of any $\mathfrak{s} \in \mathfrak{S}_{\infty}^{\Omega\Gamma}$ must have $-$ spin on the north and east edges so the claim follows. \square

Proposition 7.4. *In every state $\mathfrak{s} \in \mathfrak{S}_{\infty}^{\Omega\Gamma}$ we have the following properties: The $-$ spins appearing in the top half of the model on horizontal edges anti-interleave. More specifically if β_1, \dots, β_l are the column indices of the $-$ spins on the vertical edges above the x_i row and $\alpha_1, \dots, \alpha_{l'}$ are the $-$ spins on the vertical edges below the x_i row then we have $l' = l + 1$ and $\beta_1 \geq \alpha_1 \geq \beta_2 \geq \alpha_2 \geq \dots$. Furthermore, the $-$ spins appearing in the bottom half of the model on horizontal edges interleave. More specifically if β_1, \dots, β_l are the column indices of the $-$ spins on the vertical edges above the y_i row and $\alpha_1, \dots, \alpha_{l'}$ are the $-$ spins on the vertical edges below the y_i row then we have $l' + 1 = l$ and $\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \dots$*

Proof. From Proposition 7.2 there exists a $r \in \mathbb{N}$ such that each $\mathfrak{s} \in \mathfrak{S}_{\infty}^{\Omega\Gamma}$ may be identified with a state of $\mathfrak{S}_r^{\Omega\Gamma}$ in a way that is weight preserving and that preserves the assignment of spins at each comparable vertex. Hence, these properties are immediately deduced from Lemma 5.3 and [3, Lemma 2]. \square

Corollary 7.5. *The number of $-$ spins appearing between rows x_{i+1} and x_i is given by $m - i$. The number of $-$ spins appearing between the x_1 and y_1 rows is given by $n + k = n + m - n = m$. The number of $-$ spins appearing between rows y_i, y_{i-1} is given by $m - i$. Moreover, the number of $-$ spins appearing left of the $(k - 1)$ th column between the x_{i+1} and x_i rows is given by $n - i$, between the x_1 and y_1 rows there are exactly n , and finally between rows y_i and y_{i-1} there are $n - i$.*

8 The Cauchy Identity

In this section we prove that $Z(\mathfrak{S}_{\infty}^{\Omega\Gamma})$ realizes one side of the Cauchy identity up to a simple monomial factor. We then discuss the difficulties in applying the Yang-Baxter equation to $\mathfrak{S}_{\infty}^{\Omega\Gamma}$ to evaluate the partition function in a different way.

Theorem 8.1. For two finite alphabets of variables $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_m)$ with $m \geq n$ we have

$$Z(\mathfrak{S}_\infty^{\Omega\Gamma}) = \mathbf{x}^{\rho+\kappa} \mathbf{y}^\rho \sum_{\lambda} s_{\lambda}(\mathbf{x}) s_{\lambda}(\mathbf{y}). \quad (8.1)$$

With $\kappa = \underbrace{(k, k, \dots, k)}_n$ with $k = m - n$. The sum is over all partitions λ with at most $\min(n, m)$ parts.

Proof. The proof will consist of evaluating the partition function of $\mathfrak{S}_\infty^{\Omega\Gamma}$ in two different ways. The first way is by partitioning the set of all states according to the set of spins that occur between the rows with parameters x_1 and y_1 . Such an arrangement of spins corresponds to a partition in the usual way, i.e. the negative spins are in the columns labeled by $\lambda_i + m - i$. Note that all spins appearing to the left of the column labeled by $\lambda_1 + m - 1$ are positive and hence must be type a_1 which has weight 1 in both Gamma and Omega weight systems. So we may write the partition function as a sum of partition functions of finite lattice models

$$Z(\mathfrak{S}_\infty^{\Omega\Gamma}) = \sum_{\lambda} Z_{\lambda}^{\text{top}} Z_{\lambda}^{\text{bottom}}.$$

Here we have Z_{λ}^{top} is the partition function of the system with n rows and $\lambda_1 + m$ columns with parameters x_n, \dots, x_1 and boundary conditions of negative spins on the right, the first $k = m - n$ spins on top negative. The remaining spins on the top boundary are positive and we have λ boundary conditions for the spins on the bottom. Likewise, $Z_{\lambda}^{\text{bottom}}$ is the partition function of the model with m rows and $\lambda_1 + m - 1$ columns with parameters y_1, \dots, y_m and boundary conditions of λ on top, positive spins along the bottom and negative spins along the right.

By [Theorem 3.2](#) (we apply it with $t = 0$) we have

$$Z_{\lambda}^{\text{bottom}} = \mathbf{y}^\rho s_{\lambda}(\mathbf{y}),$$

To evaluate Z_{λ}^{top} we partition the model further. By [Proposition 7.3](#) the right $k \times n$ block of Z_{λ}^{top} consists of vertices of type a_2 only. Hence, it is easy to see that $Z_{\lambda}^{\text{top}} = x^{\kappa} Z(\mathfrak{S}_{\lambda}^{\Omega})$. Note that with the given setup the last k entries of λ are all zero, so although it is an abuse of notation, it makes sense to identify λ and the partition obtained from λ by removing the last k parts. Hence, by [Proposition 5.2](#) we conclude that

$$Z_{\lambda}^{\text{top}} = \mathbf{x}^{\rho+\kappa} s_{\lambda}(\mathbf{x}).$$

So in total we have

$$Z(\mathfrak{S}_\infty^{\Omega\Gamma}) = \mathbf{x}^{\rho+\kappa} \mathbf{y}^\rho \sum_{\lambda} s_{\lambda}(\mathbf{x}) s_{\lambda}(\mathbf{y}).$$

□

To evaluate the partition function of $\mathfrak{S}_\infty^{\Omega\Gamma}$ in a second way and realize a lattice model proof of the Cauchy identity we would like to be able to apply the Yang-Baxter equation through the usual train argument (cf. [\[3, Theorem 5\]](#)) to our half-infinite model $\mathfrak{S}_\infty^{\Omega\Gamma}$. The difficulty is that $\mathfrak{S}_\infty^{\Omega\Gamma}$ is half infinite so we cannot simply apply the train argument because there is no boundary on the left. To get around this issue we show that the partition function of $\mathfrak{S}_\infty^{\Omega\Gamma}$ may be written as an infinite sum of partition functions of finite rectangular models.

Proposition 8.2. We have

$$Z(\mathfrak{S}_\infty^{\Omega\Gamma}) = \sum_{r \geq m} Z(\mathfrak{S}_r^{\Omega\Gamma}) - Z(\mathfrak{S}_{r-1}^{\Omega\Gamma}).$$

Note that $Z(\mathfrak{S}_{m-1}^{\Omega\Gamma}) = 0$.

Proof. Let $\mathfrak{s} \in \mathfrak{S}_\infty^{\Omega\Gamma}$. Then by [Proposition 7.2](#) there exists some minimal $r \in \mathbb{N}$ with the property that for all columns with index greater than r all vertices appearing are of type a_1 and hence contribute weight 1 to the weight of \mathfrak{s} . So \mathfrak{s} may be uniquely identified, in a weight preserving way, with a state of $\mathfrak{S}_r^{\Omega\Gamma}$. Moreover since r is chosen to be minimal then \mathfrak{s} is identified with a state of $\mathfrak{S}_r^{\Omega\Gamma}$ which has no trivial columns on the left, i.e. no columns with only type a_1 vertices. By [Proposition 7.4](#) the number of $-$ spins appearing in the row between the x_1 and y_1 spectral parameters is exactly m . Hence, the minimal r which may be associated to a state $\mathfrak{s} \in \mathfrak{S}_\infty^{\Omega\Gamma}$ as described before is $r = m$ and moreover every state of $\mathfrak{S}_m^{\Omega\Gamma}$ has no trivial columns on the left. Identifying states of two finite models if one may be obtained from the other by adding trivial columns we have fashioned a weight preserving bijection between $\mathfrak{S}_\infty^{\Omega\Gamma}$ and $\mathfrak{S}_m^{\Omega\Gamma} \cup (\bigcup_{r>m} \mathfrak{S}_r^{\Omega\Gamma} \setminus \mathfrak{S}_{r-1}^{\Omega\Gamma})$. The claim then follows from taking partition functions of both sides. \square

So evaluating the partition function of $\mathfrak{S}_\infty^{\Omega\Gamma}$ reduces to evaluating the partition function of $\mathfrak{S}_r^{\Omega\Gamma}$ and computing the relevant summation. We may apply the Yang-Baxter equation to $\mathfrak{S}_r^{\Omega\Gamma}$ in order to evaluate the partition function via the usual train argument. While we fully expect that it should be possible to evaluate the partition function of $\mathfrak{S}_r^{\Omega\Gamma}$ using the Yang-Baxter equation we have not yet been able to so in full generality. Since our model involves two-types of Boltzmann weights, Omega and Gamma ice, we have four different Yang-Baxter equations [Figures 1, 4 and 6](#). Via the train argument the first two give a recurrence on the partition function for when we interchange a row of Omega ice with a row of Omega ice (i.e. permuting x_i and x_j), and a recurrence on the partition function for when we interchange a row of Gamma ice with a row of Gamma ice (i.e. permuting y_i and y_j). The last two mixed Yang-Baxter equations [Figure 6](#) give recursions for when we interchange or row of Omega ice with a row of Gamma ice and vice versa (i.e. permuting x_i and y_j). We anticipate that these mixed Yang-Baxter equations are the key tool one needs to show that $Z(\mathfrak{S}_\infty^{\Omega\Gamma})$ agrees with the right hand side of [\(1.1\)](#) and realize a lattice model proof of the Cauchy identity.

The main issue we have encountered in our attempts to apply the Yang-Baxter equation to $\mathfrak{S}_\infty^{\Omega\Gamma}$ is finding a tractable way to organize the computation. Since the right hand side of [\(1.1\)](#) involves a product of geometric series the complexity arises because essentially using the Yang-Baxter equation allows one to compute $Z(\mathfrak{S}_\infty^{\Omega\Gamma})$ as a sum of all the monomials which come about when you expand a product of geometric series. It is not yet clear how to organize the computation so one can easily recognize this fact. To demonstrate how this argument works in simplest terms we take up the rank one case of the Cauchy identity, i.e. when $n = m = 1$.

Proposition 8.3. *If $n = m = 1$ then the partition function of the two row half-infinite lattice model $\mathfrak{S}_\infty^{\Omega\Gamma}$, with spectral parameters (x, y) , is given by*

$$Z(\mathfrak{S}_\infty^{\Omega\Gamma}) = \frac{1}{1 - xy}.$$

Proof. From [Proposition 8.2](#) to evaluate $Z(\mathfrak{S}_\infty^{\Omega\Gamma})$ we may evaluate the partition function of a finite truncation and then compute the necessary infinite sum. We modify the $\mathfrak{S}_r^{\Omega\Gamma}$ model by attaching a Gamma-Omega R-vertex (i.e. a vertex from the top row of [Figure 6](#)) to the right of the x and y rows obtaining the new lattice model

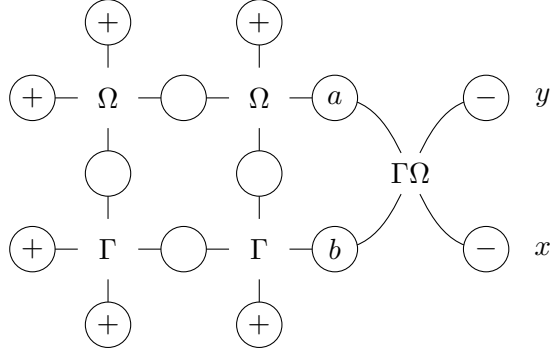


Figure 8: A diagram of our modified $\mathfrak{S}_r^{\Omega\Gamma}$ model when $n = m = 1$. For illustrative purposes we have taken $r = 2$.

Consulting [Figure 6](#) we see that the possibilities for a, b are that either both a and b carry $+$ or that both carry $-$. It is not difficult to see that all remaining vertices in the model when a and b are both $+$ spins must be of type a_1 . Hence, the partition function of our augmented model is given by

$$1 + (xy - 1)Z(\mathfrak{S}_r^{\Omega\Gamma}).$$

Remark 8.4. We remark that the fact that there are two possible completions of the attached R -vertex is in contrast to usual applications of the train argument in the literature. Usually there is only one valid completion (cf. [\[3, 8\]](#)). While this behavior is exactly what we expect as we are essentially expanding the a product of geometric series, this is exactly what causes the complexity in the computation for higher rank examples.

Applying [Theorem 6.2](#) r times we conclude that the partition function of the model appearing in [Figure 9](#) is equal to the partition function of our original augmented $\mathfrak{S}_r^{\Omega\Gamma}$ model.

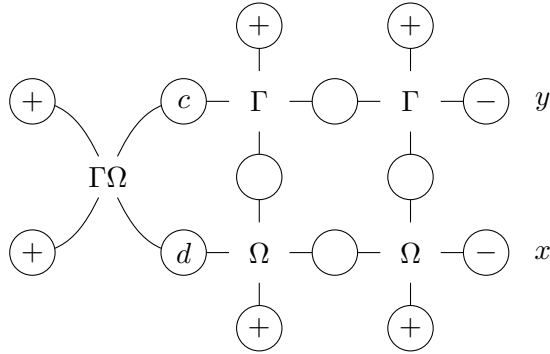


Figure 9: A diagram of our modified $\mathfrak{S}_r^{\Omega\Gamma}$ model when $n = m = 1$ after applying [Theorem 6.2](#) r times. For illustrative purposes we have taken $r = 2$.

Again consulting [Figure 6](#) we see that the possibilities for c, d are that either both c and d carry $+$ or that both carry $-$. If both c and d carry $+$ spin consulting [Figures 1](#) and [4](#) it is not hard to see that there is no filling of the remaining edges which has non-zero Boltzmann weight. If both c and d carry $-$ spin consulting [Figures 1](#) and [4](#) it is not hard to see that all vertices must be type b_2 . Hence, the partition function of the model appearing in [Figure 9](#) is given by $(xy)^r$. Since this is equal to the partition function of the model appearing in [Figure 8](#) we conclude that

$$Z(\mathfrak{S}_r^{\Omega\Gamma}) = \frac{(xy)^r - 1}{xy - 1} = \sum_{j=0}^{r-1} (xy)^j.$$

So then by [Proposition 8.2](#) we have

$$\begin{aligned} Z(\mathfrak{G}_\infty^{\Omega\Gamma}) &= 1 + \sum_{r>1} \left(\sum_{j=0}^{r-1} (xy)^j - \sum_{j=0}^{r-2} (xy)^j \right) \\ &= \sum_{r \geq 0} (xy)^r \\ &= \frac{1}{1-xy}. \end{aligned}$$

□

As a consequence of [Proposition 8.3](#) since $\rho = (0)$ and $\kappa = (0)$ in the case $n = m = 1$ by [Theorem 8.1](#) we deduce the rank one Cauchy identity:

$$\sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y) = \frac{1}{1-xy}.$$

where the sum is over all partitions λ with at most one non-zero part. To extend this argument to higher rank examples and eventually to a general case we anticipate that one should be able to augment the model by increasingly complicated configurations of Gamma-Omega vertices. For example in the rank 2 case consider [Figure 10](#).

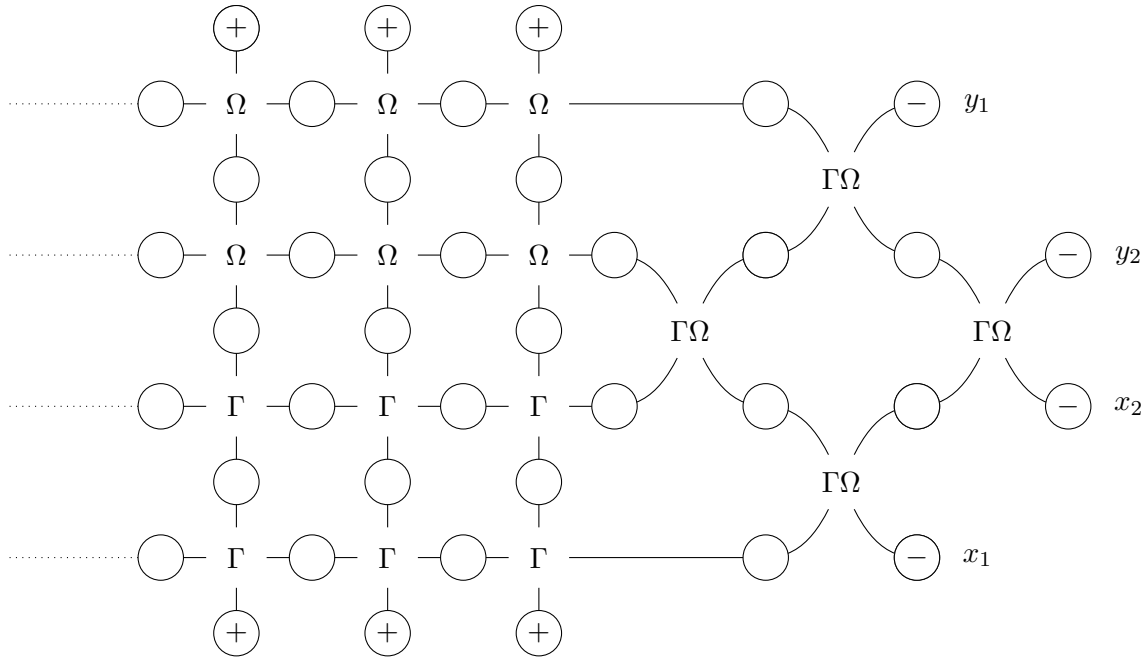


Figure 10: A diagram of the proposed augmented model in the rank 2 case, i.e. $n = m = 2$.

The picture for arbitrary n, m is analogous although not symmetric with $n \neq m$. One can quickly see why the computation becomes unwieldy in higher rank examples.

Remark 8.5. In our attempts to organize the computation it seemed that it might be easier to understand these augmented models in the case when $n = m$ because the augmented model is symmetric. In fact to prove the Cauchy identity it is enough to prove it in the case when $n = m$. One may then specialize the necessary variables to zero to recover the case $n \neq m$. So while our model seems to be able to handle the $n \neq m$ case explicitly it may be easier to understand how to organize the computation when $n = m$ which would be sufficient.

Conjecture 8.6. *Using the Yang-Baxter equation one may prove that*

$$Z(\mathfrak{S}_\infty^{\Omega\Gamma}) = \mathbf{x}^{\rho+\kappa} \mathbf{y}^\rho \prod_{i=1}^n \prod_{j=1}^m \frac{1}{1 - x_i y_j}.$$

9 Acknowledgments

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