F-Polynomials of the r-Kronecker Quiver

Feiyang Lin

Mentor: Gregg Musiker
TA: Elizabeth Kelley
UMN Combinatorics REU 2020

August 7, 2020

Formula for the F-Polynomials

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([Gupta, 2018]) Given $r \geq 2$, we care about a family of polynomials indexed by ℓ:

$$F_\ell(y_1, y_2) = \sum_{(m_1,\ldots,m_\ell)\in\mathbb{Z}_{\geq 0}^\ell} \prod_{i=1}^{\ell} \binom{a_{\ell-i+1} - r\sum_{j=i+1}^{\ell} a_{j-i} m_j}{m_i} y_1^M y_2^N,$$

where

$$a_1 m_1 + a_2 m_2 + \cdots + a_\ell m_\ell = M,$$
$$a_1 m_2 + a_2 m_3 + \cdots + a_{\ell-1} m_\ell = N.$$

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Motivating Question: Why are there eventually only finitely many terms (why does this turn out to be a polynomial)? Why are the coefficients positive?

Coefficients of the F-Polynomial

$$F_\ell(y_1, y_2) = \sum_{M \geq N \geq 0} C^{(\ell,r)}_{M,N} y_1^M y_2^N,$$

where

$$C^{(\ell,r)}_{M,N} = \sum_{\substack{(m_1,\dots,m_\ell)\in\mathbb{Z}^\ell_{\geq 0} \\ a_1 m_1 + a_2 m_2 + \cdots + a_\ell m_\ell = M \\ a_1 m_2 + a_2 m_3 + \cdots + a_{\ell-1} m_\ell = N}} \prod_{i=1}^{\ell} \binom{a_{\ell-i+1} - r \sum_{j=i+1}^{\ell} a_{j-i} m_j}{m_i}.$$

By pulling out the factor $\binom{a_\ell - rN}{m_1}$ from the product of ℓ binomials, we get

$$C^{(\ell,r)}_{M,N} = \sum_{k \geq 0} \binom{a_\ell - rN}{k} C^{(\ell-1,r)}_{N,rN-M+k}.$$

Coefficients of the F-Polynomial

Example: $r = 3$, $\ell = 4$,

$$C_{M,N}^{(4,3)} = \sum_{\substack{(m_1,\dots,m_4)\in\mathbb{Z}_{\geq 0}^4 \\ m_1+3m_2+8m_3+21m_4=M \\ m_2+3m_3+8m_4=N}} \binom{21-3(m_2+3m_3+8m_4)}{m_1}\binom{8-3(m_3+3m_4)}{m_2}\binom{3-3m_4}{m_3}\binom{1}{m_4}$$

Coefficients of the F-Polynomial

Example: $r = 3$, $\ell = 4$,

$$C^{(4,3)}_{M,N} = \sum_{\substack{(m_1,\ldots,m_4)\in\mathbb{Z}^4_{\geq 0}\\ m_1+3m_2+8m_3+21m_4=M\\ m_2+3m_3+8m_4=N}} \binom{21-3(m_2+3m_3+8m_4)}{m_1}\binom{8-3(m_3+3m_4)}{m_2}\binom{3-3m_4}{m_3}\binom{1}{m_4}$$

$M = 9, N = 3$. Then need to find tuples such that

$$m_1 + 3m_2 + 8m_3 + 21m_4 = 9,$$

$$m_2 + 3m_3 + 8m_4 = 3.$$

Two choices:

$m_1 = 1, m_2 = 0, m_3 = 1, m_4 = 0$. $\binom{21-9}{1}\binom{8-3}{0}\binom{3}{1}\binom{1}{0} = 36$

$m_1 = 0, m_2 = 3, m_3 = 0, m_4 = 0$; $\binom{21-9}{0}\binom{8}{3}\binom{3}{0}\binom{1}{0} = 56$

So $C^{(4,3)}_{9,3} = 92$.

Coefficients of the F-Polynomial

Example: $r = 3$, $\ell = 4$,

$$C_{M,N}^{(4,3)} = \sum_{\substack{(m_1,\ldots,m_4)\in\mathbb{Z}_{\geq 0}^4 \\ m_1+3m_2+8m_3+21m_4=M \\ m_2+3m_3+8m_4=N}} \binom{21-3(m_2+3m_3+8m_4)}{m_1}\binom{8-3(m_3+3m_4)}{m_2}\binom{3-3m_4}{m_3}\binom{1}{m_4}$$

$M = 22, N = 8$. Then need to find tuples such that

$$m_1 + 3m_2 + 8m_3 + 21m_4 = 22,$$

$$m_2 + 3m_3 + 8m_4 = 8.$$

$m_1 = 1, m_2 = 0, m_3 = 0, m_4 = 1$; $\binom{21-24}{1}\binom{8-9}{0}\binom{3-3}{0}\binom{1}{1} = -3$.
$m_1 = 0, m_2 = 2, m_3 = 2, m_4 = 0$; $\binom{21-24}{0}\binom{8-6}{2}\binom{3}{2}\binom{1}{0} = 3$.
So $C_{22,8}^{(4,3)} = 0$.

Grid of Coefficients $C_{M,N}^{(4,3)}$

M, N	0	1	2	3	4	5	6	7	8
0	1								
1	21								
2	210								
3	1330	8							
4	5985	144							
5	20349	1224							
6	54264	6528	28						
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15	54264	148512	140140	54120	7770	236			
16	20349	68544	84084	45837	10710	786	3		
17	5985	24480	38220	28380	10080	1440	39		
18	1330	6528	12740	12518	6420	1570	127		
19	210	1224	2940	3732	2655	1020	177	6	
20	21	144	420	675	645	366	114	15	
21	1	8	28	56	70	56	28	8	1

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Conjecture 1 (Polynomiality)

$$C_{M,N}^{(\ell,r)} \neq 0 \text{ if and only if } 0 \leq M \leq a_\ell,\ 0 \leq N \leq \frac{a_{\ell-1}}{a_\ell} M.$$

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What we know by cheating:

- $C_{M,N}^{(\ell,r)} \neq 0$ only if $0 \leq M \leq a_\ell$
  ($F_{\ell+1}F_{\ell-1} = F_\ell^r + y_1^{a_{\ell+1}} y_2^{a_\ell}$, induction);
- $C_{M,N}^{(\ell,r)} \geq 0$ always (F-polynomials have positive coefficients).

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We can show $C_{M,N}^{(\ell,r)} \neq 0$ only if $0 \leq N \leq \frac{a_{\ell-1}}{a_\ell} M$.

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Conjecture 2 (Positivity)

When $0 \leq M \leq a_\ell$, $0 \leq N \leq \frac{a_{\ell-1}}{a_\ell} M$, $C_{M,N}^{(\ell,r)} > 0$.

Conjectures on $C_{M,N}^{(\ell,r)}$

We can also show:

- For $0 \leq N < r$,

$$C_{M,N}^{(\ell,r)} = \binom{a_\ell - rN}{M - rN}\binom{a_{\ell-1}}{N}$$

- When $a_\ell - rN > 0$ or $N = a_{\ell-1}$,

$$C_{a_\ell,N}^{(\ell,r)} = \binom{a_{\ell-1}}{N}.$$

Assuming Conjecture 1, this is true for $0 \leq N \leq a_{\ell-1}$.

Conjectures on $C_{M,N}^{(\ell,r)}$

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Figure 1: Green: what we know; Red: conjectural

F-Polynomial when $r = 2$

From now on we focus on the case where $r = 2$.
When $r = 2$, $a_\ell = \ell$.
The recurrence becomes

$$C^{(\ell,2)}_{M,N} = \sum_{k\geq 0} \binom{\ell - 2N}{k} C^{(\ell-1,2)}_{N,2N-M+k}.$$

Theorem 1.

$$C^{(\ell,2)}_{M,N} = \binom{\ell - N}{\ell - M}\binom{M-1}{N}.$$

F-Polynomial when $r = 2$

The recurrence becomes

$$C_{M,N}^{(\ell,2)} = \sum_{k \geq 0} \binom{\ell - 2N}{k} C_{N,2N-M+k}^{(\ell-1,2)}.$$

Theorem 1.

$$C_{M,N}^{(\ell,2)} = \binom{\ell - N}{\ell - M} \binom{M-1}{N}.$$

Equivalently,

$$\binom{\ell - N}{M - N} \binom{M-1}{N} = \sum_{k \geq 0} \binom{\ell - 2N}{k} \binom{\ell - 1 - 2N + M - k}{\ell - 1 - N} \binom{N-1}{2N - M + k}$$

We developed two proofs:

- Hypergeometric series identity;
- Combinatorial interpretation of $\binom{\ell-N}{\ell-M}\binom{M-1}{N}$.

Combinatorial Proof of Theorem 1

Let $[n] = \{1, \ldots, n\}$.

Theorem 2 ([Musiker and Propp, 2006]).

Let $\Omega^{\ell}_{M,N}$ be the set of subsets $S \subset [2\ell - 1]$ such that S contains $\ell - M$ odd elements, N even elements and no consecutive elements. Then

$$|\Omega^{\ell}_{M,N}| = \binom{\ell - N}{\ell - M}\binom{M-1}{N}.$$

So it suffices to show that

$$|\Omega^{\ell}_{M,N}| = \sum_{k \geq 0} \binom{\ell - 2N}{k} |\Omega_{N,2N-M+k}|$$

Example

$\ell = 6,\ M = 4,\ N = 2$, then $\Omega^{\ell}_{M,N}$ is the set of subsets $S \subset [11]$ such that S contains 2 odd elements, 2 even elements and no consecutive elements.

$$
\begin{array}{rccc}
\Omega^{\ell}_{M,N} = & 2479 & 247J & 249J \\
& 4619 & 461J & 469J \\
& 6813 & 681J & 683J \\
& 8T13 & 8T15 & 8T35 \\
& 269J & & \\
& 285J & & \\
& 2T57 & & \\
& 481J & & \\
& 4T17 & & \\
& 6T13 & &
\end{array}
$$

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Indeed,

$$|\Omega^{\ell}_{M,N}| = \binom{\ell - N}{\ell - M}\binom{M-1}{N} = \binom{4}{2}\binom{3}{2} = 18.$$

Example

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Let X be a subset of N even numbers and let

$$O_X = |\{\text{odd numbers between } 1 \text{ and } 2\ell - 1 \text{ not adjacent to } X\}|.$$

Observe that

$$|\Omega^{\ell}_{M,N}| = \sum_X \binom{O_X}{\ell - M}$$

Combinatorial Proof of Theorem 1

$$|\Omega_{M,N}^{\ell}| = \sum_{k\geq 0} \binom{\ell - 2N}{k} |\Omega_{N,2N-M+k}^{\ell-1}|$$

Combinatorial Proof of Theorem 1

$$|\Omega^{\ell}_{M,N}| = \sum_{k\geq 0} \binom{\ell - 2N}{k} |\Omega^{\ell-1}_{N,2N-M+k}|$$

On the left hand side, we are choosing N even elements from $[2\ell - 1]$; on the right hand side we are choosing $\ell - 1 - N$ odd elements from $[2\ell - 3]$.

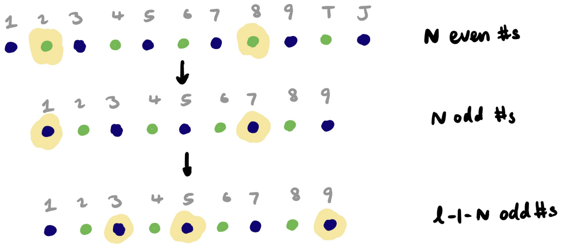

Figure 2: Correspondence of N even elements and $\ell - 1 - N$ odd elements; $\ell = 6$, $N = 2$.

Combinatorial Proof of Theorem 1

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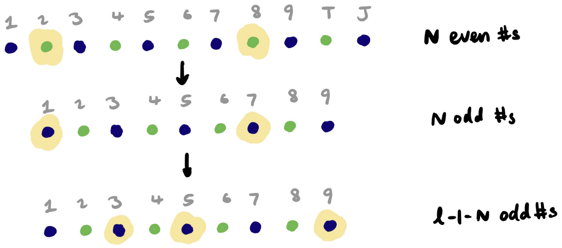

Figure 2: Correspondence of N even elements and $\ell - 1 - N$ odd elements; $\ell = 6, N = 2$.

If the N even elements we started with is X, let $\widetilde{X}$ be the corresponding set of $\ell - 1 - N$ odd elements.

Combinatorial Proof of Theorem 1

Lemma 3.

Let X be a set consisting of even numbers between 1 and $2\ell - 1$ such that $|X| = N$. Let

$$O_X = |\{\textit{odd numbers between } 1 \textit{ and } 2\ell - 1 \textit{ not adjacent to } X\}|$$

and

$$E_X = |\{\textit{even numbers between } 1 \textit{ and } 2\ell - 3 \textit{ not adjacent to } \widetilde{X}\}|,$$

then $O_X - E_X = \ell - 2N$.

Examples

$\ell = 6, M = 4, N = 2$, so $\ell - 2N = 2$.

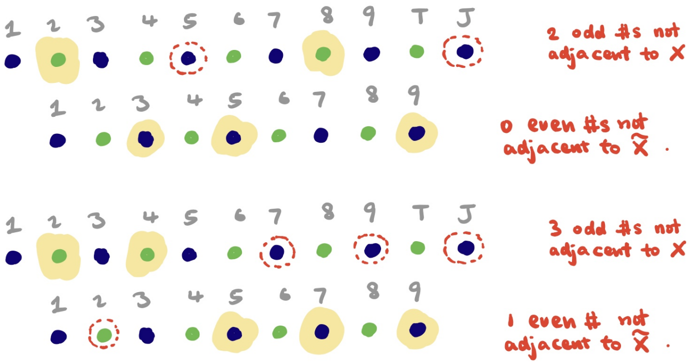

Figure 3: Examples of $O_X - E_X = \ell - 2N$; $X = \{2, 8\}$ and $X = \{2, 4\}$

Combinatorial Proof of Theorem 1

By Vandermonde's identity:

$$\binom{O_X}{\ell - M} = \sum_{k \geq 0} \binom{O_X - E_X}{k} \binom{E_X}{E_X - (O_X - \ell + M - k)}$$

$O_X - E_X = \ell - 2N$:

$$\binom{O_X}{\ell - M} = \sum_{k \geq 0} \binom{\ell - 2N}{k} \binom{E_X}{2N - M + k}.$$

Summing over all X:

$$|\Omega^{\ell}_{M,N}| = \sum_{k \geq 0} \binom{\ell - 2N}{k} |\Omega^{\ell-1}_{N,2N-M+k}|.$$

Future Directions

- Look more into showing that coefficients are positive inside the contour we outlined.
- Explore patterns of cancellation.
- Does $C_{M,N}^{(\ell,r)}$ for a fixed M or a fixed N form Pólya frequency sequences?

References

Gupta, M. (2018).
A formula for f-polynomials in terms of c-vectors and stabilization of f-polynomials.

Musiker, G. and Propp, J. (2006).
Combinatorial interpretations for rank-two cluster algebras of affine type.