REU Problem 6: Frieze Patterns from Dissections

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06/22/20

Finite Frieze Pattern of Positive Integers

	0		0		0		0		0		0		0		0		0		
		1		1		1		1		1		1		1		1		1	
	1		3		1		3		2		2		1		5		1		
		2		2		2		5		3		1		4		4		2	
···	7		1		3		3		7		1		3		3		7		···
		3		1		4		4		2		2		2		5		3	
	2		2		1		5		1		3		1		3		2		
		1		1		1		1		1		1		1		1		1	
	0		0		0		0		0		0		0		0		0		

Finite Frieze Pattern of Laurent Polynomials

0		0		0		0		0		0
	1		1		1		1		1	
x_1		$\frac{x_2+1}{x_1}$		$\frac{x_1+1}{x_2}$		x_2		$\frac{x_1+x_2+1}{x_1x_2}$		x_1
	x_2		$\frac{x_1+x_2+1}{x_1x_2}$		x_1		$\frac{x_2+1}{x_1}$		$\frac{x_1+1}{x_2}$	
1		1		1		1		1		1
	0		0		0		0		0	

Frieze Patterns

Definition ((Finite) Frieze Pattern)

- First row (and last row) consists of all zeroes
- Second row (and second to last row) consists of all ones.
- All diamonds $\begin{matrix} & a & \\ b & & c \\ & d & \end{matrix}$ satisfy the diamond condition $bc - ad = 1$.

We introduce the following indexing.

$$\begin{matrix}
0 & & 0 & & 0 & & 0 & & 0 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 \\
m_{-1,1} & & m_{0,2} & & m_{1,3} & & m_{2,4} & & m_{3,5} & \\
 & m_{-1,2} & & m_{0,3} & & m_{1,4} & & m_{2,5} & & m_{3,7} \\
m_{-2,2} & & m_{-1,3} & & m_{0,4} & & m_{1,5} & & m_{2,6} & \\
 & m_{-2,3} & & m_{-1,4} & & m_{0,5} & & m_{1,6} & & m_{2,7}
\end{matrix}$$

Basic Terms and Exercise 1

- The number of rows of a finite frieze pattern is the *width*.
- The first nontrivial row is called the *quiddity row*.
- If the quiddity row is the first row, then $m_{i,j}$ is in row $j - i - 1$.

Basic Terms and Exercise 1

- The number of rows of a finite frieze pattern is the *width*.
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REU Exercise 6.1

a. Show that for $j \geq i+2$, $m_{i,j} = m_{i,i+2}m_{i+1,j} - m_{i+2,j}$. Let $m_{i,i+1} = 1$ and $m_{i,i} = 0$ for all i.

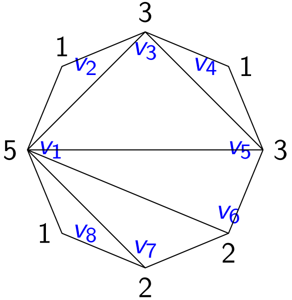
b. From part a, deduce

$$m_{i,j} = \det \begin{pmatrix} m_{i,i+2} & 1 & & & & & \\ 1 & m_{i+1,i+3} & 1 & & & & \\ & 1 & m_{i+2,i+4} & 1 & & & \\ & & & \ddots & & & \\ & & & 1 & m_{j-3,j-1} & 1 \\ & & & & 1 & m_{j-2,j} \end{pmatrix}$$

Frieze Patterns - Conway Coxeter bijection

Theorem (Conway-Coxeter)

Frieze patterns of positive integers with width n are in bijection with triangulations of an $(n+3)$-gon.




```
0     0     0     0     0     0     0     0     0
   1     1     1     1     1     1     1     1     1
1     3     1     3     2     2     1     5     1
   2     2     2     5     3     1     4     4     2
7     1     3     3     7     1     3     3     7
   3     1     4     4     2     2     2     5     3
1     2     1     5     1     3     1     3     1
   1     1     1     1     1     1     1     1     1
0     0     0     0     0     0     0     0     0
```

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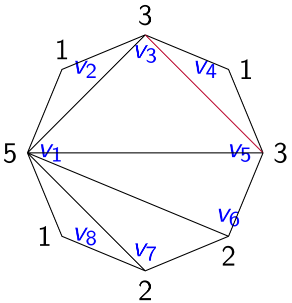

0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1
1 3 1 3 2 2 1 5 1
2 2 2 5 3 1 4 4 2
7 1 3 3 7 1 3 3 7
3 1 4 4 2 2 2 5 3
1 2 1 5 1 3 1 3 1
1 1 1 1 1 1 1 1 1
0 0 0 0 0 0 0 0 0

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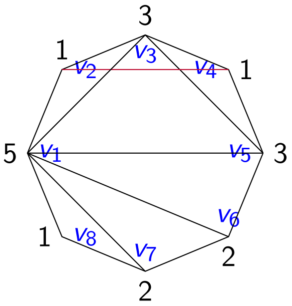

0		0		0		0		0		0		0		0		0	
	1		1		1		1		1		1		1		1		1
1		3		1		3		2		2		1		5		1	
	2		2		2		5		3		1		4		4		2
7		1		3		3		7		1		3		3		7	
	3		1		4		4		2		2		2		5		3
1		2		1		5		1		3		1		3		1	
	1		1		1		1		1		1		1		1		1
0		0		0		0		0		0		0		0		0	

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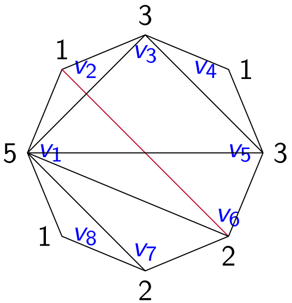

0 0 0 0 0 0 0 0 0
1 1 1 1 1 1 1 1 1
1 3 1 3 2 2 1 5 1
2 2 2 5 3 1 4 4 2
7 1 3 3 7 1 3 3 7
3 1 4 4 2 2 2 5 3
1 2 1 5 1 3 1 3 1
1 1 1 1 1 1 1 1 1
0 0 0 0 0 0 0 0 0

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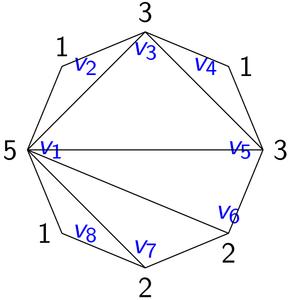

0		0		0		0		0		0		0		0		0	
	1		1		1		1		1		1		1		1		1
1		3		1		3		2		2		1		5		1	
	2		2		2		5		3		1		4		4		2
7		1		3		3		7		1		3		3		7	
	3		1		4		4		2		2		2		5		3
1		2		1		5		1		3		1		3		1	
	1		1		1		1		1		1		1		1		1
0		0		0		0		0		0		0		0		0	

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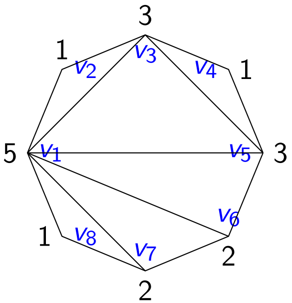
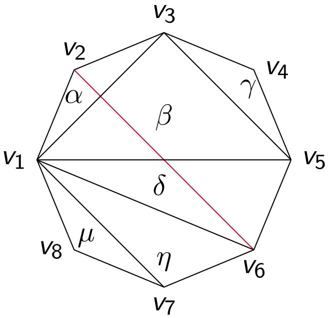


0		0		0		0		0		0		0		0		0	
	1		1		1		1		1		1		1		1		1
1		3		1		3		2		2		1		5		1	
	2		2		2		5		3		1		4		4		2
7		1		3		3		7		1		3		3		7	
	3		1		4		4		2		2		2		5		3
1		2		1		5		1		3		1		3		1	
	1		1		1		1		1		1		1		1		1
0		0		0		0		0		0		0		0		0	

- The entries of a frieze pattern can be interpreted as counting certain combinatorial objects.
- We can also think about them as giving the length of arcs in the triangulated surface.

BCI Tuples



There are three BCI tuples going clockwise from v_2 to v_6.

- α, γ, β
- α, γ, δ
- β, γ, δ

Cluster Algebras to the rescue!

- Coxeter defined frieze patterns in the 70's and soon after proved the bijection with Conway.
- Besides work by BCI, frieze patterns hibernated until the development of cluster algebras.
- Cluster algebras of type A and frieze patterns are both related to triangulated polygon, hence can be connected to each other.

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- Given a triangulated polygon, each chord corresponds to a cluster variable. Filling in the variable from chord (i, j) in spot $m_{i,j}$ gives a frieze pattern.
- Then, the $\mathbf{Z}_{>0}$ frieze pattern arises from specializing all intial cluster variables to 1.

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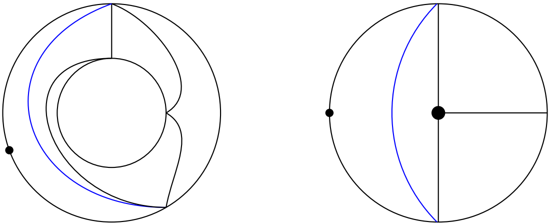
- Besides work by BCI, frieze patterns hibernated until the development of cluster algebras.
- Cluster algebras of type A and frieze patterns are both related to triangulated polygon, hence can be connected to each other.
- Given a triangulated polygon, each chord corresponds to a cluster variable. Filling in the variable from chord (i, j) in spot $m_{i,j}$ gives a frieze pattern.
- Then, the $\mathbf{Z}_{>0}$ frieze pattern arises from specializing all intial cluster variables to 1.
- We can also interpret these finite $\mathbf{Z}_{>0}$ frieze patterns as counting subrepresentations of quiver representations!

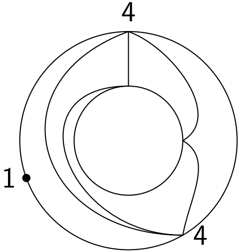
Annuli and Once-Punctured Discs

- Let S_n be a once-punctured disc with n marked points on the boundary.
- Let $A_{n,m}$ be an annulus with n marked points on the outer boundary and m marked points on the inner boundary.
- A *peripheral arc* goes between marked points on the same boundary.
- A *bridging arc* goes between marked points on different boundaries.



An example of triangulated $A_{3,2}$ and S_4. The blue arcs are peripheral.

Infinite Frieze Pattern example



0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1

1 4 4 1 4 4 1 4 4 1

3 15 3 3 15 3 3 15 3 3

8 11 11 8 11 11 8 11 11 8

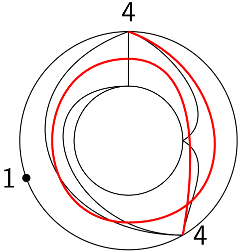
29 8 29 29 8 29 29 8 29

105 21 21 105 21 21 105 21 21 105

76 55 76 76 55 76 76 55 76

⋮

Infinite Frieze Pattern example



0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1

1 4 4 1 4 4 1 4 4 1

3 15 3 3 15 3 3 15 3 3

8 11 11 8 11 11 8 11 11 8

29 8 29 29 8 29 29 8 29

105 21 21 105 21 21 105 21 21 105

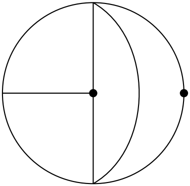
76 55 76 76 55 76 76 55 76

⋮

Infinite Frieze Pattern - Big Theorem

Theorem (Baur-Parsons-Tschabold)

Periodic infinite frieze patterns of positive integers are in bijection with triangulations of annuli and once-punctured discs.



```
0     0     0     0     0     0     0     0
   1     1     1     1     1     1     1
1     3     2     3     1     3     2     3
   2     5     5     2     2     5     5
3     3    12     3     3     3    12     3
   4     7     7     4     4     7     7
                  ⋮
```

Proof of Infinite Frieze Bijection

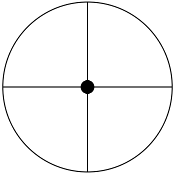
Claim

Every infinite periodic frieze pattern arises from a triangulated annulus or once-punctured disc.

Observation

The quiddity sequence $\cdots 2, 2, 2 \cdots$ comes from a punctured disc.

$$
\begin{array}{ccccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 \\
& 3 & & 3 & & 3 & & 3 & & 3 & & 3 & & 3 & & 3 & \\
4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 & & 4 \\
& & & & & & & \vdots & & & & & & & & &
\end{array}
$$



Proof of Infinite Frieze Bijection

Claim

Every infinite periodic frieze pattern arises from a triangulated annulus or once-punctured disc.

Theorem

If $(a_1, \ldots, a_n)$ *is the quiddity sequence of an infinite frieze pattern, so is* $(a_1, \ldots, a_i + 1, \ldots, a_n)$ *for any* i.

Proposition

If $(a_1, \ldots, a_n)$ is realizable, so is $(a_1, \ldots, a_i + 1, \ldots, a_n)$ for any i.

Proof of Infinite Frieze Bijection

Claim

Every infinite periodic frieze pattern arises from a triangulated annulus or once-punctured disc.

Cutting

We can *cut* at a 1 in a quiddity sequence by the following

$$(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_n) \rightarrow (a_1, \ldots, a_{i-1} - 1, a_{i+1} - 1, \ldots a_n)$$

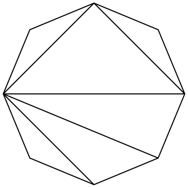
Note that unless all $a_j = 1$, we cannot have adjacent values of 1 in a quiddity sequence.

Gluing

Gluing is the reverse operation of cutting. We can glue as many times as we want to any quiddity sequence.

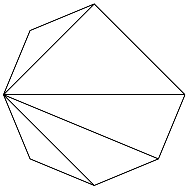
Example of Cutting

$$
\begin{array}{ccccccccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 1 & & 3 & & 1 & & 3 & & 2 & & 2 & & 1 & & 5 & & 1 & & & \\
 & & 2 & & 2 & & 2 & & 5 & & 3 & & 1 & & 4 & & 4 & & 2 & & \\
\cdots & 7 & & 1 & & 3 & & 3 & & 7 & & 1 & & 3 & & 3 & & 7 & & & \cdots \\
 & & 3 & & 1 & & 4 & & 4 & & 2 & & 2 & & 2 & & 5 & & 3 & & \\
 & 2 & & 2 & & 1 & & 5 & & 1 & & 3 & & 1 & & 3 & & 2 & & & \\
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & &
\end{array}
$$



Example of Cutting

	0		0		0		0		0		0		0		0		0		
		1		1		1		1		1		1		1		1		1	
	1		2		2		2		2		1		5		1		2		
		1		3		3		3		1		4		4		1		3	
...	3		1		4		4		1		3		3		3		1		...
		2		1		2		1		2		2		2		2		1	
	1		1		1		1		1		1		1		1		1		
		0		0		0		0		0		0		0		0		0	



Proof of Infinite Frieze Bijection

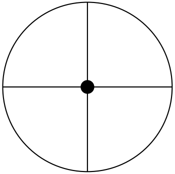
Claim

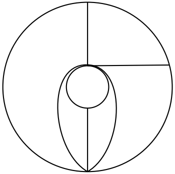
Every infinite periodic frieze pattern arises from a triangulated annulus or once-punctured disc.

Proof

Let $(a_1, \ldots, a_n)$ be the quiddity sequence of an infinite frieze pattern.

- If all $a_i = 2$, this is from a triangulated once-punctured disc.
- If all $a_i \geq 2$ and for some j, $a_j > 2$, this is from a triangulated annulus.
- Otherwise, there exists j such that $a_j = 1$. Cut at a_j and check again.





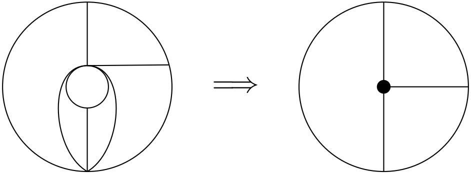
Proof of Infinite Frieze Bijection

Claim

Every triangulated once-punctured disc or annulus gives rise to an infinite frieze pattern.

Proof

- Every triangulated once-punctured disc is either a wheel or has a cuttable triangle.
- Given a triangulated annulus with quiddity sequence $(a_1, \ldots, a_n)$, we can construct a triangulated once-punctured disc with quiddity sequence $(b_1, \ldots, b_n)$ such that $b_i \leq a_i$ for all i.



Growth Coefficients

Theorem (Growth Coefficient Theorem, Baur-Fellner-Parsons-Tschabold)

Given an infinite frieze pattern with period n, $m_{i,i+kn+1} - m_{i+1,i+kn}$ is constant for each $k \geq 1$.

Informally, think difference between row kn and row $kn - 2$ is constant. This gives a sequence $\{s_k\}$ of *growth coefficients* from each infinite frieze pattern.

Theorem (BFPT)

The growth coefficients of a frieze pattern are related by

$$s_{k+1} = s_1 s_k - s_{k-1}$$

where $s_0 = 2$.

Growth Coefficient Examples

0		0		0		0		0		0		0	
	1		1		1		1		1		1		1
1		4		4		1		4		4		1	
	3		15		3		3		15		3		3
8		11		11		8		11		11		8	
	29		8		29		29		8		29		29
105		21		21		105		21		21		105	
	76		55		76		76		55		76		76
55		199		199		55		199		199		55	

$$\vdots$$

- $s_1 = 7, s_2 = 47 = 7 \cdot 7 - 2, s_3 = 7 \cdot 47 - 7$

Progression Formulas

Theorem (Gunawan-Musiker-Vogel)

Let $\mathcal{F}$ be an infinite frieze pattern with n-periodic rows. Then, for all $1 \leq \ell \leq k$

$$m_{i,j+kn} = s_\ell m_{i,j+(k-\ell)n} + m_{j+(k-\ell)n,i+\ell n}$$

where, if $a > b$, $m_{a,b} = -m_{b,a}$

Take-Away

Once we compute the first n non-trivial rows of an n-periodic frieze pattern and the first growth coefficient, we know a lot! In particular, if all of these are positive, the whole frieze pattern is positive.

Growth Coefficient Examples

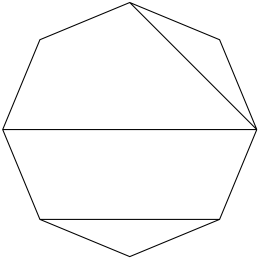
	-1		-1		-1		-1		-1		-1		
0		0		0		0		0		0		0	
	1		1		1		1		1		1		1
1		4		4		1		4		4		1	
	3		15		3		3		15		3		3
8		11		11		8		11		11		8	
	29		8		29		29		8		29		29
105		21		21		105		21		21		105	
	76		55		76		76		55		76		76
55		199		199		55		199		199		55	

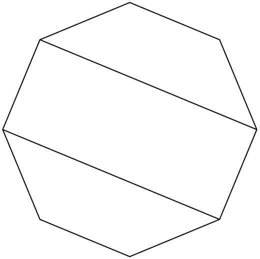
$$\vdots$$

- $s_1 = 7, s_2 = 47$
- $m_{0,7} = s_1 m_{0,4} + m_{4,3} = 7 \cdot 11 + -1.$
- Also, $m_{0,7} = s_2 m_{0,1} + m_{1,6} = 47 \cdot 1 + 29$

Polygon Dissections

We can generalize these constructions to *polygon dissections*.





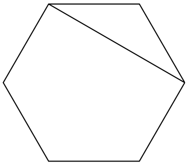
A *p-angulation* is a polygon dissection that breaks the surface into p-gons.

Frieze Patterns from Polygon Dissections

If a vertex is adjacent to polygons of size $p_1, \ldots, p_n$, we associate to it $\sum_{i=1}^{n} \lambda_{p_i}$ where $\lambda_{p_i} = 2\cos(\pi/p_i)$.

p	λ_p
3	1
4	$\sqrt{2}$
5	$\frac{1+\sqrt{5}}{2}$
6	$\sqrt{3}$

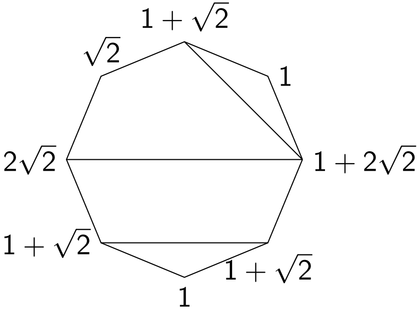
Note that λ_p is the ratio of the length of the shortest diagonal of a regular p-gon and the length of a side.



Frieze Patterns from Polygon Dissections

Theorem (Holm-Jørgensen)

Each polygon dissection of an n-gon produces a frieze pattern of width $n-3$ with entries in the ring of algebraic integers of the field $\mathbf{Q}(\lambda_{p_1}, \ldots, \lambda_{p_s})$



Frieze Patterns from Polygon Dissections

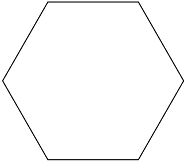
0		0		0		0		0		0		0		0		0
	1		1		1		1		1		1		1		1	
$1+2\sqrt{2}$		1		$1+\sqrt{2}$		$\sqrt{2}$		$2\sqrt{2}$		$1+\sqrt{2}$		1		$1+\sqrt{2}$		$1+2\sqrt{2}$
	$2\sqrt{2}$		$\sqrt{2}$		$1+\sqrt{2}$		3		$3+2\sqrt{2}$		$\sqrt{2}$		$\sqrt{2}$		$4+3\sqrt{2}$	
$3+2\sqrt{2}$		3		1		$3+\sqrt{2}$		$3+2\sqrt{2}$		3		1		$3+\sqrt{2}$		$3+2\sqrt{2}$
	$3+2\sqrt{2}$		$\sqrt{2}$		$\sqrt{2}$		$4+3\sqrt{2}$		$2\sqrt{2}$		$\sqrt{2}$		$1+\sqrt{2}$		3	
$2\sqrt{2}$		$1+\sqrt{2}$		1		$1+\sqrt{2}$		$1+2\sqrt{2}$		1		$1+\sqrt{2}$		$\sqrt{2}$		$2\sqrt{2}$
	1		1		1		1		1		1		1		1	
0		0		0		0		0		0		0		0		0

This has the width we expect and still has glide symmetry.

Proof of Injection

- First, establish a frieze pattern on an empty-dissected n-gon.

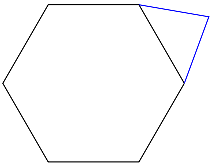
0		0		0		0		0		0		0	
	1		1		1		1		1		1		1
$\sqrt{3}$		$\sqrt{3}$		$\sqrt{3}$		$\sqrt{3}$		$\sqrt{3}$		$\sqrt{3}$		$\sqrt{3}$	
	2		2		2		2		2		2		
$\sqrt{3}$		$\sqrt{3}$		$\sqrt{3}$		$\sqrt{3}$		$\sqrt{3}$		$\sqrt{3}$		$\sqrt{3}$	
	1		1		1		1		1		1		1
0		0		0		0		0		0		0	



Proof of Injection

- First, establish a frieze pattern on an empty-dissected n-gon.
- Then, show that these elementary frieze patterns can be glued together.

$$
\begin{array}{cccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
\sqrt{3} & & \sqrt{3} & & 1+\sqrt{3} & & 1 & & 1+\sqrt{3} & & \sqrt{3} & & \sqrt{3} & \\
 & 2 & & 2+\sqrt{3} & & \sqrt{3} & & \sqrt{3} & & 2+\sqrt{3} & & 2 & & 2 \\
\sqrt{3} & & 2+\sqrt{3} & & 2 & & 2 & & 2 & & 2+\sqrt{3} & & \sqrt{3} & \\
 & 1+\sqrt{3} & & \sqrt{3} & & \sqrt{3} & & \sqrt{3} & & \sqrt{3} & & 1+\sqrt{3} & & 1 \\
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$



1-periodic Frieze Patterns

$$
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & \\
x & & x & & x & & x \\
& x^2-1 & & x^2-1 & & x^2-1 & \\
x^3-2x & & x^3-2x & & x^3-2x & & x^3-2x \\
& x^4-3x+1 & & x^4-3x+1 & & x^4-3x+1 & \\
& & & \vdots & & &
\end{array}
$$

The polynomials in this frieze pattern are *normalized Chebyshev polynomials of the second kind*, $U_k(x)$, which satisfy $U_k(x) = xU_{k-1}(x) - U_{k-2}(x)$. Let $U_1(x) = x$ and $U_0(x) = 1$.

1-periodic Frieze Patterns

0		0		0		0
	1		1		1	
x		x		x		x
	$x^2 - 1$		$x^2 - 1$		$x^2 - 1$	
$x^3 - 2x$		$x^3 - 2x$		$x^3 - 2x$		$x^3 - 2x$
	$x^4 - 3x + 1$		$x^4 - 3x + 1$		$x^4 - 3x + 1$	
			$\vdots$			

The polynomials in this frieze pattern are *normalized Chebyshev polynomials of the second kind*, $U_k(x)$, which satisfy $U_k(x) = xU_{k-1}(x) - U_{k-2}(x)$. Let $U_1(x) = x$ and $U_0(x) = 1$.

Proposition

The length of a k-diagonal in a regular p-gon with sides length s is $U_k(\lambda_p) \cdot s$.

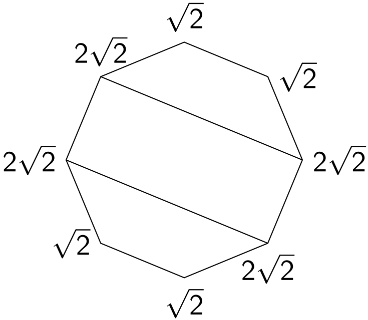
A k-diagonal bypasses k vertices. Sanity check: $U_1(\lambda_p) \cdot s = \lambda_p \cdot s$

Frieze Patterns from p-angulations

Theorem (Holm-Jørgensen)

There is bijection between p-angulations of an n-gon and frieze patterns of width $n-3$ whose quiddity row consists of multiples of λ_p.

The frieze patterns in this theorem are said to be *type* Λ_p.



Frieze Patterns from p-angulations

$$
\begin{array}{ccccccccccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
2\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & & 2\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & & 2\sqrt{2} \\
& 3 & & 1 & & 3 & & 7 & & 3 & & 1 & & 3 & & 7 & \\
5\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & 5\sqrt{2} & & 5\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & 5\sqrt{2} & & 5\sqrt{2} \\
& 3 & & 1 & & 3 & & 7 & & 3 & & 1 & & 3 & & 7 & \\
2\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & & 2\sqrt{2} & & \sqrt{2} & & \sqrt{2} & & 2\sqrt{2} & & 2\sqrt{2} \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

Frieze Patterns from p-angulations

The proof that each finite frieze pattern of type Λ_p uses the theory of Hecke groups. This is an optional presentation for one of you to give!

Frieze Patterns from p-angulations

The proof that each finite frieze pattern of type Λ_p uses the theory of Hecke groups. This is an optional presentation for one of you to give!

Let a frieze pattern whose quiddity row consists of positive linear combinations of $\lambda_{p_1}, \ldots, \lambda_{p_s}$ be type $\Lambda_{p_1,\ldots,p_s}$.

REU Problem 6.1

Does every finite frieze pattern of type $\Lambda_{p_1,\ldots,p_s}$ come from a dissected polygon? If not, describe the subset of such frieze patterns which do arise from dissected polygons.

Why do we care?

- Triangulated surfaces in general encode cluster algebras. In fact, most "nice" cluster algebras come from surfaces.

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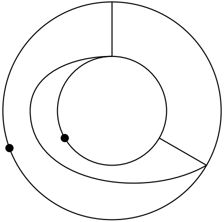
- Triangulated surfaces in general encode cluster algebras. In fact, most "nice" cluster algebras come from surfaces.
- There is interest to find cluster algebra-like structure for p-angulations (and general dissections?).
- In some very special cases, polygon dissections correspond to "generalized cluster algebras."

Dissected Annuli and Once-Punctured discs

	Finite Frieze Patterns	Infinite Frieze Patterns
p-angulations	Bijection by HJ Finite Type Λ_p Polygon	Conjecture: Bijection Infinite Type Λ_p $A_{n,m}$ and S_n **REU Problem 6.2**
Polygon Dissections	Injection by HJ Surjection?? **REU Problem 6.1**	Not Injective Not Surjective? **REU Problem 6.3**

REU Exercise 2

Which annuli $A_{n,m}$ can be p-angulated? Verify that if we set $m = 0$, we get the correct result for S_n.



0		0		0		0		0		0		0		0
	1		1		1		1		1		1		1	
$\sqrt{2}$		$2\sqrt{2}$		$3\sqrt{2}$		$\sqrt{2}$		$2\sqrt{2}$		$3\sqrt{2}$		$\sqrt{2}$		$2\sqrt{2}$
	3		11		5		3		11		5		3	
$7\sqrt{2}$		$8\sqrt{2}$		$9\sqrt{2}$		$7\sqrt{2}$		$8\sqrt{2}$		$9\sqrt{2}$		$7\sqrt{2}$		$8\sqrt{2}$
	37		13		25		37		13		25		37	

$\vdots$

Conjectured Bijection

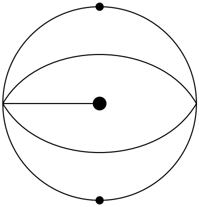
REU Problem 6.2

Conjecture: There is a bijection between p-angulations of S_n and $A_{n,m}$ and periodic, infinite frieze patterns of type Λ_p.

The following steps should help prove the conjecture.

- Establish a base set of infinite frieze patterns of type Λ_p (like $\cdots 2, 2, 2 \cdots$)
- Show that you can add multiples of λ_p to a realizable quiddity sequence and remain realizable.
- Verify cutting and gluing for infinite frieze patterns from p-angulations.

Dissections of $A_{n,m}$ and S_n



0		0		0		0		0
	1		1		1		1	
1		$2+2\sqrt{2}$		1		$2+\sqrt{2}$		1
	$1+2\sqrt{2}$		$1+2\sqrt{2}$		$1+\sqrt{2}$		$1+\sqrt{2}$	
$4+3\sqrt{2}$		$2\sqrt{2}$		$4+3\sqrt{2}$		$\sqrt{2}$		$4+3\sqrt{2}$
	$3+2\sqrt{2}$		$3+2\sqrt{2}$		$3+\sqrt{2}$		$3+\sqrt{2}$	

$\vdots$

Conjecture (Part of REU Problem 6.3?)

Every frieze pattern from a dissection of S_n has $s_1 = 2$. (Known for triangulations by Tschabold)

Frieze Patterns without corresponding dissection

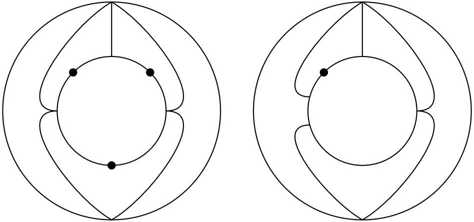
It is surprisingly easy to find infinite frieze patterns of type $\Lambda_{p_1,\dots,p_s}$ which do not *seem* to come from dissections.

$$\begin{array}{ccccccccccccc}
0 & & 0 & & 0 & & 0 \\
& 1 & & 1 & & 1 & \\
2 & & 1+\sqrt{2} & & 2 & & 1+\sqrt{2} \\
& 1+2\sqrt{2} & & 1+2\sqrt{2} & & 1+2\sqrt{2} & \\
4+2\sqrt{2} & & 4\sqrt{2} & & 4+2\sqrt{2} & & 4\sqrt{2} \\
& 7+2\sqrt{2} & & 7+2\sqrt{2} & & 7+2\sqrt{2} & \\
14 & & 7+7\sqrt{2} & & 14 & & 7+7\sqrt{2} \\
& & & \vdots & & &
\end{array}$$

Frieze Patterns with multiple Dissections

0		0		0		0
	1		1		1	
$2+2\sqrt{2}$		$2+\sqrt{2}$		$2+2\sqrt{2}$		$2+\sqrt{2}$
	$7+6\sqrt{2}$		$7+6\sqrt{2}$		$7+6\sqrt{2}$	
$24+18\sqrt{2}$		$36+24\sqrt{2}$		$24+18\sqrt{2}$		$36+24\sqrt{2}$
	$113+78\sqrt{2}$		$113+78\sqrt{2}$		$113+78\sqrt{2}$	
$502+358\sqrt{2}$		$358+251\sqrt{2}$		$502+358\sqrt{2}$		$358+251\sqrt{2}$

$$\vdots$$



Patterns in Failure?

Consider a 2-periodic frieze pattern with quiddity sequence (a, b).

$a \backslash b$	3	$2+\sqrt{2}$	$1+2\sqrt{2}$	$3\sqrt{2}$
3	$\checkmark$	$\checkmark$	?	$\times$
$2+\sqrt{2}$	$\checkmark$	$\star$	$\checkmark$	?
$1+2\sqrt{2}$	?	$\checkmark$	$\star$	$\checkmark$
$3\sqrt{2}$	$\times$	?	$\checkmark$	$\checkmark$

- $\checkmark$ means there is a unique dissection corresponding to this frieze pattern.
- ? means this is a valid frieze pattern but no clear dissection.
- $\star$ means there are multiple dissections corresponding to this frieze pattern.
- $\times$ means this frieze pattern is not valid (has negatives)

REU Exercise 6.3

Find the dissections from this table. Compute the table with $(3, 4)$.

REU Problem 6.3

Investigate infinite frieze patterns from dissections of S_n and $A_{n,m}$. Can you characterize which are drawable? When do we have ambiguity?

	Finite Frieze Patterns	Infinite Frieze Patterns
p-angulations	Bijection by HJ Finite Type Λ_p Polygon	Conjecture: Bijection Infinite Type Λ_p $A_{n,m}$ and S_n **REU Problem 6.2**
Polygon Dissections	Injection by HJ Surjection?? **REU Problem 6.1**	Not Injective Not Surjective? **REU Problem 6.3**

Thank You

Thank you for listening!
Check out the Google Drive for papers and these slides, and please contact me or Libby (or the rest of the Cluster Algebra Posse) with any questions!