VIRTUAL RESOLUTIONS OF MONOMIAL IDEALS

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Abstract. We explore the relationship between the multi-graded regularity of resolutions and resolution regularity of virtual resolutions of square-free monomial ideals in $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \mathbb{P}^2$.

1. Introduction

Virtual resolutions are a generalization of free graded resolutions of products of projective spaces introduced in [BES17]. The minimal graded free resolution of an ideal of product of projective spaces contains unimportant algebraic structure coming from the irrelevant ideal, so we instead consider it's free graded virtual resolution which is shorter, thinner, and contains all the geometrically relevant information. The geometric notion of removing the irrelevant part is known as saturation.

We recall two notions of regularity. The first is multi-graded regularity introduced in [MS04] which generalizes Castelnuovo-Mumford regularity. The second is resolution regularity introduced in [Sid04] also generalizes Castelnuovo-Mumford regularity but is not the same as multi-graded regularity. In [Mac03] the authors show that the multigraded regularity of a minimal graded free resolution is bounded below by it's resolution regularity. If we were to instead consider the virtual resolution of the same module, it's muttigraded regularity would strictly non-increase, thus it is natural to ask the following question.

Question 1.1. What's the relationship between multigraded regularity of a B-saturated module and resolution-regularity of it's virtual resolutions?

Let $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ and $\mathbb{P}^{\mathbf{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}$. Define S as

$$S := R[x_{1,0}, x_{1,1}, \dots, x_{1,n_1} \dots x_{i,0} \dots x_{i,n_i}, \dots, x_{d,0}, \dots, x_{d,n_d}]$$

so that S is the coordinate ring of $\mathbb{P}^{\mathbf{n}}$. For a toric varity I, let $\mathbb{N}\mathcal{C}$ denote the semigroup such that $S \setminus I = k[\mathbb{N}\mathcal{C}]$. Let $(\mathbf{c}_1, \dots, \mathbf{c}_l)$ denote the unique Hilbert basis of $\mathbb{N}\mathcal{C}$. We discuss several gradings on $\mathbb{P}^{\mathbf{n}}$.

Definition 1.2. The *standard grading* on S is the $\mathbb{Z}$-grading so that $\deg x_{i,j} = 1$. The *toric grading* on S is the $\mathbb{Z}^d$-grading so that $\deg x_{i,j}$ is the i-th unit vector in $\mathbb{Z}^d$. The *fine grading* on S is the $\mathbb{Z}^{(\sum(n_i+1))}$-grading where $\deg x_{i,j}$ is the $\sum_{k<i}(n_k+1)+j+1$-unit vector in $\mathbb{Z}^{(\sum(n_i+1))}$

We now discuss saturation.

Definition 1.3. The *irrelevant ideal* B of $\mathbb{P}^{\mathbf{n}}$ is the following.

$$B := \bigcap_{i=1}^{d} \langle x_{i,0}, \dots, x_{i,i_m} \rangle$$

Notice that the ideals $\langle x_{i,0}, \dots, x_{i,i_m} \rangle$ correspond nothing in their respective projective space.

Definition 1.4. The *saturation* $I : J^\infty$ of I by another S-ideal J is

$$\{r \in S | rJ^k \subset I \text{ for } k \gg 0\}$$

We will define the two notions of regularity.

Definition 1.5. For $\mathbf{m} \in \mathbb{N}^d$, a standard graded S-module M is said to be $\mathbf{m}$-*regular* if

(1) $H^i_B(M)_{\mathbf{p}} = 0$ for all $p \in \bigcup(\mathbf{m} - \lambda_1\mathbf{c}_1 - \cdots - \lambda_l\mathbf{c}_l + \mathbb{N}\mathcal{C})$ where the union indexes over all $(\lambda_1, \dots, \lambda_l)$ where $\lambda_i \in \mathbb{N}$ such that $\lambda_1 + \cdots + \lambda_l = i - 1$
(2) $H^0_B(M))\mathbf{p} = 0$ for all $p \in \bigcup_{1 \leq j < l}(\mathbf{m} + \mathbf{c}_j + \mathbb{N}\mathcal{C})$

The *multi-graded regularity* $\operatorname{reg}(M)$ is $\{\mathbf{m} | M \text{ is } \mathbf{p} - \text{regular}\}$

Definition 1.6. The *resolution-regularity* $\operatorname{rreg} M$ of M is $(r_1, \dots, r_d)$ where

$$r_j = \max\{a_l | Tor^S_j(M, k)_{(a_1, \dots, a_j + i, \dots, a_d} \neq 0\}$$

We restate Question 1.1 in the following question.

Question 1.7. What can we say about $\operatorname{reg}(I : B^\infty)$ and $\operatorname{rreg}(I)$ where I is a square-free monomial ideal?

In section 2 we outline some of the code we used in Macaulay2 to fuel our results in the following sections. In sections 3 we present our findings for ideals in $\mathbb{P}^1 \times \mathbb{P}^1$. In section 5 we suggest directions to explore.

2. Code

In our project, we tested examples in Maculay2. To calculate resolution regularity, we used the code in Figure 2. To come up with random examples, we used the code in Figure 2.

To approach this question we utilize combinatorial tools for resolution-regularity.. The following is stated in [Fra14].

Lemma 2.1. *Let I be generated by the non-faces of a simplicial complex Δ. Then the Betti numbers $\beta_{i,j}$ of the minimal graded free resolution of I is given by the following formula.*

$$\beta_{i,j} = (S \setminus I_\Delta) = \sum_{|\alpha|=j} \dim \tilde{H}_{i-j-1}(\Delta_\alpha)$$

Using Lemma 2.1, we coded a function to calculate resolution-regularity as found in Figure 2.

Due to the difficult nature of enumerating simplicial complexes beyond $\mathbb{P}^1 \times \mathbb{P}^1$, we coded a function to test random square-free monomial ideals in $\mathbb{P}^n \times \mathbb{P}^m$ as found in Figure 2.

```
    resRegularityHelper = (r,l) -> (
    max for k in keys betti r list (
        k#1#l - k#0
        )
    )

resRegularity = (r) -> (
    d := degreeLength ring r;
    for l from 0 to (d-1) list (
        resRegularityHelper(r,l)
        )
    )
```

FIGURE 2.1.1. Code for calculating resolution-regularity.

```
X = toricProjectiveSpace(n)**toricProjectiveSpace(m)
R = ring X
P=newRing(R,DegreeRank=>1)
phi=map(R,P)
L={...}--degrees of minimial generators of ideal.
I = randomSquareFreeMonomialIdeal(L,P)
print resolutionInformation phi(I);
```

FIGURE 2.1.2. Code for testing a random square-free monomial ideal in $\mathbb{P}^n \times \mathbb{P}^m$

3. IDEALS ON $\mathbb{P}^1 \times \mathbb{P}^1$

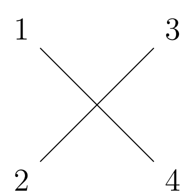
We consider $\mathbb{P}^1 \times \mathbb{P}^1 \subset S := K[x_0, x_1, y_0, y_1]$. The square-free monomial ideals of $\mathbb{P}^1 \times \mathbb{P}^1$ are of the form I_Δ where Δ is a simplicial complex on the vertices $\{x_0, x_1, y_0, y_1\}$. In Figure 3, we enumerate the simplicial complexes on $\{x_0, x_1, y_0, y_1\}$ up to permutation of the indeterminants of each individual projective space as well as the sets of indeterminants per projective space. We list the simplicial complexes associated to each ideal I, the closure of the saturation's of the simplicial complexes, the generators for the multi-graded regularities of I, and the resolution-regularities of I.

Δ_I	$\overline{\Delta_I \setminus \Delta_B}$	gens(reg $I : B^\infty$)	rreg I
$\{x_0\}$	$\{\emptyset\}$	{{0,0}}	{0,0}
$\{x_0x_1\}$	$\{\emptyset\}$	{{0,0}}	{0,0}
$\{x_0y_0\}$	$\{x_0y_0\}$	{{0,0}}	{0,0}
$\{x_0x_1,x_0y_0\}$	$\{x_0y_0\}$	{{0,0}}	{0,0}
$\{x_0y_0,x_0y_1\}$	$\{x_0y_0,x_0y_1\}$	{{0,1}}	{0,1}
$\{x_0y_0,y_0y_1\}$	$\{\emptyset\}$	{{0,0}}	{0,0}
$\{x_0y_0,x_1y_1\}$	$\{x_0y_0,x_1y_1\}$	{{0,1},{1,0}}	{1,1}
$\{x_0x_1,x_0y_0,x_0y_1\}$	$\{x_0y_0,x_0y_1\}$	{{0,1}}	{0,1}
$\{x_0x_1,x_0y_0,x_1y_0\}$	$\{x_0y_0,x_1y_0\}$	{{1,0}}	{1,0}
$\{x_0x_1,x_0y_0,x_1y_1\}$	$\{x_0y_0,x_0y_0\}$	{{0,1},{1,0}}	{0,1}
$\{x_0x_1,x_0y_0,y_0y_1\}$	$\{x_0y_0\}$	{{0,0}}	{0,0}
$\{x_0x_1,x_0y_0,x_0y_1,x_1y_0,\}$	$\{x_0y_0,x_0y_1,x_1y_0,\}$	{1, 1}	{1, 1}
$\{x_0x_1,x_0y_0,x_1y_1,y_0y_1\}$	$\{x_0y_0,x_1y_1\}$	{{0,1},{1,0}}	{0,0}
$\{x_0x_1,x_0y_0,x_1y_0,,y_0y_1\}$	$\{x_0y_0,x_1y_0,\}$	{{1,0}}	{1,0}
$\{x_0y_0,x_0y_1,x_1y_0,x_1y_1\}$	$\{x_0y_0,\ x_0y_1,\ x_1y_0,,\ x_1y_1\}$	{{1,1}}	{1,1}
$\{x_0y_0,x_1y_0,x_1y_1,\ y_0y_1\}$	$\{x_0y_0,x_1y_0,x_1y_1\}$	{{1,1}}	{1,1}
$\{x_0x_1,x_0y_0,x_0y_1,x_1y_0,x_1y_1\}$	$\{x_0y_0,x_0y_1,x_1y_0,,x_1y_1\}$	{{1,1}}	{1,1}
$\{x_0x_1,x_0y_0,x_0y_1,x_1y_0,y_0y_1\}$	$\{x_0y_0,x_0y_1,x_1y_0,\}$	{{1,1}}	{1,1}
$\{x_0x_1,x_0y_0,x_0y_1,x_1y_0,x_1y_1,y_0y_1\}$	$\{x_0y_0,x_0y_1,x_1y_0,x_1y_1\}$	{{1,1}}	{1,1}
$\{x_0x_1y_0\}$	$\{x_0x_1y_0\}$	{{0,0}}	{0,0}
$\{x_0x_1y_0,x_0y_1\}$	$\{x_0x_1y_0,x_0y_1\}$	{{0,1}}	{0,1}
$\{x_0x_1y_0,y_0y_1\}$	$\{x_0x_1y_0\}$	{{0,0}}	{0,0}
$\{x_0x_1y_0,x_0y_1,y_0y_1\}$	$\{x_0x_1y_0,\ x_0y_1\}$	{{0,1}}	{0,1}
$\{x_0x_1y_0,x_0y_1,x_1y_1\}$	$\{x_0x_1y_0,\ x_0y_1,\ x_1y_1\}$	{{1,1}}	{1,1}
$\{x_0x_1y_0,x_0x_1y_1\}$	$\{x_0x_1y_0,\ x_0x_1y_1\}$	{{0,1}}	{0,1}
$\{x_0x_1y_0,x_0x_1y_1,y_0y_1\}$	$\{x_0x_1y_0,\ x_0x_1y_1\}$	{{0,1}}	{0,1}
$\{x_0x_1y_0,x_0y_0y_1\}$	$\{x_0x_1y_0,\ x_0y_0y_1\}$	{{0,0}}	{0,0}
$\{x_0x_1y_0,x_0y_0y_1,x_1y_1\}$	$\{x_0x_1y_0,\ x_0x_1y_1,\ x_1y_1\}$	{{0,1}}	{0,1}
$\{x_0x_1y_0,x_0x_1y_1,x_0y_0y_1\}$	$\{x_0x_1y_0,\ x_0x_1y_1,\ x_0y_0y_1\}$	{{0,0}}	{0,1}
$\{x_0x_1y_0,x_0x_1y_1,x_0y_0y_1,x_1y_0y_1\}$	$\{x_0x_1y_0,x_0x_1y_1,x_0y_0y_1,x_1y_0y_1\}$	{{1,1}}	{1,1}
$\{x_0x_1y_0y_1\}$	$\{x_0x_1y_0y_1\}$	{{0,0}}	{0,0}

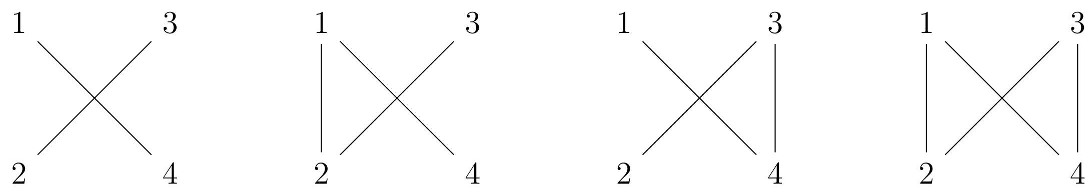
FIGURE 3.0.3. Ideals of $\mathbb{P}^1 \times \mathbb{P}^1$ enumerated

Let's look at an example from the table.

Example 3.1. Let's consider the following simplicial complex whose ideal I is B-saturated.



It's resolution-regularity is generated by $(1,0)$ and $(0,1)$. The following simplicial complexes B-saturate to I.



The resolution-regularities of these ideals are $(1,1)$, $(0,1)$, $(1,0)$, and $(0,0)$ respectively. Notice that the multi-graded regularity of I is a strict subset of the span of the resolution-regularities of the ideals that saturate to I. ◇

Let's look at another example.

Example 3.2. The ideal I associated to the complex $\{x_0x_1y_0, x_0x_1y_1, x_0y_0y_1\}$ saturated to itself, has a multi-graded regularity generated by $(0,0)$, and has a resolution-regularity of $(0,1)$. This means the span of it's resolution-regularity is a strict subset of it's multi-graded regularity. ◇

4. Further Directions

We know that resolution-regularity has a combinatorial approach in the case of square-free monomial ideals. It is given by Hotcher's Formula.

Proposition 4.1. *Let I be generated by the non-faces of a simplicial complex Δ. Then the Betti numbers $\beta_{i,j}$ of the minimal graded free resolution of I is given by the following formula.*

$$\beta_{i,j} = (S \setminus I_\Delta) = \sum_{|\alpha|=j} \dim \tilde{H}_{i-j-1}(\Delta_\alpha)$$

It is possible that a combinatorial approach for multi-graded regularity would lead to something interesting. One may consider the following proposition to figure this out. The following in stated in [Rei01].

Proposition 4.2. *Let $\Sigma \subset \Delta$ be simplicial complexes, and let $\mathbf{a} \in \mathbb{Z}$, $F_+ = supp_+(\mathbf{a})$ and $F_- = supp_-(\mathbf{a})$ Then*

$$H^i_J(k[\Delta]] \cong \tilde{H}^{i-1}(||star_\Delta(F_+) - ||\Sigma|, ||del_{star_\Delta(F_+)}(F_-))|| - ||\Sigma|)$$

where $||\Delta||$ denotes the geometric realization of Δ.

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