Formulas for Birational R-Matrix Action

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The Birational R-Matrix, η

Why we care:

- Relates to networks on a cylinder;
- Describes relations between matrix factorizations;
- Occurs in the study of geometric crystals;
- The tropicalization is the combinatorial R-matrix of affine crystals;
- Has applications to discrete Painlevé dynamical systems.

The Birational R-Matrix, η

Let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ be two sets of formal variables, where $n \geq 1$. For $1 \leq i \leq n$, let

$$\kappa_i(\mathbf{a}, \mathbf{b}) = \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k,$$

where the indices k are taken mod n. Then

$$\eta : (\mathbf{a}, \mathbf{b}) \mapsto (\mathbf{b}', \mathbf{a}')$$

where $\mathbf{a}' = (a'_1, \ldots, a'_n), \mathbf{b}' = (b'_1, \ldots, b'_n)$, and

$$a'_i = \frac{a_{i-1}\kappa_{i-1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})}$$

$$b'_i = \frac{b_{i+1}\kappa_{i+1}(\mathbf{a}, \mathbf{b})}{\kappa_i(\mathbf{a}, \mathbf{b})}.$$

Example of η

$$\kappa_i(\mathbf{a},\mathbf{b}) = \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k,$$

$$a_i' = \frac{a_{i-1}\kappa_{i-1}(\mathbf{a},\mathbf{b})}{\kappa_i(\mathbf{a},\mathbf{b})}.$$

For example, for $n = 4$,

$$a_2' = a_1 \frac{\kappa_1(\mathbf{a},\mathbf{b})}{\kappa_2(\mathbf{a},\mathbf{b})} = a_1 \frac{a_2 a_3 a_4 + b_2 a_3 a_4 + b_2 b_3 a_4 + b_2 b_3 b_4}{a_3 a_4 a_1 + b_3 a_4 a_1 + b_3 b_4 a_1 + b_3 b_4 b_1}.$$

η_i and its properties

Let $\mathbf{x}_i = (x_i^{(1)}, \ldots, x_i^{(n)})$. Now for $1 \leq i < m$, let

$$\eta_i(\mathbf{x}_1, \ldots, \mathbf{x}_m) = (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \eta(\mathbf{x}_i, \mathbf{x}_{i+1}), \mathbf{x}_{i+2}, \ldots, \mathbf{x}_m).$$

Theorem 1. [Lam–Pylyavskyy, 2008]
The birational R-matrix has the following properties:

- η is an involution: $\eta^2 = 1$;
- η satisfies the braid relations: for $1 \leq i < m-1$,

$$\eta_i \eta_{i+1} \eta_i(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \eta_{i+1} \eta_i \eta_{i+1}(\mathbf{x}_1, \ldots, \mathbf{x}_m).$$

$\Rightarrow$ Action of S_m on $(\mathbf{x}_1, \ldots, \mathbf{x}_m)$.

Main Problem

To refer to specific variables after applying a permutation s, we write $s(x_i^{(r)})$ to denote the r-th variable in the resultant i-th vector. When indices are in parentheses, they are taken mod n.

Main Problem. For any $s \in S_m$, $1 \leq i \leq m$ and $1 \leq r \leq n$, we would like to write $s(x_i^{(r)})$ explicitly as a rational function of the original variables.

Outline

Let $j > 1$. Write s_i for the transposition switching i and $i+1$.

- s is shifting by 1: $s = s_{j-1}s_{j-2}\ldots s_i$ and $s = s_i s_{i+1}\ldots s_{j-1}$;
- s is a transposition: $s = s_i s_{i+1}\ldots s_{j-2}s_{j-1}s_{j-2}\ldots s_i$;
- Combinatorial interpretation of functions that appear.

The τ, σ, $\bar{\sigma}$ Functions

Let n be a positive integer, k a nonnegative integer, and let $1 \leq r \leq n$. Then $\tau_k^{(r)}$ is defined as follows:

$$\tau_k^{(r)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) = \sum_{1 \leq i_i \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1}^{(r)} x_{i_2}^{(r-1)} \ldots x_{i_k}^{(r-k+1)}$$

where no index appears more than $n-1$ times in the sum. The σ and $\bar{\sigma}$ functions are defined using τ:

$$\sigma_k^{(r)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) = \sum_{i=0}^{k} x_1^{(r)} x_1^{(r-1)} \ldots x_1^{(r-i+1)} \tau_{k-i}^{(r-i)}(\mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_m),$$

$$\bar{\sigma}_k^{(r)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) = \sum_{i=0}^{k} \tau_{k-i}^{(r)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{m-1}) x_m^{(r-k+i)} x_m^{(r-k+i-1)} \ldots x_m^{(r-k)}.$$

The τ, σ, $\bar{\sigma}$ Functions

Let $n = 4$. Write $\mathbf{a} = (a_1, \ldots, a_4), \mathbf{b} = (b_1, \ldots, b_4), \mathbf{c} = (c_1, \ldots, c_4)$ in place of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. Then

$$\begin{aligned}
\tau_5^{(3)}(\mathbf{b}, \mathbf{c}) &= b_3 b_2 b_1 c_4 c_3 + b_3 b_2 c_1 c_4 c_3, \\
\sigma_6^{(4)}(\mathbf{a}, \mathbf{b}, \mathbf{c}) &= \tau_6^{(4)}(\mathbf{b}, \mathbf{c}) + a_4 \tau_5^{(3)}(\mathbf{b}, \mathbf{c}) + a_4 a_3 \tau_4^{(2)}(\mathbf{b}, \mathbf{c}) + a_4 a_3 a_2 \tau_3^{(1)}(\mathbf{b}, \mathbf{c}) \\
&\quad + a_4 a_3 a_2 a_1 \tau_2^{(4)}(\mathbf{b}, \mathbf{c}) + a_4 a_3 a_2 a_1 a_4 \tau_1^{(3)}(\mathbf{b}, \mathbf{c}) + a_4 a_3 a_2 a_1 a_4 a_3 \\
\bar{\sigma}_6^{(4)}(\mathbf{a}, \mathbf{b}, \mathbf{c}) &= \tau_6^{(4)}(\mathbf{a}, \mathbf{b}) + \tau_5^{(4)}(\mathbf{a}, \mathbf{b}) c_3 + \tau_4^{(4)}(\mathbf{a}, \mathbf{b}) c_4 c_3 + \tau_3^{(4)}(\mathbf{a}, \mathbf{b}) c_1 c_4 c_3 \\
&\quad + \tau_2^{(4)}(\mathbf{a}, \mathbf{b}) c_2 c_1 c_4 c_3 + \tau_1^{(4)}(\mathbf{a}, \mathbf{b}) c_3 c_2 c_1 c_4 c_3 + c_4 c_3 c_2 c_1 c_4 c_3
\end{aligned}$$

1-Shifts

We call permutations of the form $s_{j-1}\dots s_i$ and $s_i\dots s_{j-1}$ 1-shifts. For example, when $i=1, j=4$, in cycle notation, $s_3s_2s_1=(4321)$ and $s_1s_2s_3=(1234)$.

Theorem 2 ([Lam–Pylyavskyy, 2010]; [Chepuri–L., 2020+])

$$s_{j-1}\dots s_i(x_j^{(r)}) = x_i^{(r-j+i)}\frac{\sigma_{(n-1)(j-i)}^{(r-j+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_j)}{\sigma_{(n-1)(j-i)}^{(r-j+i)}(\mathbf{x}_i,\dots,\mathbf{x}_j)},$$

and for $i\le k<j$,

$$s_{j-1}\dots s_i(x_k^{(r)}) = \frac{x_{k+1}^{(r+1)}\sigma_{(n-1)(k+1-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_{k+1})\sigma_{(n-1)(k-i)}^{(r-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_k)}{\sigma_{(n-1)(k+1-i)}^{(r-k+i-1)}(\mathbf{x}_i,\dots,\mathbf{x}_{k+1})\sigma_{(n-1)(k-i)}^{(r-k+i)}(\mathbf{x}_i,\dots,\mathbf{x}_k)}.$$

1-Shifts

We call permutations of the form $s_{j-1} \ldots s_i$ and $s_i \ldots s_{j-1}$ 1-shifts. For example, when $i = 1, j = 4$, in cycle notation, $s_3 s_2 s_1 = (4321)$ and $s_1 s_2 s_3 = (1234)$.

Theorem 2 (Dual) [Chepuri–L. 2020+]

$$s_i \ldots s_{j-1}(x_i^{(r)}) = x_j^{(r+j-i)} \frac{\bar{\sigma}^{(r)}_{(n-1)(j-i)}(\mathbf{x}_i, \ldots, \mathbf{x}_j)}{\bar{\sigma}^{(r-1)}_{(n-1)(j-i)}(\mathbf{x}_i, \ldots, \mathbf{x}_j)},$$

and for $i < k \leq j$,

$$s_i \ldots s_{j-1}(x_k^{(r)}) = \frac{x_{k-1}^{(r-1)} \bar{\sigma}^{(r-2)}_{(n-1)(j-k+1)}(\mathbf{x}_{k-1}, \ldots, \mathbf{x}_j) \bar{\sigma}^{(r)}_{(n-1)(j-k)}(\mathbf{x}_k, \ldots, \mathbf{x}_j)}{\bar{\sigma}^{(r-1)}_{(n-1)(j-k+1)}(\mathbf{x}_{k-1}, \ldots, \mathbf{x}_j) \bar{\sigma}^{(r-1)}_{(n-1)(j-k)}(\mathbf{x}_k, \ldots, \mathbf{x}_j)}.$$

Combinatorial Interpretation of τ Functions

Cylindrical networks $N(n, m)$:

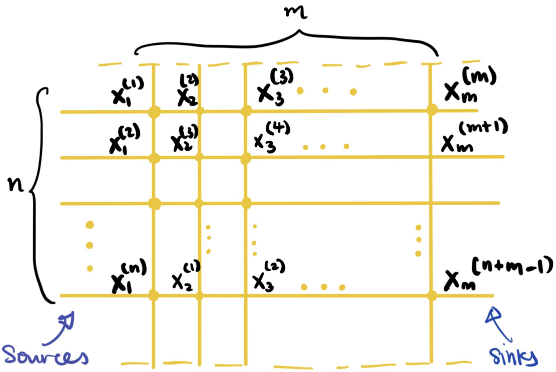

Figure 1: Illustration of $N(n, m)$

Combinatorial Interpretation of τ Functions

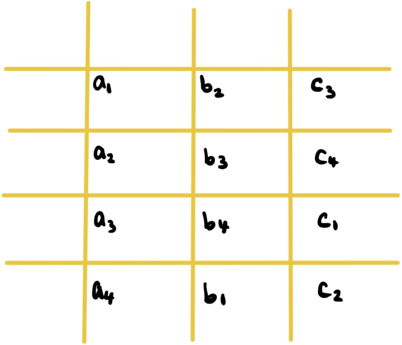

Figure 2: Illustration of $N(3,4)$

Combinatorial Interpretation of τ Functions

Highway paths:

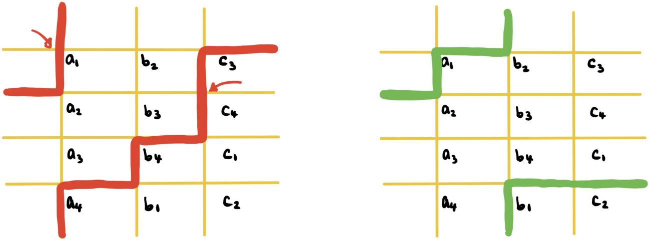

Figure 3: A non-example and an example of a highway path

Combinatorial Interpretation of τ Functions

Highway paths and $\tau_3^{(1)}(\mathbf{a}, \mathbf{b}, \mathbf{c})$:

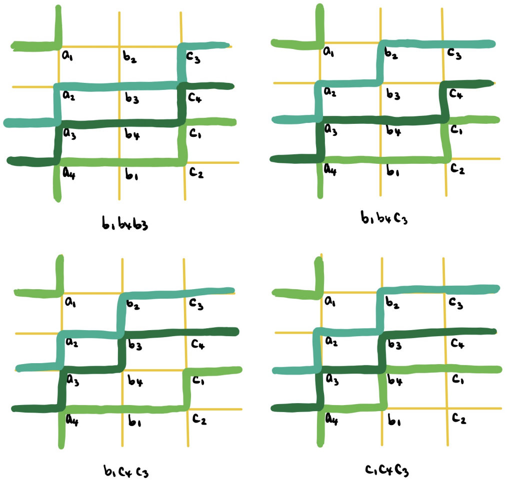

Figure 4: All terms in $\tau_3^{(1)}(\mathbf{a}, \mathbf{b}, \mathbf{c})$ that use only **b** and **c**

Combinatorial Interpretation of σ and $\bar{\sigma}$ Functions

$$\begin{aligned}
\sigma_6^{(4)}(\mathbf{a},\mathbf{b},\mathbf{c}) &= \tau_6^{(4)}(\mathbf{b},\mathbf{c}) + a_4\tau_5^{(3)}(\mathbf{b},\mathbf{c}) + a_4a_3\tau_4^{(2)}(\mathbf{b},\mathbf{c}) + a_4a_3a_2\tau_3^{(1)}(\mathbf{b},\mathbf{c}) \\
&\quad + a_4a_3a_2a_1\tau_2^{(4)}(\mathbf{b},\mathbf{c}) + a_4a_3a_2a_1a_4\tau_1^{(3)}(\mathbf{b},\mathbf{c}) + a_4a_3a_2a_1a_4a_3 \\
&= (\tau_6^{(4)}(\mathbf{b},\mathbf{c}) + a_4\tau_5^{(3)}(\mathbf{b},\mathbf{c}) + a_4a_3\tau_4^{(2)}(\mathbf{b},\mathbf{c}) + a_4a_3a_2\tau_3^{(1)}(\mathbf{b},\mathbf{c})) \\
&\quad + a_4a_3a_2a_1(\tau_2^{(4)}(\mathbf{b},\mathbf{c}) + a_4\tau_1^{(3)}(\mathbf{b},\mathbf{c}) + a_4a_3) \\
&= \tau_6^{(4)}(\mathbf{a},\mathbf{b},\mathbf{c}) + a_4a_3a_2a_1\tau_2^{(4)}(\mathbf{b},\mathbf{c})
\end{aligned}$$

Transpositions

A transposition that switches i and j can be written as $s_i s_{i+1} \ldots s_{j-1} \ldots s_{i+1} s_i$. For example $(14) = s_1 s_2 s_3 s_2 s_1 = s_3 s_2 s_1 s_2 s_3$.

The Ω Functions

For $i \le k \le j-1$, define

$${}^{(k)}\Omega^{(r)}_{(n-1)(j-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_j) = \sum_{\ell=0}^{n-1} \sigma^{(r)}_{(n-1)(k-i)+\ell}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\bar{\sigma}^{(r+k-i-\ell)}_{(n-1)(j-k)-\ell}(\mathbf{x}_{k+1},\ldots,\mathbf{x}_j).$$

Specializes to $\bar{\sigma}$ when $k=i$ and σ when $k=j-1$. Example: $j=4, i=1, k=2, n=4$,

$$\begin{aligned} {}^{(2)}\Omega^{(r)}_9(\mathbf{a},\ldots,\mathbf{d}) &= \sigma^{(r)}_3(\mathbf{a},\mathbf{b})\bar{\sigma}^{(r+1)}_6(\mathbf{c},\mathbf{d}) + \sigma^{(r)}_4(\mathbf{a},\mathbf{b})\bar{\sigma}^{(r)}_5(\mathbf{c},\mathbf{d}) \\ &\quad + \sigma^{(r)}_5(\mathbf{a},\mathbf{b})\bar{\sigma}^{(r-1)}_4(\mathbf{c},\mathbf{d}) + \sigma^{(r)}_6(\mathbf{a},\mathbf{b})\bar{\sigma}^{(r-2)}_3(\mathbf{c},\mathbf{d}) \end{aligned}$$

Transpositions

Conjecture 1. [Chepuri–L. 2020+] Let $s = s_i \ldots s_{j-2} s_{j-1} s_{j-2} \ldots s_i$. For $i < k < j$,

$$s(x_k^{(r)}) = x_k^{(r)} \frac{{}^{(k)}\Omega^{(r-k+i)}_{(n-1)(j-i)}(\mathbf{x}_i, \ldots, \mathbf{x}_j)\ {}^{(k-1)}\Omega^{(r-k+i-1)}_{(n-1)(j-i)}(\mathbf{x}_i, \ldots, \mathbf{x}_j)}{{}^{(k-1)}\Omega^{(r-k+i)}_{(n-1)(j-i)}(\mathbf{x}_i, \ldots, \mathbf{x}_j)\ {}^{(k)}\Omega^{(r-k+i-1)}_{(n-1)(j-i)}(\mathbf{x}_i, \ldots, \mathbf{x}_j)}.$$

Identity of Ω Functions

Conjecture 2. [Chepuri–L. 2020+] For $i < k \le j-1$, the following identity of ${}^{(k-1)}\Omega$ and ${}^{(k)}\Omega$ holds:

$$\left[\prod_{t=1}^{n-1} \sigma^{(r-k+i+t)}_{(n-1)(k-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_k)\right] \; {}^{(k-1)}\Omega^{(r-k+i)}_{(n-1)(j-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_j)$$
$$= \sum_{s=0}^{n-1} \prod_{t=r+1}^{r+s} x_j^{(t+j-k)} \prod_{t=r+s+1}^{r+n-1} x_k^{(t+1)} \prod_{t=s+2}^{s+n-1} \sigma^{(r-k+i+t)}_{(n-1)(k-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_k)$$
$${}^{(k)}\Omega^{(r-k+i+s)}_{(n-1)(j-i)}(\mathbf{x}_i,\ldots,\mathbf{x}_j)\sigma^{(r-k+i+s+1)}_{(n-1)(k-i-1)}(\mathbf{x}_i,\ldots,\mathbf{x}_{k-1}).$$

We proved this in the $n=2$ case.

Future Directions

- Resolve the conjectures;
- Other permutations;
- Combinatorial interpretation of the Ω functions;
- Is there an easy way of interpreting the identities we are getting using a graphical calculus of cylindrical networks?

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