REAL-ROOTEDNESS OF POLYNOMIALS FROM PLANAR GRAPHS ON A CYLINDER

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Abstract. We investigate a certain sequence arising from the dimer covers of an arbitrary edge-weighted planar bipartite graph G embedded on a cylinder. We first show that this sequence is always log-concave. Then, we show that the sequence is a Pólya frequency sequence if G is an unweighted grid graph.

1. Introduction and Preliminaries

A sequence $a_0, a_1, \ldots$ of real numbers is said to be log-concave if $a_i^2 \geq a_{i-1}a_{i+1}$ for all i. A sequence of nonnegative numbers $a_0, a_1, \ldots$ with finitely many nonzero terms is called a *Pólya frequency sequence (PFS)* if its generating function $\sum a_i x^i$ has only real roots. It is classical that a PFS is always log-concave. The following result is useful in detecting the PFS property:

Theorem 1.1 ([Sta98]). *A finite sequence $a_0, \ldots, a_n$ is a PFS if and only if the associated Aissen–Schoenberg–Whitney matrix*

$$\mathrm{ASW}(a_0, \ldots, a_n) \coloneqq \begin{pmatrix} a_0 & a_1 & \cdots & a_n & & & \\ & a_0 & a_1 & \cdots & a_n & & \\ & & a_0 & a_1 & \cdots & a_n & \\ & & & \ddots & \ddots & & \ddots \end{pmatrix}$$

is totally nonnegative.

This replaces the analytic condition of real-rootedness with infinitely many algebraic conditions given by the nonnegativity of the minors of an infinite matrix.

We can now define analogues of the log-concavity and PFS properties in a more general setting.

Definition 1.2. Let $R \coloneqq \mathbb{Z}[y_1, \ldots, y_r]$ be a polynomial ring and put a partial ordering on R by saying that $f \succeq 0$ if the coefficient of every monomial term of f is nonnegative. In this case, we say that f is *monomial-nonnegative*.

Definition 1.3. We say that a sequence $a_0, a_1, \ldots, a_n \in R$ is *log-concave* if $a_i^2 \succeq a_{i-1}a_{i+1}$ for all i. We say it is a *Pólya frequency sequence (PFS)* if all minors of $\mathrm{ASW}(a_0, \ldots, a_n)$ are monomial-nonnegative.

In this paper we investigate certain sequences associated to planar bipartite graphs embedded on a cylinder. All graphs will be finite. Throughout, we write $\mathcal{O}$ for a cylinder. We fix an isomorphism $H_1(\mathcal{O}, \mathbb{Z}) \cong \mathbb{Z}$ and use the words "counterclockwise" and "clockwise" to describe cycles with positive and negative images in $\mathbb{Z}$, respectively.

Date: August 2020.

Recall that a *dimer cover* (or *perfect matching*) on a graph G is a subgraph which uses every vertex of G and every vertex is incident to exactly one edge.

Definition 1.4. Let $G \subset \mathcal{O}$ be a planar bipartite graph on vertex sets V and W, where $|V| = |W|$. Let π_1, π_2 be two dimer covers of G. Define $\pi_1 \cup \pi_2^\vee$ to be the directed graph on $\mathcal{O}$ with vertices $V \cup W$, a directed edge from $v \in V$ to $w \in W$ whenever $\{v, w\}$ is an edge of π_1, and a directed edge from $w \in W$ to $v \in V$ whenever $\{v, w\}$ is an edge of π_2.

Henceforth, all bipartite graphs on vertex sets V and W will have $|V| = |W|$, so that they admit dimer covers.

Definition 1.5. Let π_1, π_2 be two dimer covers of G. Then it is easy to show that $\pi_1 \cup \pi_2^\vee$ is a union of vertex-disjoint directed simple cycles, which can then be viewed as singular 1-cycles on $\mathcal{O}$. We define the *relative height* $\operatorname{ht}(\pi_1, \pi_2)$ to be the image in $H_1(\mathcal{O}, \mathbb{Z}) \cong \mathbb{Z}$ of the sum of these cycles.

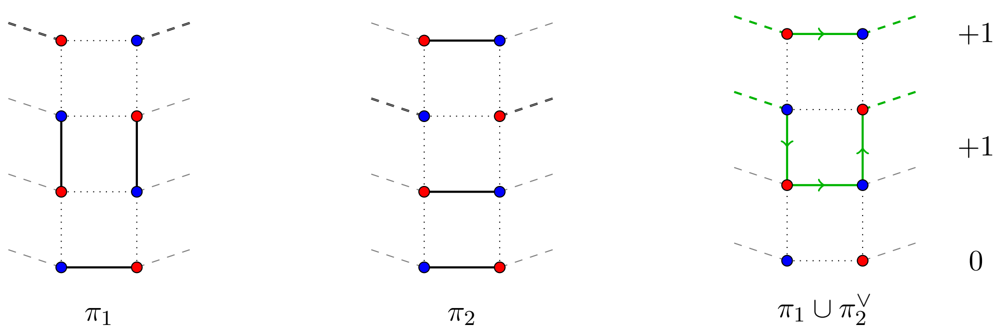

FIGURE 1. The graph G is an $n \times 2$ grid embedded on a cylinder; dashed edges signify "looping around the back" of the cylinder. The set V consists of the blue vertices and the set W consists of the red vertices. Here we have dimer covers π_1, π_2 with relative height $\operatorname{ht}(\pi_1, \pi_2) = 2$, since there are 2 counterclockwise cycles and no clockwise cycles in $\pi_1 \cup \pi_2^\vee$.

This definition of relative height makes the following properties clear:

Lemma 1.6. *Given three dimer covers π_1, π_2, π_3 on G, we have*

- $\operatorname{ht}(\pi_1, \pi_2) = -\operatorname{ht}(\pi_2, \pi_1)$
- $\operatorname{ht}(\pi_1, \pi_3) = \operatorname{ht}(\pi_1, \pi_2) + \operatorname{ht}(\pi_2, \pi_3)$.

Thus, dimer covers on G are totally preordered by height and it makes sense to speak of a (not necessarily unique) *minimal* dimer cover on G with respect to this preordering.

Definition 1.7. Fix a minimal-height dimer cover π_0. Let π be any dimer cover of G. We define the *absolute height*

$$\operatorname{ht}(\pi) := \operatorname{ht}(\pi, \pi_0).$$

Finally, we would like to be able to work with edge-weighted graphs.

Definition 1.8. Suppose G is edge-weighted (by positive reals, or by variables). The *weight* $\operatorname{wt}(\pi)$ of a dimer cover π is the product of the weights of its edges. The *weight* of a pair of dimer covers (π_1, π_2) is given by $\operatorname{wt}(\pi_1, \pi_2) := \operatorname{wt}(\pi_1) \operatorname{wt}(\pi_2)$.

We now introduce the main object of study in this note. Let $G \subset \mathcal{O}$ be edge-weighted (by positive reals or by variables), and construct the *height sequence* $a_0, a_1, \ldots$ via

$$a_i = \sum_{\mathrm{ht}(\pi)=i} \mathrm{wt}(\pi) \tag{1}$$

where the sum is over all dimer covers of G with absolute height i. This is a weighted count of dimer covers of a fixed height (if the graph G is unweighted, i.e. all edges have weight 1, then a_i is the actual number of dimer covers of height i).

Question 1.9. Is the sequence (a_i) log-concave? Is it a PFS?

In §2, we prove that when G has variable edge weights, the 2×2 minors of the Aissen–Schoenberg–Whitney matrix are monomial-nonnegative. This is enough to imply log-concavity. We also show that certain initial 3×3 minors are monomial-nonnegative. Then in §3, we provide additional evidence that the (a_i) form a PFS by showing real-rootedness for all *unweighted* grid graphs described in Lemma 3.1.

2. General planar bipartite graphs

Throughout this section, we assume that $G \subset \mathcal{O}$ is planar, bipartite, and with algebraically independent edge weights $y_1, \ldots, y_N$, where N denotes the number of edges of G. Thus all edges, dimer covers, pairs of dimer covers, etc. have weights in the ring $R := \mathbb{Z}[y_1, \ldots, y_N]$. Let (a_i) be the sequence defined in (1).

Proposition 2.1. *The 2×2 minors of the Aissen–Schoenberg–Whitney matrix associated to (a_i) are nonnegative. In particular, (a_i) is log-concave.*

The concept of "twisting along a cycle" will be useful in proving this.

Definition 2.2. Let π be a dimer cover of G and let C be a cycle in G consisting of $2k$ edges for some k. Suppose that the edges are indexed so that exactly k edges $e_1, e_3, \ldots, e_{2k-1}$ of C appear in π, and the edges $e_2, e_4, \ldots, e_{2k}$ do not appear in π. Then the *twist of π along C* is the dimer cover $\widetilde{\pi}$ which is obtained from π by removing $e_1, e_3, \ldots, e_{2k-1}$ and adding $e_2, e_4, \ldots, e_{2k}$.

Observe that twisting along C is an involution. If C is contractible, then twisting along C leaves height invariant. If C is a simple non-contractible cycle, then twisting along C changes height by ± 1 according to which edges of C are contained in π.

Proof of Proposition 2.1. Let T_i be the set of dimer covers of height i. We will show that for all i, j with $i > j$ there is a weight-preserving injection

$$\Phi : T_i \times T_j \hookrightarrow T_{i-1} \times T_{j+1},$$

i.e. if $\Phi(\pi_1, \pi_2) = (\pi_1', \pi_2')$, then $\mathrm{wt}(\pi_1)\,\mathrm{wt}(\pi_2) = \mathrm{wt}(\pi_1')\,\mathrm{wt}(\pi_2')$.

Given $(\pi_1, \pi_2) \in T_i \times T_j$, let $(C_1, \ldots, C_m)$ denote the non-contractible cycles of $\pi_1 \cup \pi_2^\vee$, of which there are a nonzero number since $i > j$. We suppose they are ordered in the following way: C_1 is the boundary of the unbounded face of G in the $+\infty$ direction on $\mathcal{O}$, C_2 is the boundary of the unbounded face of $G - C_1$ in the $+\infty$ direction, and so on. Let $\varepsilon_i \in \{\pm 1\}$ denote the winding number of C_i (that is, its image in $\mathbb{Z} \cong H_1(\mathcal{O}, \mathbb{Z})$) and let s_i be the running sum $\varepsilon_1 + \cdots + \varepsilon_i$. Let i_0 be the minimal value for which $s_{i_0} = +1$. Such an i_0 always exists since

$$s_m = \varepsilon_1 + \cdots + \varepsilon_m = \mathrm{ht}(\pi_1, \pi_2) \geq 1.$$

Now define (π_1', π_2') to be (π_1, π_2) each twisted along $C_1, \ldots, C_{i_0}$, and set $\Phi(\pi_1, \pi_2) = (\pi_1', \pi_2')$. This decreases the height of π_1 by 1, while increasing the height of π_2 by 1, so $\Phi(\pi_1, \pi_2) \in T_{i-1} \times T_{j+1}$. Twisting along a cycle is a weight-preserving operation for a double dimer cover, so Φ is weight-preserving.

Given $(\pi_1', \pi_2') = \Phi(\pi_1, \pi_2)$, we can recover (π_1, π_2) as follows. Let $(C_1', \ldots, C_m')$ be the cycles of $\pi_1' \cup \pi_2'^{\vee}$, with ε_i' and s_i' defined analogously to before. Let j_0 be the minimal index for which $s_{j_0}' = -1$. By construction, $j_0 = i_0$ since $s_k' = -s_k$ for all $k \leq i_0$. So twisting along $C_1', \ldots, C_{j_0}'$ recovers (π_1, π_2). □

Proposition 2.3. *Let $(\pi_1, \pi_2) \in T_i \times T_i$. Suppose that $\pi_1 \cup \pi_2^{\vee}$ consists of $2m$ non-contractible cycles and r contractible cycles, and let*

$$S_k := \{(\pi_1', \pi_2') \in T_{i+k} \times T_{i-k} : \mathrm{wt}(\pi_1', \pi_2') = \mathrm{wt}(\pi_1, \pi_2)\}.$$

Then $|S_k| = 2^r \binom{2m}{m+k}$.

Proof. Let $C_1, \ldots, C_{2m}$ denote the non-contractible cycles and $D_1, \ldots, D_r$ the contractible cycles. Each D_i can be partitioned into two sets D_i^1, D_i^2 each of which is a dimer cover for D_i. There are 2^r ways to include the edges of these sets in π_1', π_2'. So henceforth suppose that π_j' always contains the edges of D_i^j; we will show that there are $\binom{2m}{m+k}$ dimer covers with this property and of the prescribed weight.

Each C_j can be partitioned into sets C_j^+ and C_j^- such that each is a dimer cover of C_j and $\mathrm{ht}(C_j^+, C_j^-) = 1$. We see that $\mathrm{wt}(\pi_1', \pi_2') = \mathrm{wt}(\pi_1, \pi_2)$ iff for each j, either C_j^+ is contained in π_2' and C_j^- is contained in π_1' or vice versa (this statement depends heavily on the algebraic independence of the edge weights).

Since $\mathrm{ht}(\pi_1, \pi_2) = 0$, we see that there are exactly m values of j for which C_j^+ is contained in π_1. Thus if a pair (π_1', π_2') has n values of j for which C_j^+ is contained in π_1', we have $\mathrm{ht}(\pi_1, \pi_1') = m - n$. There are $\binom{2m}{n}$ to pick these values of j, and therefore exactly that many choices of π_2' of height $i + n - m$. Writing $n = m + k$, we see that there are $\binom{2m}{m+k}$ pairs (π_1', π_2') for which $\mathrm{ht}(\pi_1') = i + k$. Of course the sum of the heights is always constant at $2i$, so $\mathrm{ht}(\pi_2') = i - k$ in such a pair. □

Proposition 2.4. *We have the inequality*

$$\sum_{i=-n}^{n} (-1)^i a_{n-i} a_{n+i} \succeq 0.$$

Proof. From Proposition 2.1 (for example), we see that all monomials in this sum appear in the central term a_n^2. Let f be a given monic monomial appearing in the expansion of a_n^2, corresponding to a double dimer cover (π_1, π_2) with $2m$ non-contractible cycles and r contractible ones. Observe that $m \leq n$ since otherwise there exists a dimer cover of negative absolute height.

Now Proposition 2.3 implies that the coefficient of f in $a_{n-i}a_{n+i}$ is $2^r\binom{2m}{m+i}$. We then see that the coefficient of f in the desired sum is

$$\sum_{i=-n}^{n}(-1)^i 2^r\binom{2m}{m+i} = 2^r\sum_{k=0}^{2m}(-1)^{m-k}\binom{2m}{k} = \begin{cases}0, & m>0\\ 2^r, & m=0\end{cases}$$

In particular, the coefficient of f is nonnegative. □

Remark 2.5. We see from this proof that the monomials that remain after all cancellations are exactly those that arise from pairs (π_1', π_2') for which $\pi_1' \cup \pi_2'^{\vee}$ consists only of contractible cycles.

Remark 2.6. This explicit counting by binomial coefficients can be used to give another proof of log-concavity, in the style of Proposition 2.4.

Corollary 2.7. *We have the following two minors of the Aissen–Schoenberg-Whitney matrix are nonnegative:*

$$\det\begin{pmatrix} a_1 & a_2 & a_3\\ a_0 & a_1 & a_2\\ 0 & a_0 & a_1\end{pmatrix} \succeq 0, \qquad \det\begin{pmatrix} a_2 & a_3 & a_4\\ a_1 & a_2 & a_3\\ a_0 & a_1 & a_2\end{pmatrix} \succeq 0$$

Proof. For ease, write $a_{i_1\ldots i_m} = \prod_{k=1}^m a_{i_k}$. Expanding out the first determinant, we obtain

$$a_{111} - 2a_{012} + a_{033} = a_1(a_{11} - 2a_{02}) + a_{033}$$

Now Proposition 2.4 implies that $a_{11} - 2a_{02} \geq 0$, so the whole determinant is nonnegative.

Expanding out the second determinant, we obtain

$$a_{222} - 2a_{123} + a_{033} + a_{114} - a_{024} = a_2(a_{22} - 2a_{13} + 2a_{04}) + a_4(a_{11} - 2a_{02}) + a_0(a_{33} - a_{24})$$

By Propositions 2.4 and 2.1, each term in parentheses is nonnegative, so the whole sum is nonnegative. □

Remark 2.8. Naive expansions only using the relations from Proposition 2.4 do not seem to immediately show nonnegativity of other solid 3×3 minors of the Aissen–Schoenberg–Whitney matrix.

3. Grid graphs

The following highlights the particular importance of a certain class of grid graphs:

Lemma 3.1. *Let $G \subset \mathcal{O}$ be a planar bipartite graph. Then G has the same height sequence as a subgraph of a bipartite $n \times 2m$ grid graph of the form in Figure 2.*

Remark 3.2. Such a grid graph must have an even number of columns since we are assuming our graphs are bipartite.

Proof sketch. We define a *move* on a bipartite graph to be a transformation as shown in Figure 3.

In particular, a move replaces a vertex v with three new vertices v', v'', v'''. The edges E_v incident to v are partitioned into two sets E_v^1 and E_v^2 in any manner, with the edges of E_v^1 becoming the edges of v' and the edges of E_v^2 becoming the edges of v''. Each of v' and v'' is

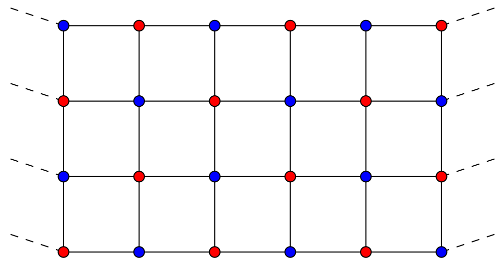

FIGURE 2. A 4×6 grid graph on a cylinder (with dashed edges looping around the cylinder).

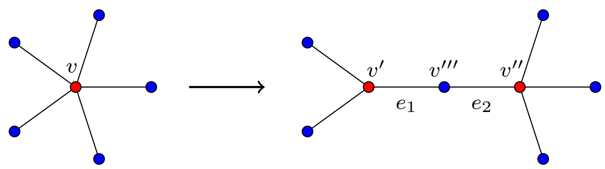

FIGURE 3. A move on a degree 5 vertex. The same diagram with colors reversed would also be a move. In the notation below, the set E_v^1 consists of the two leftmost edges of the original five, and E_v^2 consists of the three rightmost edges. The new edges e_1 and e_2 each have weight 1.

adjacent to v''' by exactly one edge of weight 1. We demand that all moves take place within a small disk around the initial vertex v, so that moves do not affect the global properties of the graph.

Let $G \subset \mathcal{O}$ be a planar bipartite graph, and let $\widetilde{G}$ be the graph obtained from G after applying a move at vertex v. We claim that the height sequence (a_i^G) is the same as the height sequence $(a_i^{\widetilde{G}})$. Indeed, any dimer cover of $\widetilde{G}$ uses exactly one of the edges e_1 and e_2. The dimer covers of $\widetilde{G}$ using e_1 correspond exactly to the dimer covers of G using an edge of E_v^2, and likewise the dimer covers of $\widetilde{G}$ using e_2 correspond exactly to the dimer covers of G using an edge of E_v^1. The height sequence remains the same, since the relative height between two dimer covers of G is not affected by the move.

It suffices to show that any bipartite graph $G \subset \mathcal{O}$, with vertices colored red and blue, after a sequence of moves, is isomorphic to a subgraph (with the same coloring) of a grid graph as in Figure 2. First, we apply as many moves as necessary to ensure that each vertex of G has degree at most 3, resulting in a graph G'. Note that G' is still planar and bipartite, and still colored in such a way that two adjacent vertices have different colors.

Now we show that up to moves, G' is a subgraph of some grid graph. To do this, we superimpose a sufficiently fine grid on $\mathcal{O}$, with the distance between parallel lines of the grid being much smaller than the distances between the vertices of G'. We also require the grid to have an even number of columns, and we color the vertices of the grid red and blue in a checkerboard fashion. Then, we perturb the vertices of G so that each vertex of G' lies on a vertex of the grid of the same color.

Let v be a vertex of G'. If possible, we replace each edge of G' incident to v with a lattice path along the grid in such a way that each pair of paths only intersects at v, and each path does not include any vertices of G' other than its endpoints; note that here, we use the fact that v has degree at most 3. After we do this, we add the grid vertices along the lattice paths to G', which we interpret as performing more moves on G'. If the edge replacement is not possible, we only need to increase the resolution of the grid by a factor of a sufficiently large even positive integer. Now we repeat the process with the other vertices that were originally part of G', with the caveat that when we increase the resolution of the grid, we must perform more moves to add the newly-formed grid vertices to the constructed lattice-path edges of G'. Since there were only finitely many vertices in G' to start with, we will be done after finitely many steps, and up to moves, we will have found a subgraph of a bipartite grid graph isomorphic to G'. □

Since real-rootedness of polynomials (and total nonnegativity of matrices) is closed under limits, to show that (a_i) is a PFS for a general planar bipartite $G \subset \mathcal{O}$, it would be sufficient to show that (a_i) has the desired property for any weighted grid graph of the form discussed above (by sending extraneous edge weights to 0). Motivated by this, we prove some particular results for grid graphs.

Definition 3.3. Real polynomials $f_1, \ldots, f_k$ are said to be *compatible* if $\sum_i c_i f_i$ has only real roots for all $c_1, \ldots, c_k \geq 0$.

Lemma 3.4 ([CS07, 2.2])**.** *If $f_1, \ldots, f_k$ are pairwise compatible real polynomials with positive leading coefficients, then the whole collection is compatible.*

Proposition 3.5. *Suppose G is the $n \times 2$ grid graph, with edges weighted by positive reals. Then (a_i) is a PFS.*

Proof. For each $1 \leq i \leq n$, the ith row of G consists of a clockwise edge of weight α_i and a counterclockwise edge of weight β_i. For $0 \leq i \leq n-1$, there is a pair of vertical edges connecting the ith row to the $(i+1)$th row, the product of whose weights will be written γ_n. All heights will be in reference to the minimal height dimer cover which uses all the clockwise edges from blue vertices to red vertices.

Let $q_{-1}(x) = 0, q_0(x) = 1$ and for $i \geq 1$, let $q_i(x)$ be the height polynomial of the subgraph of G induced by the first i rows. In adding the $(i+1)$th row, there are three possibilities:

- We use the clockwise edge of the new row, which does not change the height. This contributes an $\alpha_{i+1} q_i(x)$ to $q_{i+1}(x)$.
- We use the counterclockwise edge of the new row, which adds 1 to the height. This contributes a $\beta_{i+1} x q_i(x)$ to $q_{i+1}(x)$.
- We use the pair of vertical edges connecting to the ith row. This introduces a counterclockwise cycle and allows for anything in the first $i-1$ rows, so contributes an $\gamma_i x q_{i-1}(x)$.

So we obtain a recurrence of the form

$$q_{i+1}(x) = (\alpha_{i+1} + \beta_{i+1} x) q_i(x) + \gamma_i x q_{i-1}(x). \tag{2}$$

Using the terminology of [CS07], we now define the following 3 statements:

- A_i: The polynomial $q_i(x)$ is real-rooted.
- B_i: The polynomials $q_i(x)$ and $q_{i-1}(x)$ are compatible.

- C_i: The polynomials $q_i(x)$ and $xq_{i-1}(x)$ are compatible.

A_1, B_1, C_1 are verified directly. Assume that A_i, B_i, C_i hold. We see directly from (2) and Lemma 3.4 that $A_i, B_i, C_i \implies A_{i+1}$. Now let $\kappa, \sigma \geq 0$. Then using (2) we have

$$\kappa q_{i+1}(x) + \sigma q_i(x) = (\kappa\alpha_{i+1} + \sigma)q_i(x) + \kappa\beta_{i+1}xq_i(x) + \kappa\gamma_i xq_{i-1}(x)$$

so $A_i, B_i, C_i \implies B_{i+1}$. Similarly one finds that $A_i, B_i, C_i \implies C_{i+1}$. It follows that

$$A_i \wedge B_i \wedge C_i \implies A_{i+1} \wedge B_{i+1} \wedge C_{i+1}.$$

In particular, A_i always holds. □

To prove results on larger grid graphs, we introduce some new machinery (see [Ken08] for more on these concepts).

Definition 3.6. Let E denote the edge set of a planar bipartite graph G. A *Kasteleyn weighting* of G is a function $\Psi : E \to \{\pm 1\}$ such that for any face of G bounded by edges $e_1, \ldots, e_{2k}$, we have

$$\prod_{i=1}^{2k} \Psi(e_i) = (-1)^{k-1}.$$

Definition 3.7. Let $\Psi : E \to \{\pm 1\}$ be a Kasteleyn weighting of an unweighted graph G. Fix a vertical line ℓ on the cylinder $\mathcal{O}$. For an edge $\{v, w\} \in E$ with $v \in V, w \in W$, define the sign ε_{vw} to be $+1$ if the directed edge from v to w cross ℓ in the counterclockwise direction, -1 if in the clockwise direction, and 0 otherwise. Then we define the *Kasteleyn matrix* $K^{\Psi,\ell}$ to be the matrix defined over $\mathbb{Z}[x, x^{-1}]$ with rows indexed by V, columns indexed by W, and with (v, w)th entry given by the following formula:

$$K^{\Psi,\ell}_{v,w} = \begin{cases} \Psi(\{v, w\})x^{\varepsilon_{vw}}, & \{v, w\} \text{ is an edge of } G \\ 0, & \text{otherwise} \end{cases}$$

Lemma 3.8. *Up to multiplication by a power of x, the height polynomial $\sum a_i x^i$ of G coincides with $\pm \det K^{\Psi,\ell}$ for any Kasteleyn weighting Ψ and any choice of line ℓ on $\mathcal{O}$.*

Proof. This is a simple modification of [Ken08, Theorem 3 of §4.2]. □

Proposition 3.9. *Suppose G is the unweighted $n \times 2m$ grid graph, $m \geq 2$. Then (a_i) is a PFS.*

Proof. Label the vertices as in Figure 4.
Assign a Kasteleyn weighting to G by dictating that edges in the 1st, 3rd, 5th, ... columns of the grid be given weight -1, as well as edges looping around the back of the cylinder. First we deal with $m > 2$. The Kasteleyn matrix for G becomes

$$K := \begin{pmatrix} A & -I & & & \\ I & B & I & & \\ & -I & A & -I & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

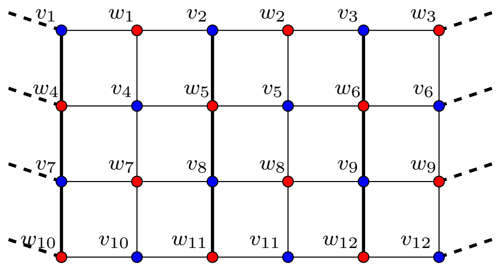

FIGURE 4. Choice of labeling in the case of the 4×6 grid graph. Bold edges are those given Kasteleyn weight -1.

with matrices A and B alternating along the diagonal and I and $-I$ alternating along the subdiagonal and superdiagonal, where A and B are $m \times m$ matrices given by

$$A = \begin{pmatrix} 1 & & & & -x^{-1} \\ 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ -x & & & & 1 \end{pmatrix}.$$

Let q_k be the determinant of the upper left $mk \times mk$ submatrix of K, so that $q_n = \det K$ By induction, it is not hard to compute that $q_k = \det P_k$ where $P_0 = I$, $P_1 = A$, and

$$P_k = \begin{cases} P_{k-1}A + P_{k-2}, & k \text{ odd} \\ P_{k-1}B + P_{k-2}, & k \text{ even} \end{cases}$$

We now claim there are real polynomials $p_k(t)$ for which $P_k = p_k(AB)$ if k is even and $P_k = p_k(AB)A$ if k is odd. Indeed, $p_0(t) = p_1(t) = 1$ works. Now suppose that $P_{2k} = p_{2k}(AB)$ and $P_{2k+1} = p_{2k+1}(AB)A$. Then the recurrence for $\{P_k\}$ tells us

$$P_{2k+2} = P_{2k+1}B + P_{2k} = p_{2k+1}(AB)AB + P_{2k}$$

so $p_{2k+2} = tp_{2k+1} + p_{2k}$ works. Similarly, we see that $p_{2k+3} = p_{2k+2} + p_{2k+1}$ works. So we get

$$p_k = \begin{cases} p_{k-1} + p_{k-2}, & k \text{ odd} \\ tp_{k-1} + p_{k-2}, & k \text{ even} \end{cases}$$

Now define the statements

- A_i: The polynomials p_{2i} and p_{2i+1} are real-rooted.
- B_i: The polynomials p_{2i} and tp_{2i+1} are compatible.
- C_i: The polynomials p_{2i} and tp_{2i-1} are compatible.

Now an induction nearly the same as that of Proposition 3.5 shows that these statements always hold; in particular p_k is always real rooted. We want to show that $\det p_k(AB)$ is real-rooted, so it suffices to show that $\det(AB - \lambda I)$ is real-rooted for each root λ of p_k. Observe that since all coefficients of p_k are positive, such a λ is necessarily negative. We have explicitly that

$$AB = \begin{pmatrix} 2 & 1 & & & & -x^{-1} \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ -x & & & & 1 & 2 \end{pmatrix}$$

Since $\lambda < 0$, we can write $2 - \lambda = r + r^{-1}$ for some $r > 0$, so we have

$$\det(AB - \lambda I) = \det \begin{pmatrix} r + r^{-1} & 1 & & & & -x^{-1} \\ 1 & r + r^{-1} & 1 & & & \\ & 1 & r + r^{-1} & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ -x & & & & 1 & r + r^{-1} \end{pmatrix}$$

By induction on m, it is now straightforward to show that

$$\det(AB - \lambda I) = (-1)^k (x + x^{-1}) + r^m + r^{-m}$$

which has real roots since $r^m + r^{-m} \geq 2$. In the case $m = 2$, the same proof works using the matrices

$$A = \begin{pmatrix} 1 & -x^{-1} \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ -x & 1 \end{pmatrix}.$$

□

Acknowledgments

This research was conducted at the 2020 University of Minnesota, Twin Cities REU, supported by NSF RTG grant DMS-1745638. We would like to thank Chris Fraser and Eric Stucky for their mentorship and feedback.

References

[AESW51] Michael Aissen, Albert Edrei, I.J. Schoenberg, and Anne Whitney, *On Generating Functions of Totally Positive Sequences*, Proc. N.A.S. **37** (1951), no. 5, 303–307. ↑

[CS07] Maria Chudnovsky and Paul Seymour, *The roots of the independence polynomial of a clawfree graph*, Journal of Combinatorial Theory, Series B **97** (2007), 350-357. ↑7

[Ken08] Robert Kenyon, *Exact Methods in Low-dimensional Statistical Physics and Quantum Computing: lectures on the dimer model*, Vol. 89, 2008. ↑8

[Sta98] Richard Stanley, *Graph colorings and related symmetric functions*, Discrete Math. **93** (1998), 267–286. ↑1