q-Continued Fractions

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Abstract

We study the conjectured unimodality property of q-analogous of rational numbers via their combinatorial interpretation of counting lattice paths in a snake graphs. We derive recurrence relations for the height polynomial of a general snake graph and give a geometric interpretation of the height polynomial in the special case of snake graphs which we call snake graphs with isolated U's.

1 Introduction

The main aim of this report is to investigate Morier-Genoud and Ovsienko's unimodality conjecture for q-rationals [1] by studying the combinatorial interpretation of q-rationals which uses lattice paths in snake graphs. The structure of this report is as follows: After introducing the notion of q-rationals and the combinatorial interpretation we investigate we proceed, in Section 2, to derive recurrence relations for the height polynomial of the poset of lattice paths in a snake graph. Then in Section 3 we narrow our focus to a specific class of snake graphs: snake graphs from words with isolated U's (Definition 3.1), which are a slight generalization of the class for which unimodality is proven in [2]. We give a geometric interpretation in Section 3.2 of the height polynomial for snake graphs with isolated U's which allows us to derive an explicit, but unwieldy, formula for the height polynomial. Then in Section 3.1 we use the recurrence relations developed in Section 2 to write the height polynomial of words with isolated U's as an expression involving products of certain q-integers. We then discuss elementary symmetry properties of snake graphs in Section 4 which allow one to reduce the question of unimodality. Then in Section 5 we discuss several possible avenues one might try to prove the unimodal conjecture. Finally, we conclude with a few conjectures on properties of q-rationals.

The classical q-analogue of integers, also known as q-integers, are defined as:

Definition 1.1. Let q be a formal parameter. Then for $n \in \mathbb{N}$, the q-integer $[n]_q \in \mathbb{Z}[q]$ is defined as

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$$

We say $[n]_q$ is the q-integer corresponding to n.

Recently, Morier-Genoud and Ovsienko gave a new definition of q-deformed continued fractions and rational numbers [1]. The q-deformation of a the rational $\frac{r}{s}$ denoted $\left[\frac{r}{s}\right]_q$ is a rational function in q defined as a continued fraction. To introduce this notion we first recall the definition of continued fractions.

Definition 1.2. Given a rational number $\frac{r}{s} \in \mathbb{Q}_{>1}$ greater than 1 such that r and s are positive relatively prime, there are unique finite sequences $(a_1, \ldots, a_{2m})$ and $(c_1, \ldots, c_k)$ such that

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_{2m}}}} = c_1 - \cfrac{1}{c_2 - \cfrac{1}{\ddots - \cfrac{1}{c_k}}}$$

We denote these expressions by $[a_1, \ldots, a_{2m}]$ and $[[c_1, \ldots, c_k]]$ respectively. We call these the regular and negative continued fractions of $\frac{r}{s}$ respectively.

We are prepared to define the q-analogue of rational numbers.

Definition 1.3. Given a continued fraction $[a_1, \ldots, a_{2m}]$ its q-deformation is defined as

$$[a_1, \ldots, a_{2m}]_q := [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{\ddots \cfrac{\ddots}{[a_{2m-1}]_q + \cfrac{q^{a_{2m-1}}}{[a_{2m}]_{q^{-1}}}}}}$$

Given a negative continued fraction $[[c_1, \ldots, c_k]]$ its q-deformation is defined as

$$[[c_1, \ldots, c_k]]_q = [c_1]_q - \cfrac{q^{c_1-1}}{[c_2]_q - \cfrac{q^{c_2-1}}{\ddots \cfrac{\ddots}{[c_k-1]_q - \cfrac{q^{c_k-1}}{[c_k]_q}}}}$$

It is not hard to check that if $[a_1, \ldots, a_{2m}] = [[c_1, \ldots, c_k]]$ then $[a_1, \ldots, a_{2m}]_q = [[c_1, \ldots, c_k]]_q$. In light of this we define the q-analogue of a rational $\frac{r}{s} = [a_1, \ldots, a_{2m}] = [[c_1, \ldots, c_k]]$ by

$$\left[\frac{r}{s}\right]_q = [a_1, \ldots, a_{2m}]_q$$

Notice that $[\frac{r}{s}]_q$ takes the form $\frac{\mathcal{R}(q)}{\mathcal{S}(q)}$ where $\mathcal{R}(q), \mathcal{S}(q) \in \mathbb{Z}[q]$. By requiring their leading coefficients to be positive, they become unique.

Example 1.4. *For a first interesting example consider*

$$\left[\frac{5}{2}\right]_q = \frac{1+2q+q^2+q^3}{1+q}, \quad \left[\frac{5}{3}\right]_q = \frac{1+q+2q^2+q^3}{1+q+q^2}$$

Here one observes that the quantized 5 *appearing in the numerator is not always the same.*

The reason why Definition 1.3 is made for q-rationals instead of the more naive $\left[\frac{r}{s}\right]_q = \frac{[r]_q}{[s]_q}$ is because the former satisfies the following interesting combinatorial properties while the latter does not. Firstly there is a total positivity statement [1, Theorem 2]:

Theorem 1.5. *For every pair of q-rationals, $\left[\frac{r}{s}\right]_q$ and $\left[\frac{r'}{s'}\right]_q$, the polynomial in q*

$$\mathcal{X}_{\frac{r}{s},\frac{r'}{s'}} := \mathcal{R}\mathcal{S}' - \mathcal{S}\mathcal{R}' \tag{1}$$

has positive integer coefficients, provided $\frac{r}{s} \geq \frac{r'}{s'}$

One can quickly see that the naive definition of q-rational does not have this total positivity property.

Example 1.6. *We have $\frac{1}{2} > \frac{2}{5}$ but*

$$[1]_q[5]_q - [2]_q[2]_2 = 1 + q + q^2 + q^3 + q^4 - 1 - 2q - q^2 = -q + q^3 + q^4$$

The second important property is that the coefficients of $\mathcal{R}$ and $\mathcal{S}$ admit several combinatorial descriptions. There are a few different combinatorial interpretations of the numerator and denominator of a q-rational defined via continued fractions. The first was described [1] where it is shown that the coefficients of the numerator and denominator of a q-rational count the closure sets of a certain graphs or equivalently they count subrepresentations of the maximal indecomposable quiver representation. Other equivalent combinatorial objects include perfect matching on snake graphs [3, 4], angle matchings [4, 5], T-paths [4, 6], and lattice paths in snake graphs [4]. The interested reader is referred to the literature for proofs of these combinatorial interpretations. In this report we have the modest goal of exploring the lattice path interpretation of q-rationals. To introduce this perspective let us first develop the notion of lattice paths in a snake graph.

Definition 1.7. A binary word W on $\{U, R\}$ is a finite string composed only of the letters U and R. For shorthand if $a \in \mathbb{N}$ then let R^a denote the word consisting of a R's and let U^a denote the word consisting of a U's.

Definition 1.8. If W is a binary word then $\ell(W)$ denotes the length of the word, i.e. $\ell(URRR) = 4$.

Definition 1.9. If W is a binary word on $\{U, R\}$ then define W^T, the transpose, to be the word formed from interchanging R with U in W.

Example 1.10. *If* $W = RURU$ *then* $W^T = URUR$.

Definition 1.11. A tile is a square in the plane whose sides are either parallel or orthogonal to the fixed basis. Due to the orientation of a tile, we may refer to it's edges by north, east, south, or west.

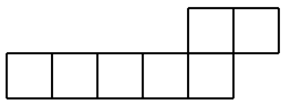


Figure 1: Example snake graph

A snake graph is a planar graph constructed in the following manner. Let $(G, G_1 \ldots, G_n)$ be a sequence of tiles. Suppose $G, \ldots, G_m$ are placed on the plane where $m < n$. We place G_{m+1} on the plane in one of the following 2 ways:

1. The south boundary of G_{m+1} is the north boundary of G_m.
2. The west boundary of G_{m+1} is the east boundary of G_m.

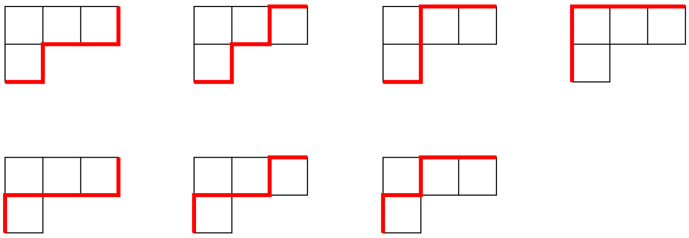
Each snake graph is represented uniquely as a binary word on the alphabet $\{U, R\}$, that is, a unique sequence $(a_1, \ldots, a_n)$ where each a_i is R or U. The description of a snake graph in terms of its binary word is as follows: start with a tile. For each letter in the binary word, add another tile either above (if you see a U in the word) or to the right (if you see a R in the word). For example the word for Figure 1 is $W = RRRRUR$.

Remark 1.12. In the literature, "snake graph" refers to a slightly different, but related, construction. The construction above is called the "dual snake graph" corresponding to a triangulation.

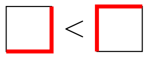
Definition 1.13. We associate two snake graphs, $G_{\frac{r}{s}}$ and $\hat{G}_{\frac{r}{s}}$ to each $\frac{r}{s} = [a_1, \ldots, a_{2m}] \in \mathbb{Q}$. The binary word specifying $G_{\frac{r}{s}}$ is $U^{a_1-1}R^{a_2}U^{a_3} \cdots R^{a_{2m}-1}$. The binary word specifying $\hat{G}_{\frac{r}{s}}$ is $R^{a_2-1}U^{a_3} \cdots R^{a_{2m-1}}$.

Definition 1.14. If G is a snake graph then a lattice path in G is a path starting on the lower left corner and ending at the top right corner which only goes up or right at each juncture.

Example 1.15. *The 7 lattice paths in* $G_{7/3}$ *are*

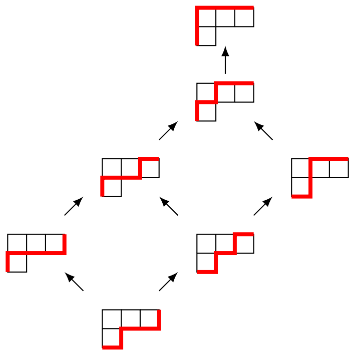


Definition 1.16. There is a partial order on the lattice paths in a snake graph so that locally



In this way the lattice path in a snake graph form a graded poset. If G is a snake graph let $L(G)$ denote the poset of lattice paths on G.

Example 1.17. $L(G_{7/3})$



Definition 1.18. Define the *height* or *rank* of a lattice path as how many steps it takes to get to it from the minimal path.

Definition 1.19. If W is a binary word on $\{U, R\}$ define the height polynomial by $H(W) := \sum_i h_i q^i$ with h_i is the number of paths of height i in the snake graph for W. Also let $H(G_{\frac{r}{s}})$ denote the height polynomial of the word associated to the snake graph $G_{\frac{r}{s}}$.

The combinatorial interpretation of Morier-Genoud and Ovsienko's q-deformed rational we investigate is in terms of lattice paths in snake graphs [4, Theorem 9.1]. In our notation this result reads:

Theorem 1.20. *If $\frac{r}{s} = [a_0, \dots, a_n]$ then we have*

$$\left[\frac{r}{s}\right]_q = \frac{H(G_{\frac{r}{s}})}{H((\hat{G}_{\frac{r}{s}})^T)}$$

In particular it is immediate that $\left[\frac{r}{s}\right]_q$ is a rational function with positive integer coefficients.

Definition 1.21. A sequence of integers $a_0, a_1, \dots, a_n$ is unimodal if there exits an $s \in \mathbb{N}$ such that

$$a_0 \leq \dots \leq a_s \geq a_{s+1} \geq \dots \geq a_n$$

A polynomial $p(q) = \sum_i p_i q^i$ is said to be unimodal if the sequence p_i is unimodal.

Conjecture 1.22. *It is conjectured [1] that the numerator and denominator of any q-rational are unimodal, i.e. the coefficients of $\mathcal{R}(q)$ and $\mathcal{S}(q)$ form a unimodal sequence. In terms of the lattice path interpretation of q-rationals this is the statement that the height polynomial of lattice paths in any snake graph is unimodal.*

The unimodality of the height polynomial of a snake graph associated to a binary word W is know in some special cases:

1. W consists only of U's or only of R's. It is easy to see that the height polynomial is $[\ell(W)+2]_q$ which is clearly unimodal.
2. W is a zigzag word, i.e. there are no consecutive R's or U's in W (Fibonacci cubes are unimodal [7]).
3. W is a word with isolated U's (Definition 3.1) with constant row length (up-down posets are unimodal [2]).

2 Recurrence Relations

In this section we derive some recurrence relations for the height polynomial of a snake graph. We begin by setting some notation:

Definition 2.1. If W is a binary word on $\{U, R\}$ then WR denotes the word W with a R appended on the right and WU the word W with a U appended on the right, i.e. if $W = URR$ then $WR = URRR$.

Definition 2.2. Let W_R denote the word obtained from W by removing the first section of R's on the right. Similarly let W_U denote the word obtained from W by removing the first section fo U's on the right hand side.

Example 2.3. *For example if $W = RURRRR$ then $W_R = RU$ and if $W = RUUUUU$ then $W_U = R$.*

Definition 2.4. If W is a binary word on $\{U, R\}$ then $\widehat{W}$ is defined to be the word formed from W by removing the right most letter in W. Alternatively, if G is the snake graph for W then $\widehat{W}$ is the word corresponding to the snake graph obtained from G by removing the last box. With this second definition it makes sense to talk about $\widehat{W}$ when W is the empty word, this "word" should correspond to the empty snake graph. By convention the height polynomial of the poset corresponding to the empty snake graph is just 1. In other words if W is the empty word then $H(\widehat{W}) = 1$. For convention if W is the empty word then $\ell(\widehat{W}) = -1$.

Example 2.5. *For example if $W = RURUR$ then $\widehat{W} = RURU$.*

We will now state and prove recurrence relations for the polynomials $H(WR)$ and $H(WU)$.

Theorem 2.6. *If W is a binary word on $\{U, R\}$ then we have the following recurrences for the height polynomial*

$$H(WU) = H(W) + q^{\ell(W)-\ell(\widehat{W_U})+1} H(\widehat{W_U}) \tag{2}$$

and

$$H(WR) = H(\widehat{W_R}) + qH(W) \tag{3}$$

Proof. There are four cases to consider: appending a R or a U to a word ending in either a R or a U. These cases are handled individually in Lemma 2.8, Lemma 2.10, Lemma 2.9, Lemma 2.11. □

Corollary 2.7. *If W is a binary word on $\{U, R\}$ if $n \in \mathbb{N}$ then we have the following recurrences for the height polynomial:*

$$H(WR^n) = [n]_q H(\widehat{W_R}) + q^n H(W) \tag{4}$$

and

$$H(WU^n) = H(W) + [n]_q q^{\ell(W)-\ell(\widehat{W_U})+1} H(\widehat{W_U}) \tag{5}$$

where $[n]_q$ is the q-integer.

Proof. The proof of both formula is by induction using Theorem 2.6. Consider first (5). The case $n = 1$ is just Theorem 2.6. So assume the nth case. Then we have

$$\begin{aligned} H(WR^{n+1}) &= H(\widehat{WR^n_R}) + qH(WR^n) && \text{(Theorem 2.6)} \\ &= H(\widehat{W_R}) + qH(WR^n) && (\widehat{WR^n_R} = \widehat{W_R}) \\ &= H(\widehat{W_R}) + q([n]_q H(\widehat{W_R}) + q^n H(W)) && \text{(inductive hypothesis)} \\ &= (1 + q[n]_q)H(\widehat{W_R}) + q^{n+1}H(W) \\ &= [n+1]_q H(\widehat{W_R}) + q^{n+1}H(W) \end{aligned}$$

We proceed in the same way to prove (4). The case $n = 1$ is again the content of Theorem 2.6. So assume then nth case. Then we have

$$\begin{aligned} H(WU^{n+1}) &= H(WU^n) + q^{\ell(WU^n)-\ell(\widehat{WU^n_U})+1} H(\widehat{WU^n_U}) && \text{(Theorem 2.6)} \\ &= H(WU^n) + q^{\ell(WU^n)-\ell(\widehat{W_U})+1} H(\widehat{W_U}) && (\widehat{WU^n_U} = \widehat{W_U}) \\ &= H(WU^n) + q^{\ell(W)-\ell(\widehat{W_U})+1+n} H(\widehat{W_U}) && (\ell(WU^n) = \ell(W) + n) \\ &= H(W) + [n]_q q^{\ell(W)-\ell(\widehat{W_U})+1} H(\widehat{W_U}) + q^{\ell(W)-\ell(\widehat{W_U})+1+n} H(\widehat{W_U}) \\ & && \text{(inductive hypothesis)} \\ &= H(W) + ([n]_q + q^n) q^{\ell(W)-\ell(\widehat{W_U})+1} H(\widehat{W_U}) \\ &= H(W) + [n+1]_q q^{\ell(W)-\ell(\widehat{W_U})+1} H(\widehat{W_U}) \end{aligned}$$

□

Lemma 2.8. *If W is a binary word ending in R then we have the following recurrence for the height polynomial of WR:*

$$H(WR) = H(\widehat{W_R}) + qH(W) \tag{6}$$

Proof. First suppose that W is a word ending in R which contains at least one U. Then we consider the word WR which has snake graph that looks like

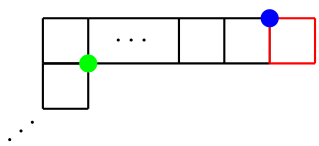


Figure 2: Top right section of snake graph for WR with W ending in R and containing at least one U

The lattice paths that pass through the blue circle are in bijection with the lattice paths in the snake graph for the word W. The height of a lattice path passing through the blue circle is then clearly its height as a lattice path in the snake graph for W plus 1. The height polynomial for the lattice paths passing through the blue circle is then $qH(W)$.

Any lattice path that does not pass through the blue circle must pass through the green circle. The lattice paths passing through the green circle and not passing through the blue circle are in bijection with lattice paths in the snake graph for the word $\widehat{W_R}$. The height of a lattice path passing through the green and not the blue circle is then clearly its height as a lattice path in the snake graph for $\widehat{W_R}$ because any such path passes along the bottom of the second row in Figure 2. The height polynomial for the lattice paths passing through the green circle and not the blue circle is then $H(\widehat{W_R})$. Since a lattice path either passes through the blue circle or not the claim is shown.

Now assume that W is a word ending in R which does not contain at least one U, i.e. W consists only of R's. Then the snake graph for WR looks like

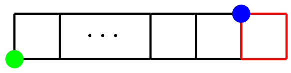


Figure 3: Snake graph for WR with W consisting only of R's

It is easily seen that for such a word we have $H(W) = [\ell(W)+2]_q$ and clearly $[\ell(WR)+2]_q = 1 + q[\ell(W)+2]_q$ so the claim is shown.

□

Lemma 2.9. *If W is a binary word ending in R then we have the following recurrence for the height polynomial of WU:*

$$H(WU) = H(W) + q^2 H(\widehat{W}) \tag{7}$$

Proof. Let W be a word ending in R and consider the word WU which has snake graph that looks like

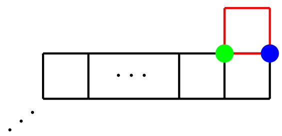


Figure 4: Top right section of snake graph for WU with W ending in R

The lattice paths that pass through the blue circle are in bijection with the lattice paths in the snake graph for the word W. The height of a lattice path passing through the blue circle is then clearly its height as a lattice path in the snake graph for W. The height polynomial for the lattice paths passing through the blue circle is then $H(W)$.

Any lattice path that does not pass through the blue circle must pass through the green circle. The lattice paths passing through the green circle and not passing through the blue circle are in bijection with lattice paths in the snake graph for the word $\widehat{W}$. The height of a lattice path passing through the green and not the blue circle is then clearly its height as a lattice path in the snake graph for $\widehat{W}$ plus 2. The height polynomial for the lattice paths passing through the green circle and not the blue circle is then $q^2 H(\widehat{W})$. Since a lattice path either passes through the blue circle or not the claim is shown. □

Lemma 2.10. *If W is a binary word ending in U then we have the following recurrence for the height polynomial of WR:*

$$H(WR) = H(\widehat{W}) + qH(W) \tag{8}$$

Proof. Let W be a word ending in U and consider the word WR which has snake graph that looks like

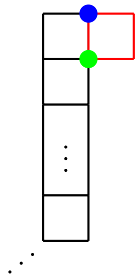


Figure 5: Top right section snake graph for WR with W ending in U

The lattice paths that pass through the blue circle are in bijection with the lattice paths in the snake graph for the word W. The height of a lattice path passing through the blue circle is then clearly its height as a lattice path in the snake graph for W plus 1. The height polynomial for the lattice paths passing through the blue circle is then $qH(W)$.

Any lattice path that does not pass through the blue circle must pass through the green circle. The lattice paths passing through the green circle and not passing through the blue circle are in bijection with lattice paths in the snake graph for the word $\widehat{W}$. The height of a lattice path passing through the green and not the blue circle is then clearly its height as a lattice path in the snake graph for $\widehat{W}$. The height polynomial for the lattice paths passing through the green circle and not the blue circle is then $H(\widehat{W})$. Since a lattice path either passes through the blue circle or not the claim is shown. $\square$

Lemma 2.11. *If W is a binary word ending in U then we have the following recurrence for the height polynomial of WU:*

$$H(WU) = H(W) + q^{\ell(W)-\ell(\widehat{W_U})+1}H(\widehat{W_U}) \tag{9}$$

Proof. First let W be a word ending in U which includes at least one R and consider the word WR which has snake graph that looks like

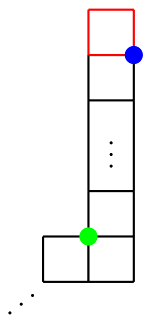


Figure 6: Top right section of snake graph for WU when W ends in U and contains at least one R

The lattice paths that pass through the blue circle are in bijection with the lattice paths in the snake graph for the word W. The height of a lattice path passing through the blue circle is then clearly its height as a lattice path in the snake graph for W. The height polynomial for the lattice paths passing through the blue circle is then $H(W)$.

Any lattice path that does not pass through the blue circle must pass through the green circle. The lattice paths passing through the green circle and not passing through the blue circle are in bijection with lattice paths in the snake graph for the word $\widehat{W_U}$. The height of a lattice path passing through the green and not the blue circle is then clearly its height as a lattice path in the snake graph for W_U plus the quantity $\ell(W) - \ell(\widehat{W_U}) + 1$. The height polynomial for the lattice paths passing through the green circle and not the blue circle is then $q^{\ell(W)-\ell(\widehat{W_U})+1} H(\widehat{W_U})$. Since a lattice path either passes through the blue circle or not the height polynomial for the word WU is given by

$$H(WU) = H(W) + q^{\ell(W)-\ell(\widehat{W_U})+1} H(\widehat{W_U})$$

Now, assume that W is a word ending in U which does not contain at least one R, i.e. W consists only of U's. Then the snake graph for W looks like

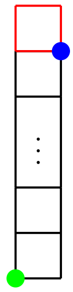


Figure 7: Snake graph for WU when W consists only of U's

It is easily seen that for such a word we have $H(W) = [\ell(W)+2]_q$ and clearly $[\ell(WU)+2]_q = [\ell(W)+2]_q + q^{\ell(W)-\ell(\widehat{W_U})+1}$ because $\ell(W) - \ell(\widehat{W_U}) + 1 = \ell(W) + 2$ since by convention $\ell(\widehat{W_U}) = -1$ since W_U is the empty word. So the claim is shown. □

3 Words with isolated U's

Definition 3.1. A binary word on $\{U, R\}$ is said to be a word with isolated U's if consecutive U's do not appear in W.

The data of a word with isolated U's is equivalent to specifying the number of consecutive blocks of R which appear and their respective length. This corresponds in the snake graph to the number of "rows" appearing in the snake graph and their length. It is shown in [2] that words with isolated U's in which the row length is constant have unimodal height sequence. Our snake graphs with isolated U's are then a slight generalization a class of snake graphs known to be unimodal. For brevity denote the word $R^{k_1} U R^{k_2} U \cdots U R^{k_n}$ by $I(k_1, k_2, \ldots, k_n)$.

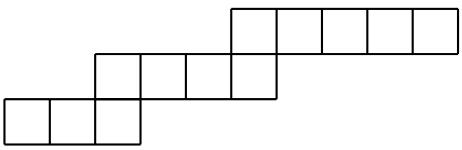


Figure 8: Example of snake graph corresponding to the word $I(2, 3, 4)$

3.1 Some explicit formulae

Proposition 3.2. *Let $k_1, k_2, k_3, k_4 \in \mathbb{N}$. Then we have*

$$H(I(k_1)) = \frac{q^{k_1+2} - 1}{q - 1} = [k_1 + 2]_q \tag{10}$$

$$H(I(k_1, k_2)) = [k_1+1]_q q^{k_2+2} + [k_1+2]_q [k_2+1]_q = \frac{-((q^3 - q^2 + q - q^{k_1+4}) q^{k_2} + q^{k_1+2} - 1)}{q^2 - 2q + 1} \tag{11}$$

$$H(I(k_1,k_2,k_3)) = [k_1{+}2]_q([k_2{+}1]_q[k_3{+}1]_q{+}q^{k_3+2}[k_2]_q){+}q^{k_2+2}[k_1{+}1]_q[k_3{+}2]_q = \frac{N_3}{q^3-3q^2+3q-1} \tag{12}$$

with

$$\begin{aligned} N_3 = &(q^3-q^2+q-q^{k_1+4})q^{k_2}+\\ &+(q^3-(q^5-q^4+q^3)q^{k_1}-(q^5-q^4+q^3-q^{k_1+6})q^{k_2}-q^2+q)q^{k_3}+\\ &+q^{k_1+2}-1 \end{aligned}$$

$$H(I(k_1,k_2,k_3,k_4)) = \frac{N_4}{q^4-4q^3+6q^2-4q+1} \tag{13}$$

$$\begin{aligned} &N_4 = -[(q^3-q^2+q-q^{k_1+4})q^{k_2}-\\ &-(q^5-q^4-(q^7-q^6+q^4-q^3)q^{k_1}-(q^7-q^6+q^4-q^3-(q^8-q^6)q^{k_1})q^{k_2}+q^2-q)q^{2k_3}+\\ &+(2q^3-(2q^5-2q^4+2q^3-q^2)q^{k_1}-(q^6-q^5+2q^4-2q^3-(q^7+q^5-q^4)q^{k_1}+2q^2-q)q^{k_2}-2q^2+2q-1)q^{k_3}-\\ &-((q^3-q^2+q-q^{k_1+4})q^{k_2}+(q^3-(q^5-q^4+q^3)q^{k_1}-(q^5-q^4+q^3-q^{k_1+6})q^{k_2}-q^2+q)q^{k_3}+q^{k_1+2}-1)q^{k_4}+\\ &+q^{k_1+2}-1] \end{aligned}$$

Proof. First we claim that

$$H(I(k_1,\dots,k_n)) = [k_n+1]_q H(I(k_1,\dots,k_{n-1})) + q^{k_n+2}H(I(k_1,\dots,k_{n-1}-1)) \tag{14}$$

which may easily be deduced from Theorem 2.6 and Corollary 2.7. Then from Corollary 2.7 it is immediate that $H(I(k_1)) = [k_1+2]_q$. So (10) is shown. To derive the next equation use (14)

$$H(I(k_1,k_2)) = [k_2+1]_q[k_1+2]_q + q^{k_2+2}[k_1+1]_q$$

so we have shown (11). It was verified using [8] that this polynomial is equal to the rational function appearing in (11). To obtain the next equation apply (14) again to obtain

$$\begin{aligned} H(I(k_1,k_2,k_3)) &= [k_3+1]_q([k_2+1]_q[k_1+2]_q+q^{k_2+2}[k_1+1]_q)+q^{k_3+2}([k_2]_q[k_1+2]_q+q^{k_2+1}[k_1+1]_q)\\ &= [k_1+2]_q([k_2+1]_q[k_3+1]_q+q^{k_3+2}[k_2]_q)+q^{k_2+2}[k_1+1]_q[k_3+2]_q \end{aligned}$$

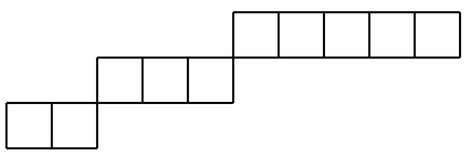
it was verified using [8] that this polynomial is equal to the rational function appearing in (12). To obtain the final equation apply the recurrence again:

$$\begin{aligned} H(I(k_1,k_2,k_3,k_4)) = &[k_4+1]_q([k_1+2]_q([k_2+1]_q[k_3+1]_q+q^{k_3+2}[k_2]_q)+q^{k_2+2}[k_1+1]_q[k_3+2]_q)+\\ &+q^{k_4+2}([k_1+2]_q([k_2+1]_q[k_3]_q+q^{k_3+1}[k_2]_q)+q^{k_2+2}[k_1+1]_q[k_3+1]_q) \end{aligned}$$

It is verified using [8] that this polynomial is indeed equal to the claimed rational expression. □

3.2 Geometric interpretation

Snake graphs with isolated U's appear to almost be graphs like the following example.



Such graphs would have height functions that can easily be calculated and are unimodal as well as symmetric. They are $\prod[n_i+1]_q$ where n_i denotes the number of squares in the ith row. One may think of a hyper-rectangle with side lengths n_i+1. The lattice paths on the graph would correspond to lattice points in the hypercube whose height is given by its coordinate sum. This is formalized in the next theorem.

Theorem 3.3. *Let $W = I(k_1,\ldots,k_n)$ and define $D \subseteq \mathbb{Z}^m$ by $D = \{(x_1,\ldots,x_n) : \exists i, 1 \le i \le m, x_{i-1} = 0, x_i = k_i\}$. Furthermore, let R be the set of all integral points in $\prod_{i=1}^n [0,k_i] \setminus D$. Then there is a height preserving bijection between lattice paths in G_W and of R where the height of $p = (x_1\ldots,x_n) \in R$ is defined as $\sum_i x_i$.*

Proposition 3.4. *The height polynomial of $W = I(k_1,\ldots,k_n)$ is given by*

$$\prod_{i=1}^{n}[k_i+1]_q-\sum_{j=1}^{n-1}\left(x^{k_{j+1}-1}\prod_{i\notin\{j,j+1\}}[k_i+1]_q\right)+\sum_{j=1}^{n-1}\left(x^{k_{j+1}-1}x^{k_{p+1}-1}\prod_{i\notin\{j,j+1,p,p+1\},|j-p|>1}[k_i+1]_q\right)-\cdots \tag{15}$$

Proof. The integral points in $\prod_{i=1}^n [0,k_i]$ is counted by $\prod_{i=1}^n [k_i+1]_q$. To subtract out D, we use the principle of inclusion-exclusion. □

4 Symmetry properties

Snake graphs posses certain symmetries which allow one to reduce the unimodality conjecture. Essentially, you really only need to check "half" of all snake graphs are unimodal to prove the conjecture.

Proposition 4.1. *If W is a binary word such that the poset $L(G_W)$ is unimodal then $L(G_{W^T})$ is also unimodal*

Proof. If we let our snake graphs live in $\mathbb{R}^2$ with lower left corner at $(0,0)$ then it is not hard to see that G_W is related to G_{W^T} by a reflection across the line e_1+e_2 where e_1 and e_2 are the standard basis vectors. So then it is clear that $L(G_W)$ is related to $L(G_{W^T})$ by inverting the order relation, i.e. $L(G_{W^T}) = L(G_W)^{\text{op}}$. Since inverting the order of the elements in a unimodal sequence preserves the unimodal property the conclusion follows. □

Corollary 4.2. *To prove that all snake graphs are unimodal it is enough to prove that if $H(W)$ is unimodal then $H(WR)$ or $H(WU)$ is also unimodal.*

Proof. An inductive proof of the unimodality of $H(W)$ for any word would go as follows: prove that $H(W)$ is unimodal when W has length zero, i.e. W is the empty word. This is clear because the height polynomial of the empty word is simply $1+q$. Then prove for W a word of length n that if $H(W)$ is unimodal then this implies that $H(WR)$ and $H(WU)$ are both unimodal. Suppose that you where able to prove that for all words of length n then $H(W)$ unimodal implies $H(WR)$ is unimodal. Then given W a word of length n we have W^T is also a word of length n. So we know that $H(W^T)$ is unimodal by assumption. Then by assumption we know that $H(W^TR)$ is unimodal so by Proposition 4.1 $H((W^TR)^T) = H(WU)$ is unimodal. The argument replacing WR with WU is analogous. □

5 Possible proof techniques

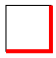
We collect here a few possible direction for proving the unimodality of snake graphs.

5.1 Twisting maps

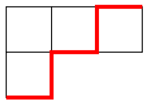
The unimodality of a sequence $a_0, \ldots, a_n$ is equivalent to the existence of an $0 \leq s \leq n$ such that we have a sequence of injections and surjections

$$T_0 \hookrightarrow T_1 \hookrightarrow \cdots \hookrightarrow T_s \twoheadrightarrow T_{s+1} \twoheadrightarrow \cdots \twoheadrightarrow T_n$$

with T_i a set with a_i elements. So to prove the unimodality of $H(W)$ if T_i is the set of lattice paths of height i in $L(G_W)$ it is enough to construct such a sequence of injections and surjections. An interesting observation is that T_i comes with some ready made maps to T_{i+1} and T_{i-1} (of course only when $0 \leq i+1, i-1 \leq n$). We call these maps twists and they are defined as follows. Given a lattice path $p \in T_i$ label all occurrences of



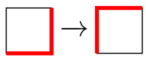
by the length of the word corresponding to the snake graph truncated at that box. We call these indices twists indices. For example the twist indices for



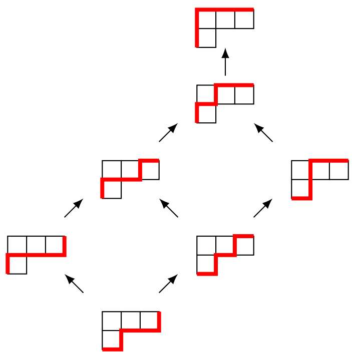
would be $(0, 2)$.

Proposition 5.1. *Each path $p \in T_i$ has at lest one twist index if i is not the maximum height.*

A choice of twist index for each $p \in T_i$ then defines a map $T_i \to T_{i+1}$ where the map changes each lattice path $p \in T_i$ by switching



at each twisting index. Such a map, one defined by a choice of twisting indices, will be called a twisting map. There is an analogous procedure for defining a map from $T_i \to T_{i-1}$ when $i \neq 0$. In all example we computed it is possible to form a sequence of injections and surjections using only twisting maps which shows that the height polynomial is unimodal. As a rather simple example of this observation consider the snake graph corresponding to $G_{7/3}$. The lattice path poset, $L(G_{7/3})$, is then given by



If one then takes the following sequence of twisting indices $\{(0), (3, 2), (2, 0), (1)\}$ starting from the minimal element and each entry arranged in correspondence with the picture one obtains the necessary sequence of injections and surjections. We remark that there is not always a unique sequence which shows unimodality. Our example indeed shows the failure of uniqueness. We have not been able to discover an algorithm which tells you which choice of twisting indices at each height one should take to get such a sequence of injections and surjections but it does not seem too unreasonable to expect that one could understand the unimodality of snake graphs via twisting maps.

5.2 Inductive proof

Given the recurrences we derived Theorem 2.6 it is natural to attempt to prove unimodality of snake graphs by induction. In fact this was the approach taken for zigzag snake graphs in [7] although the recurrences they used are in a slightly different form than ours. We will sketch the idea here. The first observation is that the recurrence for appending an R to a word is significantly simpler than the recurrence for appending a U (cf. Theorem 2.6). In light of Corollary 4.2 it is enough to prove that if $H(W)$ is unimodal then $H(WR)$ is also unimodal. Let a_i be the number of lattice paths in the snake graph for W of height i and let b_i be the number of lattice paths in the snake graph for the word $\widehat{W_R}$. Then by Theorem 2.6 with c_i defined to be the number of lattice paths of height i in the snake graph for WR we have

$$c_i = a_i + b_{i-1}$$

so then to show that $H(W)$ is unimodal it suffices to show that there exists a s such that we have

$$c_0 \leq c_1 \leq \cdots \leq c_s \geq c_{s+1} \geq \cdots \geq c_{\ell(WR)+1}$$

by inductive hypothesis for some n, m we have

$$a_0 \leq a_1 \leq \cdots \leq a_n \geq a_{n+1} \geq \cdots \geq a_{\ell(W)+1}$$
$$b_0 \leq b_1 \leq \cdots \leq b_m \geq b_{m+1} \geq \cdots \geq b_{\ell(\widehat{W_R})+1}$$

One would then attempt to prove the necessary inequalities which imply the height sequence is unimodal. There are four cases to consider. Case 1 when $i < n$ and $i < m+1$, case 2 when $i \geq n$ and $i \geq m+1$, case 3 when $i \geq n$ and $i < m+1$ and case 4 when $i < n$ and $i \geq m$.

In case 1 we have injections $a_i \leq a_{i+1}$ and $b_{i-1} \leq b_i$. Hence, we have $a_i + b_{i-1} \leq a_{i+1} + b_i$, i.e. we have $c_i \leq c_{i+1}$

In case 2 we have surjections $a_i \geq a_{i+1}$ and $b_{i-1} \geq b_i$. Hence, we have $a_i + b_{i-1} \geq a_{i+1} + b_i$, i.e. we have $c_i \geq c_{i+1}$.

In case 3 we have $a_i \geq a_{i+1}$ and $b_{i-1} \leq b_i$.

In case 4 we have $a_i \leq a_{i+1}$ and $b_{i-1} \geq b_i$.

The situation is described in the following diagram:

$$c_0 \leq \cdots \leq c_j \,?\, \cdots \,?\, c_k \geq c_{k+1} \cdots \geq c_{\ell(W)+1} \tag{16}$$

With $j = \min(n, m+1)$ and $k = \max(n, m+1)$ and where the ? indicate we are in case 3 or 4 and it is not apriori clear what is going on. With this set up to show unimodality it suffices to show that if we are at position i, i.e. considering c_i and c_{i+1} if we are in case 3 or 4 and we are again in case 3 or 4 at $i+1$ then if $c_i \geq c_{i+1}$ then also $c_{i+1} \geq c_{i+2}$. The idea would to be to use strongly the relationship between a_i and b_{i-1} (i.e. the relationship between W and $\widehat{W_R}$) plus the additional information about which case we are in to prove the necessary inequalities. We remark that one could possibly combine this inductive approach with the idea outlined in Section 5.1.

6 Conjectures

In [7] an inductive proof of the unimodality of zigzag snake graphs (i.e. those which binary word looking like $RURURUR$) was given. Besides a recurrence relation on the height polynomial of such snake graphs knowledge of where the peak in the height sequence was located was also critical to their proof. In the course of our work we computed many examples of height sequences of snake graphs and found that they all satisfied what we call the snaking property. We expect that all snake graphs have this property and we hope that it will allow one to predict the mode of the height sequence of a snake graph.

Definition 6.1. A unimodal sequence (a_i) is said to have the snaking property if it has a peak element a_m such that

$$a_m \geq a_{m+1} \geq a_{m-1} \geq a_{m+2} \geq a_{m-2} \geq \dots$$

or

$$a_m \geq a_{m-1} \geq a_{m+1} \geq a_{m-2} \geq a_{m+2} \geq \dots$$

Conjecture 6.2. *The coefficients of $H(W)$ have the snaking property*

Conjecture 6.3. *The peak of the height sequence of G_W is given by $\lfloor \frac{l(W)}{2} \rfloor$ or $\lceil \frac{l(W)}{2} \rceil$*

Establishing Conjectures 6.2 and 6.3 would enable predicting the mode of some products of unimodal sequences as seen in Proposition 6.4

Proposition 6.4. *Let $A(q) = \sum a_i q^i$ and $B(q) = \sum b_i q^i$ be polynomials such that a_i is unimodal and snaking with peak a_m and b_i is symmetric and unimodal with mode b_n. Then $A(q)B(q)$ is unimodal and snaking with peak $a_m + \lfloor \frac{l(W)}{2} \rfloor$ or $a_m + \lceil \frac{l(W)}{2} \rceil$. In particular, we are interested when $A(q) = H(W)$ for some W and $B(q)$ is a product of q-integers.*

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