q-Analogues of Rational Numbers

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Definition

For each $n \in \mathbb{N}$, define the polynomial $[n]_q \in \mathbb{Z}[q]$:

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Remark: Substituting $q = 1$ gives n.

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A first natural guess for the definition is

$$\left[\frac{a}{b}\right]_q := \frac{[a]_q}{[b]_q} = \frac{1 + q + \cdots + q^{a-1}}{1 + q + \cdots + q^{b-1}}$$

We will use a different definition which uses continued fractions

Continued Fractions

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A *continued fraction* is an expression consisting of nested fractions, like this:

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Remark: These are not unique. For example, $\frac{7}{4}$ is also equal to $[1, 1, 2, 1]$. Requiring an even number of coefficients makes it unique.

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Fact: The only time this agrees with the "naive guess" is for $\left[\frac{n+1}{n}\right]_q = \frac{[n+1]_q}{[n]_q}$.

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- **Order:** Define a relation on rational functions by $\frac{a(q)}{b(q)} \succeq \frac{c(q)}{d(q)}$ if $a(q)d(q) - b(q)c(q)$ has all non-negative coefficients.
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- **Convergence:** If $\frac{a_n}{b_n} \to \lambda \in \mathbb{R}$ irrational, then $\left[\frac{a_n}{b_n}\right]_q$ “converges” in some sense, and moreover the convergence is independent of the sequence

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Example: $\frac{7}{3} = [2, 3]$ and thus has binary word $W = URR$.

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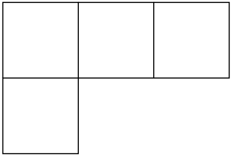

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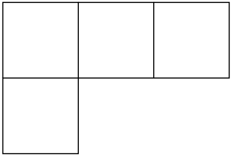
$G_{7/3} =$ 

In this way we associated a snake graph $G_{r/s}$ to a rational $\frac{r}{s}$

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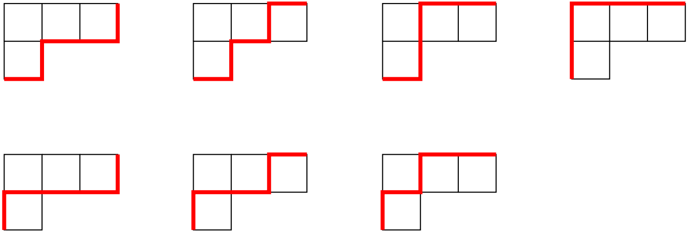
Example: The 7 lattice paths in $G_{7/3}$ are



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Theorem [Schiffler, Çanakçi]

If $\frac{r}{s} = [a_1, a_2, \ldots, a_{2m}]$ then

$$\left|L(G_{r/s})\right| = r \quad \text{and} \quad \left|L(\widehat{G}_{r/s})\right| = s$$

The notation $\widehat{G}_{r/s}$ means the snake graph from the transpose of the word associated to the continued fraction $[a_2, a_3, \ldots, a_{2m}]$.

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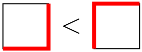

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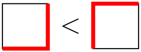
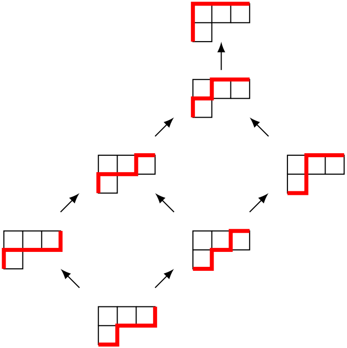


Example: $L(G_{7/3})$



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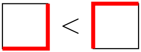

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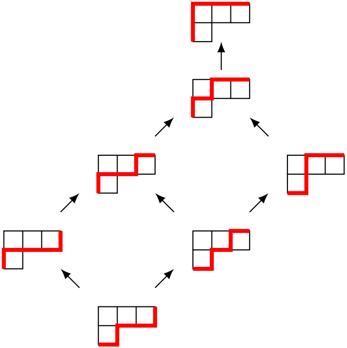

Define the *height* or *rank* of a lattice path as how many steps it takes to get to it from the minimal path. This make $L(G)$ a ranked poset.

What Do *q*-Rationals Count?

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Theorem [Claussen]

Let $\left[\frac{r}{s}\right]_q = \frac{R(q)}{S(q)}$. Then:

1. The coefficient of q^k in $R(q)$ is the number of lattice paths in $G_{r/s}$ of height k.
2. The coefficient of q^k in $S(q)$ is the number of lattice paths in $\widehat{G}_{r/s}$ of height k.

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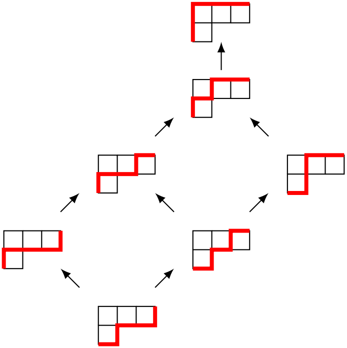

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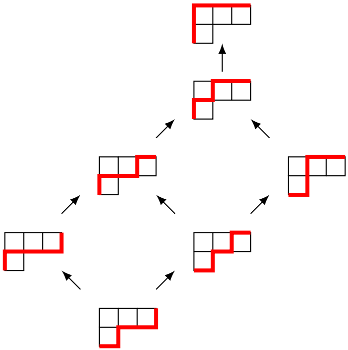

The corresponding height polynomial is $1 + 2q + 2q^2 + q^3 + q^4$ which indeed agrees with the numerator of $\left[\frac{7}{3}\right]_q$ from the continued fraction definition

Unimodal sequences

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Definition

A sequence of integers $a_0, a_1, \ldots, a_n$ is unimodal if there exits an $s \in \mathbb{N}$ such that

$$a_0 \leq \cdots \leq a_s \geq a_{s+1} \geq \ldots \geq a_n$$

A polynomial $p(q) = \sum_i p_i q^i$ is said to be unimodal if the p_i form a unimodal sequence.

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2. W is a zigzag word, i.e. there are no consecutive R's or U's in W (Fibonacci cubes are unimodal [Munarini and Salvi, 2002])
3. W is a word with isolated U's with constant row length (up-down posets are unimodal [Emden, 1982])

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If W is a word then define W^T, the transpose, to be the word formed from interchanging R with U in W.

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Theorem

If W is a binary word on $\{U, R\}$ then we have the following recurrences for the height polynomial

$$H(WU) = H(W) + q^{\ell(W)-\ell(\widehat{W_U})+1}H(\widehat{W_U})$$

and

$$H(WR) = H(\widehat{W_R}) + qH(W)$$

Code

```
R.<q> = PolynomialRing(QQ)
def word-to-num (w):
top = 1
bot = 1

for letter in w:
    if letter == 'U':
        bot = top + bot
    elif letter == 'R':
        top = top + bot
    else:
        print ("No!!!")
        raise Exception()

return top + bot
```

```
    def word_to_poly (w):
H = 1 + q
H_U_hat = 1
H_R_hat = 1
U_run = 0

for letter in w:
    if letter == 'R':
        U_run = 0
        H_U_hat = H
        H = H_R_hat + q * H
    elif letter == 'U':
        U_run += 1
        H_R_hat = H
        H = H + q^(U_run + 1)
            * H_U_hat
    else:
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Consequence: To prove that all snake graphs are unimodal it is enough to prove that if $H(W)$ is unimodal then $H(WR)$ or $H(WU)$ is also unimodal.

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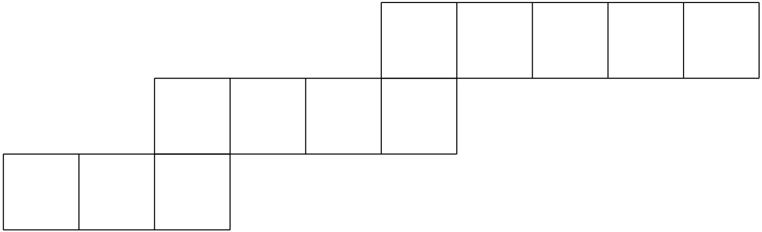

Snake graph corresponding to the word $I(2, 3, 4)$

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$$\begin{aligned} H(I(k_1, k_2)) &= [k_1 + 1]_q q^{k_2+2} + [k_1 + 2]_q [k_2 + 1]_q \\ &= \frac{-((q^3 - q^2 + q - q^{k_1+4})q^{k_2} + q^{k_1+2} - 1)}{q^2 - 2q + 1} \end{aligned}$$

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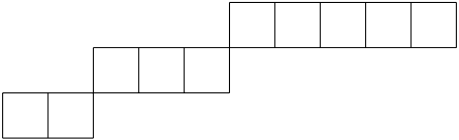
$$\begin{aligned} H(I(k_1, k_2, k_3)) &= [k_1 + 2]_q([k_2 + 1]_q [k_3 + 1]_q + q^{k_3+2}[k_2]_q) + q^{k_2+2}[k_1 + 1]_q [k_3 + 2]_q \\ &= \frac{N_3}{q^3 - 3q^2 + 3q - 1} \end{aligned}$$

with

$$\begin{aligned} N_3 = & (q^3 - q^2 + q - q^{k_1+4})q^{k_2} + \\ & + (q^3 - (q^5 - q^4 + q^3)q^{k_1} - (q^5 - q^4 + q^3 - q^{k_1+6})q^{k_2} - q^2 + q)q^{k_3} + \\ & + q^{k_1+2} - 1 \end{aligned}$$

Geometric Interpretation

Consider the following graph.



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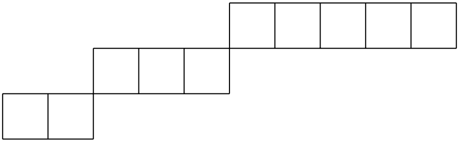
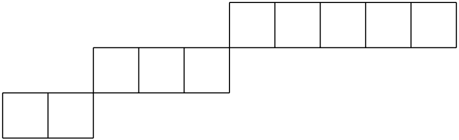


It's height polynomial counts the lattice points of a $(2+1) \times (3+1) \times (5+1)$ hyper-rectangle.

Geometric Interpretation

Consider the following graph.



It's height polynomial counts the lattice points of a $(2+1)\times(3+1)\times(5+1)$ hyper-rectangle.

Theorem

The height sequence of $R^{k_1}UR^{k_2}\cdots$ is given by

$$\prod_{i=1}^{n}[k_i+1]_q - \sum_{j=1}^{n-1}\left(x^{k_{j+1}-1}\prod_{i\notin\{j,j+1\}}[k_i+1]_q\right)+\cdots$$

Snaking Sequences

Definition

A unimodal sequence (a_i) is said to snake if it has a peak element a_m such that

$$a_m \geq a_{m+1} \geq a_{m-1} \geq a_{m+2} \geq a_{m-2} \geq \dots$$

or

$$a_m \geq a_{m-1} \geq a_{m+1} \geq a_{m-2} \geq a_{m+2} \geq \dots$$

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Conjecture

Not only are the height polynomials of lattice paths unimodal, but they also snake.

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Conclusion

Questions?