Equality in the Eisenbud–Goto Conjecture for Certain Toric Ideals

Preston Cranford, Alan Peng*, Vijay Srinivasan*

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Outline

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- $I_A := \langle \{\mathbf{x}^\mathbf{a} - \mathbf{x}^\mathbf{b} : \mathbf{a}, \mathbf{b} \in \mathbb{Z}^n_{\geq 0} \text{ and } \mathbf{a} - \mathbf{b} \in \ker A\} \rangle \subseteq k[\mathbf{x}] = k[x_1, \ldots, x_n]$ is prime
- If one of the rows of A is $\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{Z}^n$, then I_A is a *graded* (or *homogeneous*) ideal—we will assume this is the case from now on

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$$I_A = \langle x_1x_3 - x_2^2, x_2x_4 - x_3^2, x_1x_4 - x_2x_3 \rangle \subseteq k[x_1, x_2, x_3, x_4].$$

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$$0 \leftarrow M \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} \cdots \xleftarrow{d_i} F_i \xleftarrow{d_{i+1}} \cdots$$

is a *minimal graded free resolution* of M if each F_i is a finite direct sum of twists of S, each d_i preserves the degree of (nonzero) homogeneous elements, and $d_{i+1}(F_{i+1}) \subseteq \langle x_1, \ldots, x_n \rangle F_i$ for all $i \geq 0$.

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- All minimal graded free resolutions of M are isomorphic and have "finite length"

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The minimal graded free resolution of I_A is

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Castelnuovo–Mumford regularity

As before, let M be a finitely generated S-module, where $S = k[x_1, \ldots, x_n]$, and consider the minimal graded free resolution of M:

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The *Castelnuovo–Mumford regularity* of M is given by the quantity

$$\operatorname{reg} M := \max\{j : \beta_{i,i+j} \neq 0 \text{ for some } i\}.$$

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The nonzero Betti numbers are $\beta_{0,2} = 3$ and $\beta_{1,3} = 2$.
Thus, $\operatorname{reg} I_A = 2$.

The Eisenbud–Goto conjecture

Conjecture (Eisenbud–Goto 1984)

Suppose k is algebraically closed and $S = k[x_1, \ldots, x_n]$. For all graded prime ideals I that are contained in $\langle x_1, \ldots, x_n \rangle^2$, we have

$$\operatorname{reg} I \leq \deg I - \operatorname{codim} I + 1.$$

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However, the Eisenbud–Goto conjecture is still open for toric ideals.

Problem Statement

Question

When is equality achieved in the EG conjecture for toric ideals?

It makes sense to attack this in the cases where the inequality has already been proven.

- Curves ($d = 2$)
- Complete Intersections
- S/I is a simplicial affine semigroup ring
- Codimension 2 ($n - d = 2$) – this is our primary focus; the EG inequality becomes

$$\operatorname{reg} I \leq \deg I - 1$$

Outline

Peeva–Sturmfels 1998

- Proved Eisenbud–Goto for codimension 2.
- Considered the more general class of *lattice ideals*. Given a lattice $\mathcal{L} \subset \mathbb{Z}^n$, construct $I_{\mathcal{L}} \subset k[x_1, \ldots, x_n]$ via

$$I_{\mathcal{L}} := \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \mathbf{u} - \mathbf{v} \in \mathcal{L} \rangle$$

- When $\mathcal{L} = \ker A$ for some A, this is a toric ideal.

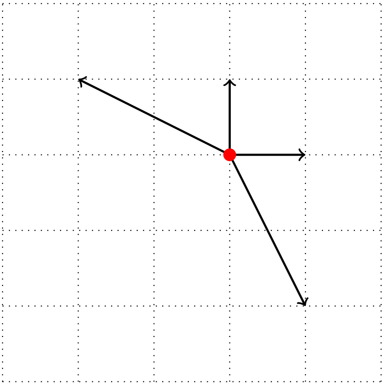
Gale diagrams

Let $\mathcal{L}$ have rank 2. Choose an $n \times 2$ matrix B whose columns span $\mathcal{L}$.

$$B := \begin{pmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}$$

The rows of B are vectors in $\mathbb{Z}^2$, which gives us the *Gale diagram* of $\mathcal{L}$.

Gale diagrams



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Peeva–Sturmfels 1998

$I_{\mathcal{L}}$ is a codimension-2 lattice ideal defining a projective variety...

Theorem (Peeva–Sturmfels 1998)

Then $\operatorname{reg} I_{\mathcal{L}} \leq \deg I_{\mathcal{L}}$. *If equality holds, any Gale diagram of* $\mathcal{L}$ *lies on two lines.*

Inequality is strict when $I_{\mathcal{L}}$ is toric! This proves EG in the codimension-2 case: $\operatorname{reg} I_{\mathcal{L}} \leq \deg I_{\mathcal{L}} - 1$.

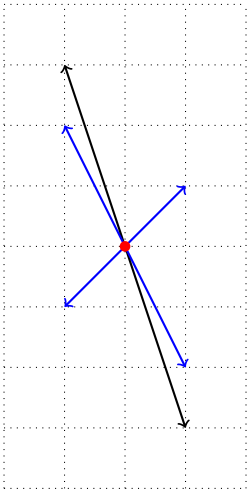
Outline

Our results

Theorem

Assume $I_{\mathcal{L}}$ is toric, codimension 2, not Cohen–Macaulay. If $I_{\mathcal{L}}$ achieves equality in Eisenbud–Goto, then there are two lines containing all but two Gale vectors. The remaining two Gale vectors are "nearest points" to one of these lines.

Example



$$B := \begin{pmatrix} -1 & 3 \\ -1 & 2 \\ 1 & -3 \\ 1 & -2 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$\operatorname{reg} I_{\mathcal{L}} = 7, \qquad \deg I_{\mathcal{L}} = 8$$

Method of proof

- Reduction to $n = 4$ (i.e., curves in $\mathbb{P}^3$).
- (P–S) Given a lattice ideal $I_{\mathcal{L}} \subset k[x_1, \ldots, x_n]$ that is not Cohen–Macaulay, there is a lattice ideal $I_{\mathcal{L}'} \subset k[y_1, \ldots, y_4]$ for which

$$\operatorname{reg} I_{\mathcal{L}} \leq \operatorname{reg} I_{\mathcal{L}'} \leq \deg I_{\mathcal{L}'} \leq \deg I_{\mathcal{L}}$$

- If $\operatorname{reg} I_{\mathcal{L}} = \deg I_{\mathcal{L}} - 1$ (i.e., equality holds in the EG conjecture), this chain of inequalities is pretty tight

Reduction to $n = 4$

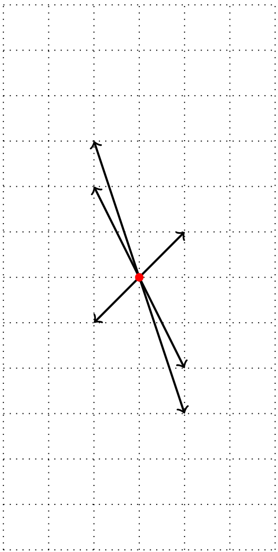
- Find a suitable partition of $\{x_1, \dots, x_n\}$ into four subsets
- Add the corresponding Gale vectors to get a new lattice

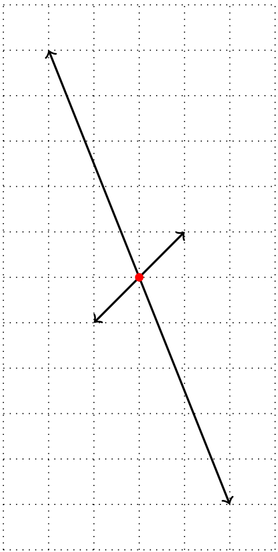
$$B := \begin{pmatrix} -1 & 3 \\ -1 & 2 \\ 1 & -3 \\ 1 & -2 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} \longrightarrow B' := \begin{pmatrix} -2 & 5 \\ 2 & -5 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$x_1, x_2 \mapsto y_1, \quad x_3, x_4 \mapsto y_2, \quad x_5 \mapsto y_3, \quad x_6 \mapsto y_4$$

Get a new lattice ideal $I_{\mathcal{L}'} \subset k[y_1, y_2, y_3, y_4]$ (after a saturation).

Reduction to $n = 4$ (cont).





Example result

Suppose that $I_{\mathcal{L}}$ is not Cohen–Macaulay.

Proposition

If $\deg I_{\mathcal{L}} = \deg I_{\mathcal{L}'}$, then $\operatorname{reg} I_{\mathcal{L}} = \operatorname{reg} I_{\mathcal{L}'}$.

Corollary

If $\operatorname{reg} I_{\mathcal{L}} = \deg I_{\mathcal{L}} - 1$, then for any reduction, we have $\operatorname{reg} I_{\mathcal{L}'} = \operatorname{reg} I_{\mathcal{L}}$.

Example result

Proposition

Suppose that $I_{\mathcal{L}}$ is not Cohen–Macaulay, and that the Gale diagram of $\mathcal{L}$ contains at least 5 nonzero vectors. If $\operatorname{reg} I_{\mathcal{L}} = \deg I_{\mathcal{L}} - 1$, then there exists a choice of reduction for which $\operatorname{reg} I_{\mathcal{L}'} = \deg I_{\mathcal{L}'}$.

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