Cyclic Group Invariants and The Minimal Free Resolution of S over S^{C_n}

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Abstract

We consider the invariant ring S^{C_n} induced by the action of the cyclic group C_n on the polynomial ring S, and consider three areas related to the study of S as an S^{C_n}-module. We first find a basis for the degree d-component of the coinvariant algebra $S/(S_+^{C_n})$ for $d \geq \frac{n+1}{2}$, as well as for the values $d = 0, 1, 2, 3$. We also analyze the minimal free resolution of S as an S^{C_n}-module, and give some asymptotic behavior on the graded Betti numbers. Additionally, we provide a concrete free resolution in the case when $n = 4$. We finish by resolving one of the two conjectures needed to apply the Garsia-Stanton method, which produces a basis for S^{C_n} as an $S^{\mathfrak{S}_n}$-module.

Contents

1 Intro

The cyclic group C_n acts diagonally on the polynomial ring $S := \mathbb{C}[y_0, y_1, \dots, y_{n-1}]$ via $g \cdot y_k = \zeta_n^k y_k$, extended multiplicatively, where g is a generator for C_n and ζ_n is a primitive n^{th} of unity. We then consider the *invariant subring* S^{C_n}, which consists of all polynomials of S that are fixed under the C_n action. Letting $S_+^{C_n}$ be the elements of S^{C_n} of positive degree, we define the *coinvariant algebra* to be the quotient ring $S/(S_+^{C_n})$.

With these definitions, we can view S as a graded S^{C_n}-module. The cyclic action gives rise to the decomposition

$$S = S^{\chi_0, C_n} \oplus S^{\chi_1, C_n} \oplus \cdots \oplus S^{\chi_{n-1}, C_n},$$

where S^{χ_k} is the k^{th} *isotypic component* defined as

$$S^{\chi_k, C_n} := \{f \in S : g \cdot f = \zeta_n^k f, \text{ where } C_n = \langle g \rangle\}.$$

Now, the coinvariant algebra is an $\mathbb{N}$-graded $\mathbb{C}$-algebra and the isotypic components are $\mathbb{N}$-graded modules. We can construct the *Hilbert series* of such objects T, given by

$$\mathrm{Hilb}\,(T, t) := \sum_{d \in \mathbb{N}} t^d \cdot \dim_{\mathbb{C}} T_d,$$

where T_d is the d^{th} graded component of T. For a little more background on the invariant theory of finite groups and associated combinatorial properties, see the survey by Stanley [8]. For our special case of C_n and some of its history, see Harris and Wehlau [3].

This paper is divided into three main sections. In section 2, we first analyze the Hilbert series of the coinvariant algebra. Theorem 2.4, the main result of the section, gives both the $\mathbb{C}$-dimension and an explicit basis for the $n-i$ graded piece of $S/(S_+^{C_n})$, where $1 \leq i \leq \frac{n-1}{2}$. In particular, we show that the coefficient of t^{n-i} is $\phi(n)\sum_{j=1}^{i-1} p(j)$, where $\phi(n)$ is Euler's totient function and $p(j)$ is the number of partitions of j; the basis elements are constructed using partitions as well. The section concludes with some results about the coefficients of lower degree terms. Data for this part of the problem can be found in the Appendix.

Section 3 discusses the free resolutions of S and S^{χ_k} over the invariant ring S^{C_n}. The first major result, Theorem 3.21, considers the minimal graded free resolution of S^{χ_k} as an S^{C_4} module:

$$\cdots \xrightarrow{\phi_3^k} F_2^k \xrightarrow{\phi_2^k} F_1^k \xrightarrow{\phi_1^k} F_0^k \longrightarrow S^{\chi_k} \longrightarrow 0.$$

The theorem gives a description of all matrices of ϕ_i^k, which is then leveraged to produce a recurrence for the Betti numbers $\beta_{i,j}$. More data on the Betti numbers, which also can be found in the appendix, was ultimately folded into the following conjecture:

Conjecture. *For the minimal free resolution of S^{χ_k, C_n}, for a fixed i, the set of j so that $\beta_{i,j} \neq 0$ is precisely the interval* $[3i+1, ni+n-1]$.

Finally, in section 4 we go over the Garsia–Stanton method to compute the a basis for S as an S^{C_n}-module from their paper [2]. Theorem 4.1 proves Question 6.1 in the preprint by Reiner and White [6]. This result, combined with a proof of Question 6.2 in the same paper, would allow us to use the Garsia–Stanton method to obtain a basis for S as an S^{C_n}-module for prime n.

2 Coefficients of the Hilbert Series of the Coinvariant Algebra

In this section, we investigate the coefficients of the $\mathbb{N}$-graded Hilbert series for the coinvariant algebra $S/(S_+^{C_n})$, which we denote by $\mathrm{Hilb}(S/(S_+^{C_n}), t)$. We begin with a basic result that takes advantage of the diagonalization of the action to expresses the elements of S^{C_n} in a more manageable way.

Proposition 2.1. *A monomial $m = y_0^{a_0} y_1^{a_1} \cdots y_{n-1}^{a_{n-1}}$ lies in S^{C_n} if and only if n divides $\sum\limits_{i=1}^{n-1} i a_i$.*

Proof. Letting g be a generator for C_n, we have that

$$g \cdot m = g \cdot y_0^{a_0} y_1^{a_1} \cdots y_{n-1}^{a_{n-1}} = (\zeta_n^{0a_0} y_0^{a_0})(\zeta_n^{1a_1} y_1^{a_1}) \cdots (\zeta_n^{(n-1)a_{n-1}} y_{n-1}^{a_{n-1}}) = \zeta_n^{\sum_{i=1}^{n-1} i a_i} m. \qquad \square$$

Remark. Notice that Proposition 2.1 also holds if our monomial m was multiplied by some nonzero element of $\mathbb{C}$. However, for the purposes of studying the coinvariant algebra, since these scalars do not affect whether a monomial has nonzero image $S_+^{C_n}$, throughout we'll only be worried about monic monomials.

Now, let $m = y_0^{a_0} y_1^{a_1} \cdots y_{n-1}^{a_{n-1}}$ be a monomial in S. Then m reduces to 0 in the coinvariant algebra $S/(S_+^{C_n})$ if and only if there exist nonnegative integers $b_i \leq a_i$ such that $y_0^{b_0} y_1^{b_1} \cdots y_{n-1}^{b_{n-1}} \in S_+^{C_n}$, or equivalently, by Proposition 2.1, if and only if there exist nonnegative integers $b_i \leq a_i$, not all zero, where n divides $\sum_{i=0}^{n-1} ib_i$.

Eventually, we will be interested in restricting to a smaller subspace of S, namely the kth isotypic component of S; see section 1 of [8], for instance, on a more thorough treatment.

2.1 Coefficients of Large Degree Terms

The main result of this section is an explicit formula for the coefficients of the terms of degree at least $\frac{n+1}{2}$ in $\text{Hilb}(S/(S_+^{C_n}), t)$. In proving this result, the corresponding bases of monomials that yields these coefficients will naturally arise.

To begin, we follow the logic of Harris and Wehlau [3] and first show that such a basis will contain no monomials of degree at least n.

Lemma 2.2. *Let $\{\{a_1, a_2, \cdots, a_n\}\}$ be a multiset of n elements from $\mathbb{Z}/n\mathbb{Z}$, so they are not necessarily the same. Then some nonempty sub-multiset sums to 0 mod n; the multiset is not "zero-sum free."*

Proof. By the pigeonhole principle, out of the $n+1$ integers $0, a_1, a_1 + a_2, \cdots, a_1 + a_2 + \cdots + a_n$, there must be two that are congruent modulo n; let these be $a_1 + \cdots + a_i$ and $a_1 + \cdots + a_j$, with $0 \leq i < j \leq n$. Subtracting yields that $a_{i+1} + \cdots + a_j$ is divisible by n, and we are done. □

We can use Lemma 2.2 to show the following statement:

Proposition 2.3. *The coinvariant algebra $S/(S_+^{C_n})$ vanishes at degrees n and higher. In other words, for $i \geq n$, the coefficient of t^i in $\text{Hilb}(S/(S_+^{C_n}), t)$ is zero.*

Remark. We observe that since $y_0 \in S_+^{C_n}$, every multiple of y_0 is zero in the coinvariant algebra. Taking this fact into account, we will sometimes ignore the y_0 term when considering an arbitrary monomial in the coinvariant algebra.

Proof. It suffices to show that all monomials $m = y_0^{a_0} y_1^{a_1} \cdots y_{n-1}^{a_{n-1}}$ with degree at least n lie in the ideal $(S_+^{C_n})$. By the previous remark, we may assume that $a_0 = 0$. Proceed by constructing the multiset A that consists of a_1 copies of 1, a_2 copies of 2, and so on. An application of Lemma 2.2 to A implies that m is divisible by an element of $S_+^{C_n}$, and hence reduces to 0 in the coinvariant algebra. □

Before moving on, we fix a bit of notation for the rest of this section. The element g will always be a generator of C_n, and the *degree vector* of a monomial $m = y_0^{a_0} \cdots y_{n-1}^{a_{n-1}}$ is the vector $(a_0, \ldots, a_{n-1})$. We will associate to a partition λ of i the tuple $(\lambda(1), \lambda(2), \ldots, \lambda(i))$, where $\lambda(j)$ is the number of size j parts in the partition. We repeat that $\lambda(1)$ is the number of size 1 parts in λ, not the size of λ's largest part. Moreover, We now state the main result of this section.

Theorem 2.4. *For $1 \leq i \leq \frac{n-1}{2}$, the coefficient of t^{n-i} in $\text{Hilb}(S/(S_+^{C_n}), t)$ is*

$$\phi(n) \sum_{j=1}^{i-1} p(j), \tag{1}$$

where $p(j)$ is the number of partitions of j.

Furthermore, the monomials of degree $n - i$ in $S/S_+^{C_n}$, for these i, are precisely those of the form $y_s^\alpha y_{2s}^{\lambda(1)} y_{3s}^{\lambda(2)} \cdots y_{is}^{\lambda(i-1)}$, where $\alpha = n - i - \sum_{j=1}^{i-1} \lambda(j)$ and $(\lambda(1), \lambda(2), \ldots, \lambda(i-1))$ is a partition of some integer $0 \leq k < i$, and where $\gcd(s, n) = 1$, taking indices modulo n.

Remark. The structure of (1) shares many features with a result from Elashvilli's conjectures found in [3], and ultimately proven in [9]. That result formulates the zero-sum free sequences of length $k \geq \lfloor \frac{n}{2} \rfloor + 1$, by considering partitions of $n-k$.

We will require two lemmas and a proposition to prove Theorem 2.4. First, to formulate a basis of $S/(S_+^{C_n})$, we need to identify which monomials in S do not vanish in the quotient. This next lemma is a first step in this direction.

Lemma 2.5. *Suppose that $m_1, m_2, \ldots, m_k$ are $\mathbb{C}$-linearly independent monomials lying in S but not in the ideal $(S_+^{C_n})$. Then their images in the coinvariant ring remain $\mathbb{C}$-linearly independent.*

Proof. Suppose for contradiction that the images are dependent. Then there exist $c_1, c_2, \ldots, c_k \in \mathbb{C}$, all nonzero, such that $\sum_{i=1}^{k} c_i m_i$ lies in $(S_+^{C_n})$. Since each monomial has a distinct degree vector, each m_i must also lie in the ideal $(S_+^{C_n})$, as this ideal is generated by monomials. This gives the desired contradiction. $\square$

We now make a brief detour to consider the action of $(\mathbb{Z}/n\mathbb{Z})^\times$ on S, which will be of central importance.

Definition 1. The group $(\mathbb{Z}/n\mathbb{Z})^\times$ acts $\mathbb{C}$-linearly on S as follows: for $c \in (\mathbb{Z}/n\mathbb{Z})^\times$, we have $c \cdot y_k = y_{ck}$ taking the index modulo n and extending multiplicatively.

The action is a permutation of the indices. It also has the following useful property, which we state as a lemma.

Lemma 2.6. *For $c \in (\mathbb{Z}/n\mathbb{Z})^\times$ and m a monomial in $S_+^{C_n}$, the monomial $c \cdot m$ is also an element of the same ideal.*

Proof. Writing $m = y_0^{a_0} y_1^{a_1} \cdots y_{n-1}^{a_{n-1}}$, we have by definition that $c \cdot m = y_0^{a_0} y_c^{a_1} \cdots y_{c(n-1)}^{a_{n-1}}$, where the indices are taken modulo n. Acting on this monomial by g scales it by ζ_n raised to the power of $\sum_{i=1}^{n-1} cia_i = c\sum_{i=1}^{n-1} ia_i$, which is equivalent to 0 mod n by assumption. $\square$

Since all monomials $m' \in (S_+^{C_n})$ can be written as $m' = km$, with $m \in S_+^{C_n}$ and $k \in S$, this lemma implies that the $(\mathbb{Z}/n\mathbb{Z})^\times$ action on S restricts to an action on the ideal $(S_+^{C_n})$. Dually, we see that the action sends a monomial not in the ideal $(S_+^{C_n})$ to another monomial not in the ideal.

We now show that the coefficient of t^{n-i} in the Hilbert series is at least as large as the value given in Theorem 2.4 by explicitly constructing a list of $\mathbb{C}$-linearly independent monomials.

Proposition 2.7. *Suppose that $1 \leq i \leq \frac{n-1}{2}$. Then there exists $\phi(n) \sum_{j=1}^{i-1} p(j)$ distinct monomials in the coinvariant algebra $S/(S_+^{C_n})$ of degree $n-i$ all of the form*

$$y_s^{n-i-\sum\limits_{j=1}^{i-1}\lambda(j)} y_{2s}^{\lambda(1)} y_{3s}^{\lambda(2)} \cdots y_{is}^{\lambda(i-1)},$$

where $(\lambda(1), \lambda(2), \ldots, \lambda(i-1))$ is a partition of some integer $0 \leq k < i$, and where $\gcd(s,n) = 1$, *taking indices modulo n.*

Proof. We will construct an injection from the set of partitions of nonnegative integers k less than i to the set of monomials of degree $n-i$ and argue that the orbit of each of these elements under the $(\mathbb{Z}/n\mathbb{Z})^\times$-action are all distinct and have size $\phi(n)$.

For a partition λ of $0 \leq k < i$, map it to the degree $n-i$ monomial m_λ in S given by

$$m_\lambda := y_1^{n-i-\sum\limits_{j=1}^{i-1}\lambda_j} y_2^{\lambda(1)} y_3^{\lambda(2)} \cdots y_i^{\lambda(i-1)}. \tag{2}$$

It is not hard to see that each partition is associated to a unique monomial with distinct degree vectors, so this mapping is injective.

We now show that such monomials are nonzero in the coinvariant algebra. Seeking to apply Proposition 2.1, observe that the monomial m_λ has a weighted sum of exponents given by

$$n - i - \sum_{j=1}^{i-1} \lambda(j) + \sum_{j=1}^{i-1} (j+1)\lambda(j) = n - i + \sum_{j=1}^{i-1} j\lambda(j) = n - i + k < n, \tag{3}$$

where the last inequality follows since we are partitioning k. Since n cannot possibly divide a nonnegative integer less than itself, m_λ cannot lie in $S_+^{C_n}$, and in fact, no element of $S_+^{C_n}$ can divide m_λ for the same reason. Hence m_λ is not in the ideal $(S_+^{C_n})$, and Lemma 2.5 allows us to conclude that all elements constructed so far are linearly independent in the coinvariant algebra.

We analyze the $(\mathbb{Z}/n\mathbb{Z})^\times$-action on our constructed monomials. Since none of them lie in $(S_+^{C_n})$, Lemma 2.6 implies that their orbits do not either. Now, let m_λ and m_μ be monomials as defined in (2), not necessarily distinct. We will write them as $m_\lambda = y_1^{a_1} y_2^{a_2} \cdots y_i^{a_i}$ and $m_\mu = y_1^{b_1} y_2^{b_2} \cdots y_i^{b_i}$. To complete the proof, it suffices to show that if $c \cdot m_\lambda = m_\mu$ for $c \in (\mathbb{Z}/n\mathbb{Z})^\times$, then $c = 1$ and $m_\lambda = m_\mu$.

Using the definition of m_λ and (3), we have that

$$a_1 + a_2 + \sum_{j=3}^{i} a_j = n - i \quad \text{and} \quad a_1 + 2a_2 + 3\sum_{j=3}^{i} a_j < n,$$

meaning that $a_1 - \sum_{j=3}^{n-1} a_j > n - 2i > 0$ as $i \leq \frac{n-1}{2}$. Thus a_1 is the largest of all non-a_2 exponents; a similar argument holds for the b_j. So, the action of c on m_λ must send y_1 to either y_1 or y_2, which only happens when $c = 1$ or $c = 2$. But if $c = 2$, notice that $2i < n$, and hence in $c \cdot m_\lambda$ we do not need to take our indices modulo n as y_i is the largest possible subscript with a nonzero exponent by construction of m_λ. In particular, the y_1 coefficient of $c \cdot m_\lambda$ must be zero, and thus cannot be equal to m_μ. Hence $c = 1$ and the proof is complete. $\square$

To complete the proof of the theorem, we now need to show that these elements m_λ and their orbits under the action $(\mathbb{Z}/n\mathbb{Z})^\times$ form a basis, where λ runs over partitions of $k < i$.

Proof of Theorem 2.4. To begin, suppose that $m = y_0^{a_0} y_1^{a_1} y_2^{a_2} \cdots y_{n-1}^{a_{n-1}}$ is a monomial of degree $n - i$ whose image is nonzero in the coinvariant algebra. We may assume that $a_0 = 0$. We know that $\sum_{j=1}^{n-1} j a_j \not\equiv 0 \pmod{n}$, and moreover that no monomial of $S_+^{C_n}$ divides m. This second property, recalling Lemma 2.2, says the multiset (we will use the term "sequence") associated with m is zero-sum free. Our goal will be to construct an element $x'' \in (\mathbb{Z}/n\mathbb{Z})^\times$ that when applied to m yields a monomial of the form m_λ for some partition λ of $k < i$.

Using Theorem 7 from [7], there exists a term x' in our sequence such that our original sequence, with a_i copies of i, is instead given by the numbers $b_1 x', b_2 x', \ldots, b_{n-i} x'$, where $b_1, b_2, \ldots, b_{n-i}$ are positive integers so that $b_1 + b_2 + \cdots + b_{n-i}$ has a sum that is less than the order of x' in $\mathbb{Z}/n\mathbb{Z}$, and every integer between 1 and $b_1 + b_2 + \cdots + b_{n-i}$ is achieved as a sum of some subsequence. Notice that $1 \leq b_j < n$ as the order of x' is at most n. We then have that $b_1 + \cdots + b_{n-i} \geq n - i$, and hence the order of x' is at least $n - i > n/2$, which implies that $x' \in (\mathbb{Z}/n\mathbb{Z})^\times$. Denote its inverse by x''.

Let $m'' = x'' \cdot m = y_{x''}^{a_1} y_{2x''}^{a_2} \cdots y_{(n-1)x''}^{a_{n-1}}$, where indices are taken modulo n. But by how we constructed our sequence and the fact that $x'' \cdot x' = 1$ in $(\mathbb{Z}/n\mathbb{Z})^\times$, we see that our monomial is in fact $m'' = y_{b_1} y_{b_2} \cdots y_{b_{n-i}}$, which follows from multiplying the sequence $b_1 x', b_2 x', \ldots, b_{n-i} x'$ by x''. Notice now if r_j is unique integer at most $n - 1$ such that $r_j \equiv j x'' \pmod{n}$, then $\sum_{j=1}^{n-1} r_j a_j = \sum_{j=1}^{n-i} b_j < n$, and that this is at least $n - i$. For ease of notation, let d_j be the exponent of y_j in m'', so then we see that $\sum_{j=1}^{n-1} j d_j = \sum_{j=1}^{n-i} b_j$.

Now, consider the partition $(d_2, d_3, \ldots, d_{n-1})$. Notice that

$$\sum_{j=1}^{n-1} (j+1) d_j = \sum_{j=1}^{n-1} j d_j - \sum_{j=1}^{n-1} d_j < n - (n - i) = i,$$

meaning that this is a partition of some nonnegative integer $k < i$, and with $d_1 = n - i - \sum_{j=2}^{n-1} d_j$. It follows that m lies in the orbit of one of the monomials constructed in Proposition 2.7, which finishes the proof. $\square$

We once again emphasize that our method of proof not only yields the coefficient of the Hilbert series for degree $n - i$, where $1 \leq i \leq \frac{n-1}{2}$ but also computes an explicit basis.

2.2 Some Coefficients of Small Degree Terms

Smaller degree terms of our Hilbert series turn out to be a lot more difficult to understand. However, the coefficients in degree $0, 1, 2$, and 3 can be given explicitly:

Proposition 2.8. *For any integer n, the first terms of the Hilbert series for the coinvariant algebra are given by*

$$\mathrm{Hilb}(S/(S_+^{C_n}), t) = 1 + (n-1)t + 2\lfloor \tfrac{n}{2} \rfloor \lfloor \tfrac{n-1}{2} \rfloor t^2 + \left(\binom{n+1}{3} - (n-1)\lfloor \tfrac{n}{2} \rfloor - 2 \sum_{j=0}^{\lfloor n/3 \rfloor} (\lfloor \tfrac{n-3j}{2} \rfloor + 1) \right) t^3 + \cdots$$

Proof. By Lemma 2.5, it suffices to check that the coefficients given above the monomials of appropriate degree in S that don't lie in the ideal $(S_+^{C_n})$.

For the t^0 coefficient, the constant 1 gives the sole basis element of degree 0.

For the t^1 coefficient, note that $y_0 \in S_+^{C_n}$. But the set of all variables except for all other y_i, forms a basis, yielding $n - 1$ elements total.

For the t^2 coefficient, all degree-two monomials have the form $y_i y_j$ with $i \leq j$. To be nonzero in $S/(S_+^{C_n})$, we need $i, j \neq 0$ and since i and j take values between 1 and $n - 1$, we seek the number of pairs such that $i + j \neq n$. There are $\binom{n}{2}$ total pairs of i and j, and $\lfloor n/2 \rfloor$ solutions to $i + j = n$, and so our desired coefficient is their difference, which one can verify is $2\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$.

For the t^3 coefficient, we require more casework. The degree-3 monomials which are not contained in the ideal $(S_+^{C_n})$ are those $y_a y_b y_c$ with $1 \leq a \leq b \leq c \leq n-1$, such that no nonempty sub-multiset of $\{a, b, c\}$ sums to a multiple of n. There are $\binom{n+1}{3}$ total monomials $y_a y_b y_c$, and so we instead subtract from $\binom{n+1}{3}$ the number of triples which do have a sub-multiset that sums to a multiple of n.

We can automatically rule out the possibility that any a, b, or c is a multiple of n, because their values lie in the range $1, \ldots, n-1$. First, suppose that a pair of elements in $\{a, b, c\}$ sum to a value divisible by n; this value must be n itself. As mentioned above, there are $\lfloor \frac{n}{2} \rfloor$ such pairs. The final element can take any of the other $n - 1$ values. After verifying that we are not overcounting, we subtract $(n-1)\lfloor \frac{n}{2} \rfloor$ from our running total, as in the formula for the coefficient.

Finally, suppose n divides $a + b + c$. It is evident that no pair of $\{a, b, c\}$ can sum to n, for then the remaining index would be 0, which is not permitted. Note that we need only double-count the triples for which $a + b + c$ is n, because $a + b + c = 2n$ if and only if $(n - a) + (n - b) + (n - c) = n$, and we note that $n - a, n - b, n - c$ also take values between 1 and $n - 1$. Now, for each value of a, we have that $(b - a) + (c - a) = n - 3a$, and if we count the cases for which $b - a$ and $c - a$ are nonnegative, we obtain $\lfloor \frac{n-3a}{2} + 1 \rfloor$. Summing over possible values of a and subtracting this from the running total twice yields the t^3 coefficient, and completes the proof. $\square$

The t^3 coefficient already starts to look a little unwieldly, so it would serve us well to analyze a smaller space. Indeed, it turns out that the χ_1-component of S is a little more well-behaved than the whole ring S.

Proposition 2.9. *The t^3 coefficient of $\mathrm{Hilb}(S^{\chi_1}/(S_+^{C_n}), t)$, is $\sum_{i=0}^{n-1} \lfloor \frac{i}{3} \rfloor$.*

Proof. We proceed by induction. The base cases $n = 0, 1, 2, 3$ can be manually verified. Now, suppose we have shown that $|A_k| = \sum_{i=0}^{k-1} \lfloor \frac{i}{3} \rfloor$, where

$$A_k = \{(a, b, c) : 1 \leq a \leq b \leq c \leq n-1 \text{ and } a + b + c \equiv 1 \pmod{n}\},$$

so that $\{y_a y_b y_c : (a, b, c) \in A_k\}$ is a basis for the degree-3 elements of $S^{\chi_1}/(S_+^{C_n})$.

For any $(a, b, c) \in A_k$, since a, b, c all lie between 1 and $k-1$ inclusive, the sum $a+b+c$ must either be $k+1$ or $2k+1$. One can then verify that in the first case, $(a, b, c+1) \in A_{k+1}$ and that in the second case, $(a, b+1, c+1) \in A_{k+1}$; note that this requires checking that no pair in either tuple can sum to $k+1$. This correspondence gives us an injection from A_k to A_{k+1}.

The elements of A_{k+1} that are not obtained via this injection from A_k are the solutions to either $a+b+c = k+2$ and $b = c$, or $a+b+c = 2k+3$ and $a = b$. We can rewrite the first case as the equation $a + 2c = k+2$, where since $a \leq c$ we see that a can vary from 2 to $\lfloor \frac{k+2}{3} \rfloor$ and must take on the same parity as k. In the second case, $(k+1-c) + 2(k+1-a) = k$; this implies that the value $k+1-c$ can vary from 1 to $\lfloor k/3 \rfloor$ and must also have the same parity as k.

Putting these two together, we see that $a-1$ ranges from 1 to $\lfloor \frac{k+2}{3} \rfloor - 1$ with values of opposite parity compared with that of $k+1-c$. When $k \equiv 1, 2 \pmod 3$, we have $\lfloor \frac{k+2}{3} \rfloor - 1 = \lfloor \frac{k}{3} \rfloor$, which implies $\lfloor \frac{k}{3} \rfloor$ new solutions. Otherwise, k is divisible by 3, and one sees that each value in the range from 1 to $\frac{k}{3}$ affords a new solution as well. So in all cases the jump from A_k to A_{k+1} yields $\lfloor \frac{k}{3} \rfloor$ new elements, which completes the induction. $\square$

3 The Free Resolution of S as an S^{C_n}-module

In this section, we consider the free resolution of the polynomial ring $S = \mathbb{C}[\underline{y}] := C[y_0, y_1, y_2, \ldots, y_{n-1}]$ as a module over the invariant subring $R = \mathbb{C}[\underline{y}]^{C_n}$. We begin by reviewing some terms related to free resolutions, following Peeva [5].

Given a finitely generated multi-graded module M over a multi-graded Noetherian ring R, a *(multi-graded) free resolution of M* is an exact sequence

$$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to M \to 0,$$

where the modules F_i are free multi-graded R-modules, and the map between F_i and F_{i-1}, denoted ϕ_i, is a multi-degree preserving R-module homomorphism.

Multi-graded free resolutions exist for any R-module M (see [5, Construction 4.2], for instance). We record for completeness some elementary facts about free resolutions in the following proposition.

Proposition 3.1. *Fix a free resolution* $\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to M \to 0$ *and homogeneous bases* $\{e_{i,j}\}$ *for each* F_i. *Then:*

(a) *All of the entries of the matrix of ϕ_i with respect to those bases are monomials.*

(b) *For a fixed row r of ϕ_i and column c of ϕ_{i+1}, each nonzero product of the k^{th} entry of r and the k^{th} entry of c has the same multi-degree.*

(c) *If two columns c, d have nonzero monomials m_1, m_2 in the k^{th} entry and m_3, m_4 in the l^{th} entry (respectively), then $m_1/m_2 = m_3/m_4$.*

Example. Suppose that we have $i = 1$, and we are looking at the free resolution of S^{χ_1, C_4}. Then, if we take the first row of ϕ_1, this is $\begin{pmatrix} y_2y_3^2 & y_1y_2^2y_3 & y_1^3y_2y_3 & 0 & 0 & 0 & y_3^4 & y_1^3y_3^3 \end{pmatrix}$, and the first column of ϕ_2, which is $\begin{pmatrix} y_1^2y_2 & -y_1y_3 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$, our pairwise products are all scalar multiples of $y_1^2y_2^2y_3^2$.

Moreover, we can see that if we pick the second column of ϕ_2, which is $\begin{pmatrix} y_1y_2^2y_3 & -y_1^2y_2 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$, we see that $\frac{y_1^2y_2}{y_1y_2^2y_3} = \frac{-y_1y_3}{-y_3^2y_2} = \frac{y_1}{y_2y_3}$.

For our purposes, all of the free resolutions we take will have ϕ_i maps that preserve multi-degree, and we will always aim to pick basis elements that homogeneous in multi-degree.

A way of characterizing these multi-graded free resolutions is through the notion of the Betti numbers.

Definition 2 (see Construction 3.2 and Definition 12.1 of [5])**.** Given a free resolution of a finitely generated $R-$module M by

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

we can write each free module F_i as the direct sum $\bigoplus_{j=1}^{n} R(-d_j)$, where $R(-d_j)$ represents a copy of R shifted by degree d_j. The *Betti number* $\beta_{i,j}$ is the number of copies of $R(-j)$ appearing in the direct sum of F_i.

We can define the multi-graded Betti number similarly, with the number of direct sum copies of $R(-\mathbf{j})$ and dealing with the Betti numbers $\beta_{i,\mathbf{j}}$.

Of particular interest to us will also be minimal free resolutions. In particular, we say that our free resolution is *minimal* if no invertible elements appear as entries in the matrices; in particular, for our example, this means that each of our ϕ_i maps will not have any elements of $\mathbb{C}$. It is also known that the following theorem holds.

Theorem 3.2 (Theorem 7.3 from [5]). *A free resolution is minimal if and only if the images under ϕ_i of any (hence, every) compatible free basis minimally generates the kernel of ϕ_{i-1}.*

Both of these equivalent definitions will be employed throughout the paper; when we discuss the free resolution in the $n = 4$ case we will usually default to this latter condition.

We can also see that a minimal generating set for the kernel of ϕ_{i-1} lifts to a free basis of F_i, and vice versa, under the action of ϕ_i. This is the following lemma.

To simplify, we will consider the free resolution of the submodule S^{χ_k}, the kth isotypic component of S. We begin with some general results. Then, after reviewing results about free resolutions when $n = 2, 3$, we proceed to a more specific description of the resolution in the case when $n = 4$. In the Appendix, we include some data and `Macaulay2` code for small values of n.

We can also simplify our work by noting that if $g \in (\mathbb{Z}/n\mathbb{Z})^\times$, we can consider the action of g by $g(y_i) = y_{gi}$ (which we first saw back in Section 2). This gives us an isomorphism between S^{χ_k} and $S^{\chi_{gk}}$ for each k, and so for our minimal free resolution it suffices to just consider a single element per orbit.

To simplify the language in this section, we introduce the following terminology, reminiscent of that in [3].

Definition 3. A monomial $m \in S$ is *indecomposable* if and only if it doesn't have a proper divisor $m' \in S_+^{C_n}$. Notice that there can be some elements in $S_+^{C_n}$ that are indecomposable, such as y_0.

3.1 Asymptotics on the Ranks of the Free Modules

We begin with some basic properties about the general free resolution for S^{χ_i} as a S^{C_n}-module, before finishing with some still-open conjectures. For our purposes we will only be concerned with the free resolutions of $n \geq 3$ in this section (which will also allow us to avoid some annoying edge cases).

For the sake of simplicity, we only concern ourselves with the minimal free resolutions of S^{χ_1, C_n} and $S^{\chi_1, n-1}$, whose spaces we denote as

$$\cdots \to F_2 \to F_1 \to F_0 \to S^{\chi_1, C_n} \to 0.$$

Using the action of $(\mathbb{Z}/n\mathbb{Z})^\times$ on S, we see that these results also hold if χ_1 is replaced by χ_k, where k is relatively prime to n.

The bounds that we have on F_i can be summarized in the following proposition. Here, we will use rank M to denote the rank of M as an S^{C_n}-module.

Proposition 3.3. *Given the free resolution of S^{χ_1, C_n} above, we have the following inequality:*

$$(\dim_{\mathbb{C}}(S/(S_+^{C_n}))^{\chi_1})(\dim_{\mathbb{C}}(S/(S_+^{C_n}))^{\chi_1} - 1)^i \leq \operatorname{rank} F_i \leq n^{(i+1)n}((i+1)!)^n.$$

First, we will demonstrate the lower bound on the rank of the free modules. To do this, we construct a subset of minimal generators for each F_i, and also for $\ker \phi_{i+1}$.

For the sake of convenience, we write $\iota(i)$ to denote 1 if i is odd and $n-1$ if i is even.

Definition 4. We label the indecomposable monomials in S^{χ_1, C_n} by $m_{1,1}, m_{2,1}, \ldots, m_{\dim_{\mathbb{C},1} S/S_+^{C_n}}$, such that $m_{1,1} = y_1$.

Definition 5. We define the sets B_i and M_{i+1}, where $i = 0, 1, 2, \dots$, as follows.

- Let $B_0 = \{\mathbf{e}_{0,1}, \dots, \mathbf{e}_{0,\dim F_0}\}$ be a free basis for F_0 such that $\phi_0(\mathbf{e}_{0,j}) = m_{j,1}$.
- Let
$$M_1 = \{cm_{j,1}\mathbf{e}_{0,1} - cm_{1,1}\mathbf{e}_{0,j} : 2 \le j \le |B_0|, \text{ and } c \in S^{\chi_{n-1},C_n} \text{ indecomposable}\}.$$
Since $cm_{j,1}\mathbf{e}_{0,1} - cm_{1,1}\mathbf{e}_{0,j}$ is in the kernel of ϕ_0, it has a preimage under ϕ_1; let this be $\mathbf{e}_{1,j,c}$. Further, since $cm_{j,1}\mathbf{e}_{0,1} - cm_{1,1}\mathbf{e}_{0,j}$ is homogeneous and ϕ_1 preserves degree, $\mathbf{e}_{1,j,c}$ is homogeneous. Note that $\deg \mathbf{e}_{1,j,c} - \deg \mathbf{e}_{1,j,d} = \deg c - \deg d$, and that $\phi_1(d\mathbf{e}_{1,j,c} - c\mathbf{e}_{1,j,d}) = 0$, for all possible c, d. Set B_1 to be the set of all of these $\mathbf{e}_{1,j,c}$. Assign a total ordering to the elements of B_1.
- Given M_i and totally ordered set B_i, where the elements of B_i are denoted as $\mathbf{e}_{i,j,c}$ with $\deg \mathbf{e}_{i,j,c} - \deg \mathbf{e}_{i,j,d} = \deg c - \deg d$ and $\phi_i(d\mathbf{e}_{i,j,c} - c\mathbf{e}_{i,j,d}) = 0$ for all possible c, d, define
$$M_{i+1} = \{c'c\mathbf{e}_{i,j,y_{\iota(i)}} - c'y_{\iota(i)}\mathbf{e}_{i,j,c} : c' \in S^{\chi_{n-\iota(i)},C_n} \text{indecomposable}, \mathbf{e}_{i,j,c} \in B_i, c \neq y_{\iota(i)}\}.$$
Note that all such elements in M_{i+1} are homogeneous and in the kernel of ϕ_i. Let $\mathbf{e}_{i+1,j',c'}$ be the homogeneous preimage of $c'c\mathbf{e}_{i,j,y_{\iota(i)}} - c'y_{\iota(i)}\mathbf{e}_{i,j,c}$, where $\mathbf{e}_{i,j,c}$ is the $(j')^{\text{th}}$ element of B_i. We also have $\deg \mathbf{e}_{i+1,j,c} - \deg \mathbf{e}_{i+1,j,d} = \deg c - \deg d$ and $\phi_1(d\mathbf{e}_{i+1,j,c} - c \cdot \mathbf{e}_{i+1,j,d}) = 0$ for all possible c, d.

  Set B_{i+1} to be the set of all $\mathbf{e}_{i+1,j',c'}$, and totally order B_{i+1}.

As we inductively show in the definition, each M_i is a subset of $\ker \phi_{i-1}$, and therefore of $\operatorname{im} \phi_i$. Also note that B_i is a subset of F_i, with $\phi_i(B_i) = M_i$. We first demonstrate the size of M_i.

Proposition 3.4. *We have*
$$M_i = (\dim_{\mathbb{C}}(S/(S_+^{C_n}))^{\chi_1})(\dim_{\mathbb{C}}(S/(S_+^{C_n}))^{\chi_1} - 1)^i.$$

Proof. For simplicity, write $N = \dim_{\mathbb{C}}(S/(S_+^{C_n}))^{\chi_1}$. Notice that this is the number of indecomposable monomials in S^{χ_1,C_n}, since the images of these indecomposable monomials under the quotient map to $(S/(S_+^{C_n}))^{\chi_1}$ form a basis of $(S/(S_+^{C_n}))^{\chi_1}$.

We proceed by induction on i to prove a stronger statement: M_i consists of $N(N-1)^i$ elements, of which $(N-1)^{i+1}$ of these are such that their images under ϕ_{i-1} have both coefficients of degree at least 3. We will later see why we care about this second quantity.

For the base case, M_1 consists of elements $cm_{j,1}\mathbf{e}_{0,1} - cm_{1,1}\mathbf{e}_{0,j}$ where there are N choices for c and $N-1$ choices for j, since $\dim_{\mathbb{C}}(S/(S_+^{C_n}))^{\chi_1} = \dim_{\mathbb{C}}(S/(S_+^{C_n}))^{\chi_{n-1}}$ due to the action of $\mathbb{Z}/n\mathbb{Z}$. Then M_1 has size $N(N-1)$, and $(N-1)^2$ of its elements have both coefficients (namely, $cm_{j,1}$ and $cm_{1,1}$) of degree at least 3 (since in S^{χ_k} the only element of degree 1 is y_k).

Now suppose that M_i consists of $N(N-1)^i$ binomials, have both coefficients of degree at least 3. By definition, we know that every element of M_{i+1} is of the form $c'c\mathbf{e}_{i,j,y_{\iota(i)}} - c'y_{\iota(i)}\mathbf{e}_{i,j,c}$. We will now count how many such binomials we have, and how many of these have coefficients $c'c, c'y_{\iota(i)}$ both of degree at least 3. To do this, recall that, as c' is allowed to vary over the indecomposables of $S^{\chi_{n-\iota(i)},C_n}$, there are N possible values of c'. We now need to count the number of elements $\mathbf{e}_{i,j,c}$ where $c \neq y_{\iota(i)}$.

To count this, recall that we have $N(N-1)^i$ elements in M_i, and hence in B_i. We claim that the number of such $\mathbf{e}_{i,j,c}$ is the number of elements in M_i where both coefficients are of degree at least 3. Indeed, first suppose that $c = y_{\iota(i)}$. Then, have that $\phi_i\mathbf{e}_{i,j,c} = y_{\iota(i-1)}y_{\iota(i)}\mathbf{e}_{i-1,j',c''} - y_{\iota(i)}c''\mathbf{e}_{i-1,j',y_{\iota(i-1)}}$ for some $c'' \in S^{\chi_{n-\iota(i-1)},C_n}, c'' \neq y_{\iota(i-1)}$. But then $\mathbf{e}_{i+1,j,c}$'s image has a coefficient of degree 2 on one of the elements. Furthermore, if $c \neq y_{\iota(i+1)}$, it's not hard to see that both cc'' and $cy_{\iota(i)}$ are degree at least 3. Therefore, the number of $\mathbf{e}_{i,j,c}$ in B_i where $c \neq y_{\iota(i)}$ is precisely the number of elements in M_i with both coefficients degree at least 3. This means that there are $(N-1)^{i+1}$ choices for $\mathbf{e}_{i,j,c}$, and gives our total of $N(N-1)^{i+1}$ elements in M_{i+1}.

Furthermore, of these choices of c', $N-1$ have both coefficients of degree at least 3, namely the one where c' just not equal to $y_{n-\iota(i)}$. ∎

The reason we care about the growth of M_i is that the elements in M_i can all be taken to be a subset of a minimal generating set for $\ker\phi_{i-1}$. Before we show this lemma, we first prove another lemma, which justifies why we are allowed to construct our set B_i that is a subset of a free basis for F_i.

Lemma 3.5. *Given a minimal free resolution for a module M, given by*

$$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to M \to 0,$$

we can lift any homogeneous minimal generating set of $\ker\phi_{i-1}$ to a homogeneous free basis of F_i. Conversely, the image of a homogeneous free basis of F_i is a homogeneous minimal generating set of $\ker\phi_{i-1}$.

Proof. Suppose $\{x_1, x_2, \ldots, x_n\}$ is a minimal generating set of $\ker\phi_{i-1}$. We know from [5, Theorem 7.3], in the construction of our free resolution, that there is some minimal generating set $\{y_1, y_2, \ldots, y_m\}$ that gets lifted by ϕ_i to a free basis, and from this construction the y_i and their preimages can be chosen to be homogeneous. Then, we can write $x_k = \sum a_{kj} y_j$.

But by part (4) of [5, Theorem 2.12], we can assume that the a_{kj} are homogeneous and of degree $\deg(x_k) - \deg(y_j)$, and furthermore this change of coordinates is invertible. Therefore, notice that, if $\mathbf{e}_j \in F_i$ is so that $\phi_i(\mathbf{e}_j) = y_j$, then $\sum a_{kj}\mathbf{e}_j$ gets mapped by ϕ_i to x_k. But as we observed, this change of coordinates is invertible, and so we have a homogeneous free basis that maps to x_k, as desired.

For the other direction, suppose that this isn't the case. It's not hard to see that the images under ϕ_i of this free basis generate $\ker\phi_{i-1} = \operatorname{im}\phi_i$; we just need to show that it is minimal. But if it isn't, then there is some smaller subset of this that is a minimal generating set.

Taking the other direction, which we've proven, yields us that the preimages of this subset can be chosen to be a free basis of F_i. But this is a contradiction, since free bases of a free module must have the same size. $\square$

As such it doesn't particularly matter, given a minimal free resolution, if we pick a different minimal generating set for $\ker\phi_{i-1}$; we will still get another free basis for the next free module. This is also related to [5, Theorem 7.5] with the uniqueness of minimal free resolutions, up to some change of basis.

Remark. In fact, it doesn't matter which homogeneous preimages are chosen in the above lemma. We can do this by adding elements in the kernel of ϕ_i to our free basis, one at a time. We start with $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ as a free basis for F_i, and add $k \in \ker\phi_1$ to $\mathbf{e}_1$.

Then, notice that if $\sum_{j=2}^{n} c_j\mathbf{e}_j + c_1(\mathbf{e}_1 + \mathbf{v}) = 0$, it follows that, considering the coefficient of $\mathbf{e}_1$ in the expansion, the coefficient of $\mathbf{e}_1$ in $\mathbf{v}$ has to be -1. But then $\phi_i(\mathbf{e}_1)$ can be written in terms of the other $\phi_i(\mathbf{e}_j)$, meaning that the image of our free basis under ϕ_i isn't a minimal generating set, contradiction.

With this in mind, we now are ready to state our lemma.

Lemma 3.6. *There exists a homogeneous minimal generating set for $\ker\phi_{i-1}$ that contains M_i, for $i = 1, 2, \ldots$.*

Proof. We proceed by induction. Our base case follows a very similar argument to the inductive step, so we present the inductive step first. Suppose that we've shown this holds for $i-1$. Notice that this statement implies that B_{i-1} is a subset of a free basis for F_{i-1}, from Lemma 3.5 (since if we extend M_{i-1} to a homogeneous minimal generating set, then we can pick out B_{i-1} as a subset of the preimages, which are a homogeneous free basis for F_i).

First, we extend M_i to a finite homogeneous generating set X of $\ker\phi_{i-1}$; for instance, take a finite homogeneous generating set for $\ker\phi_{i-1}$ and then append it to M_i. From here, we run the following process: we remove an element of X that is a linear combination of others. We repeat this process until X becomes a minimal generating set, at which point such an element no longer exists. We will show that we can arrange this removal of elements in a way so $M_i \subset X$ at every point.

To argue this, suppose at some point in this process we pick an element that lies in M_i, say

$$c'c\mathbf{e}_{i-1,j,y_{\iota(i-1)}} - c'y_{\iota(i-1)}\mathbf{e}_{i-1,j,c},$$

where $c' \in S^{\chi_{n-\iota(i-1)},C_n}$ and suppose it equals $\sum b_i \mathbf{x}_i$, where the sum is over all other elements in X. If the multi-degree of

$$c'c\mathbf{e}_{i-1,j,y_{\iota(i-1)}} - c'y_{\iota(i-1)}\mathbf{e}_{i-1,j,c}$$

is $\mathbf{d}$, then take the $\mathbf{d}-$homogeneous component of this equation. As each x_i is homogeneous, this amounts to taking the homogeneous component of b_i such that $b_i\mathbf{x}_i$ has multi-degree $\mathbf{d}$.

Now, consider the term $-c'y_{\iota(i-1)}\mathbf{e}_{i-1,j,c}$. From our inductive hypothesis, our elements $\mathbf{e}_{i-1,j,c} \subset B_{i-1}$ are a subset of a free basis of F_{i-1}. Then, in order for the equation

$$c'c\mathbf{e}_{i-1,j,y_{\iota(i-1)}} - c'y_{\iota(i-1)}\mathbf{e}_{i-1,j,c} = \sum b_i\mathbf{x}_i$$

to hold, some $b_i\mathbf{x}_i$ must have a nonzero coefficient for $\mathbf{e}_{i-1,j,c}$ which we can write as $b_i d$. Here, d is the coefficient of $\mathbf{e}_{i-1,j,c}$ in the expansion of $\mathbf{x}_i$ with respect to the our free basis for F_{i-1}. Furthermore, $b_i d$ must be a nonzero scalar multiple of $c'y_{\iota(i-1)}$, by homogenity. But as $c' \in S^{\chi_{n-\iota(i-1)},C^n}$ is indecomposable by definition of M_i, and $y_{\iota(i-1)}$ is a single variable, we can see that $c'y_{\iota(i-1)}$ is indecomposable too. Otherwise, $c'y_{\iota(i-1)}$ splits into two proper factors that lie in S^{C_n}; one of these divides c' and is a proper factor of c', which is a contradiction. Therefore, d is either in $\mathbb{C}$, or d is a $\mathbb{C}$-multiple of $c'y_{\iota(i-1)}$.

However, $\mathbf{x}_i \in X$, and so $\mathbf{x}_i \in \ker\phi_{i-1}$. Thus, since our free resolution is minimal, d cannot be a scalar. Therefore, d is a $\mathbb{C}$-multiple of $c'y_{\iota(i-1)}$. But therefore it follows that $b_i \in \mathbb{C}$. This means that we can re-arrange our linear combination so $\mathbf{x}_i$ is on the left-hand side.

This element $\mathbf{x}_i$ necessarily cannot lie in M_i, since at most one element of M_i contains a term that is a scalar multiple of $-c'y_{\iota(i-1)}\mathbf{e}_{i-1,j,c}$. We may thus replace our element in M_i and remove $\mathbf{x}_i$ instead. This can be done until we arrive at a minimal generating set, proving the lemma and showing that each element of M_i lies in our minimal generating set.

For our base case $i = 1$, we perform the above procedure, except instead of $\mathbf{e}_{i-1,j,c}$ we are considering coefficients of $\mathbf{e}_{0,j}$, which are given to be a free basis of F_0. Then, if we augment M_1 to a generating set, and remove redundant elements, again suppose at some point we end up choosing an element of M_1 to remove, say $cm_{j,1}\mathbf{e}_{0,1} - cm_{1,1}\mathbf{e}_{0,j}$. Then, this is equal to a linear combination $\sum c_j\mathbf{x}_j$ of elements $\mathbf{x}_j$ within our generating set. Again, similarly to how we argued above, there has to exist some $\mathbf{x}_j$ so that, when written in terms of the free basis B_0, the coefficient of $\mathbf{e}_{0,j}$ in $\mathbf{x}_j$ is a $\mathbb{C}$-multiple of $cm_{1,1}$, and so we may re-arrange our equation to remove this $\mathbf{x}_j$ instead.

Again, by construction, there is only one such element in M_1 containing a $cm_{1,1}\mathbf{e}_{0,j}$ term, so we have been able to remove an element not in M_1. This proves the claim. □

Because we know some of the elements that can be taken in a minimal generating set, we are now able to provide some lower bounds for the Betti numbers and the rank of the free modules. We first have the following immediate corollary, which gives us the first part of the Proposition 3.3.

Corollary 3.7. *The rank of F_i has degree at least*

$$(\dim_{\mathbb{C}}(S/(S_+^{C_n}))^{\chi_1})(\dim_{\mathbb{C}}(S/(S_+^{C_n}))^{\chi_1} - 1)^i.$$

Proof. In our free resolution, we have that the rank of F_i is equal to the rank of the kernel of ϕ_{i-1}. But we see that the minimal generating set of $\ker\phi_{i-1}$ contains M_i, meaning that the rank of F_i is at least $|M_i|$, which by Proposition 3.4 is equal to $(\dim_{\mathbb{C}}(S/(S_+^{C_n}))^{\chi_1})(\dim_{\mathbb{C}}(S/(S_+^{C_n}))^{\chi_1} - 1)^i$. □

We also can show that certain Betti numbers are nonzero.

Corollary 3.8. *In the resolution of S^{χ_1,C_n} as a S^{C_n} module, $\beta_{i,j} \neq 0$ for all $i \geq 0, j \in [3i+1, ni+n-1]$.*

Proof. We will prove by induction that all these degrees are attained by M_i; Lemma 3.6 will allow us to then conclude the corollary.

We have two base cases. First, for $i = 0$, notice that F_0 has the same degrees as the indecomposable elements of S^{χ_1}. But these range from $[1, n-1]$, and it's not hard to see that each of these are attained.

Next, for $i = 1$, notice that elements in M_1 have degree $1 + \deg(\mathbf{e}_j) + \deg(c)$. As $\deg(\mathbf{e}_{0,j})$ ranges over $[2, n-1]$ and c ranges over $[1, n-1]$, we see that all integers in $[4, 2n-1]$ are degrees of elements in M_1, and thus, by Lemma 3.6 in a minimal generating set for $\ker \phi_0$. Hence $\beta_{1,j} \neq 0$ for each $j \in [4, 2n-1]$, as desired.

From here, suppose that all degrees $[3j+1, nj+n-1]$ are attained among the elements of F_j for $j = i$; we will also need our result that this holds for $i = 0$. For each degree $d \in [3i+1, ni+n-1]$, there is a basis element in B_i of the form $\mathbf{e}_{i,j,c}$ where $c \neq y_{\iota(i)}$ such that its degree is either d or $d+1$.

Indeed, if there are no such elements for d, because by the inductive hypothesis d is attainable if we are allowed to pick any elements in M_i (and hence B_i), it follows that some element of the form $\mathbf{e}_{i,j,y_{\iota(i)}}$ has degree d. But then $\mathbf{e}_{i,j,y_{2\iota(i)}y_{n-\iota(i)}}$ has degree $d+1$ (taking indices modulo n), which is what we want. In particular, observe that there is some element $\mathbf{e}_{i,j,c}$ of degree either $3i+1$ or $3i+2$, where $c \neq y_{\iota(i)}$, and there is also such an element of degree either $ni+n-1$ or $ni+n$. So, the range $[3i+1, ni+n]$ never has two consecutive numbers that are not the degree of such an $\mathbf{e}_{i,j,c}$.

Now we consider the elements in M_{i+1}, which have degree equal to $\deg c' + 1 + \deg \mathbf{e}_{i,j,c}$ for $c \neq y_{\iota(i)}$, $c' \in S^{\chi_{n-\iota(i)}}$. Note that $\deg c'$ ranges over $[1, n-1]$, and the degree of $\mathbf{e}_{i,j,c}$ is equal to that of its corresponding element $\phi_i(\mathbf{e}_{i,j,c})$ in M_i. Then if $\mathbf{e}_{i,j,c}$ has degree d, and $c \neq y_{\iota(i)}$, the elements $c''c\mathbf{e}_{i,j,\iota(i)} - c''y_{\iota(i)}\mathbf{e}_{i,j,c}$ have degrees in $[d+2, d+n]$ (and all are attained).

Ranging over all possible values for d, we now argue that every integer in $[3i+4, ni+2n-1]$ must therefore be the degree of an element in M_{i+1}. If any d' is not attained in this way, then none of $d'-n, d'-n+1, \ldots, d'-2$ are possible degrees for $\mathbf{e}_{i,j,c}$, where $c \neq y_{\iota(i)}$. But for $d' \in [3i+3+1, ni+2n-1]$, the interval $[3i+1, ni+n]$ contains either $d'-1$ and $d'-2$, or $d'-n$ and $d'-n+1$. So at least one of $d'-n, d'-n+1, \ldots, d'-2$ is a degree of some $\mathbf{e}_{i,j,c}$ for $c \neq y_{\iota(i)}$ and we are done. $\square$

We will now also bound the rank of the free modules from above. To do this, we bound the multi-degrees of the elements in the free modules F_i, in a way that we will introduce below. For this, the following definitions will be useful:

- Given a multi-graded ring or module M and homogeneous element $\mathbf{e}_j \in M$, let $\mathrm{md}(\mathbf{e}_j)$ be the multi-degree of $\mathbf{e}_j$.
- For a multi-degree $\mathbf{a}$, we define $\mathbf{y}^{\mathbf{a}}$ to be the monic monomial with multi-degree $\mathbf{a}$.
- Given two multi-degrees $\mathbf{a}$ and $\mathbf{b}$, we use the notation $\mathbf{a} \leq \mathbf{b}$ to mean that the coordinate-wise difference $\mathbf{b} - \mathbf{a}$ has only nonnegative entries.
- Given homogeneous elements $a_1, a_2, \ldots, a_k \in M$, let $\ell(a_1, a_2, \ldots, a_k)$ be the coordinate-wise maximum of the multi-degrees of $a_1, a_2, \ldots, a_k$. Similarly, define $g(a_1, a_2, \ldots, a_k)$ to be the coordinate-wise minimum of the multi-degrees of $a_1, a_2, \ldots, a_k$.

The notation is intended to be suggestive. If we are working in a polynomial ring, then the a_i are monomials, and ℓ gives the multi-degree of the least common multiple of the monomials. In general, $\leq$ is a partial ordering on multidegrees, for which ℓ and g are the meet and join operations, respectively.

For instance, if a_1 has multi-degree $(1,2,0)$, a_2 has multi-degree $(3,1,1)$, and a_3 has multi-degree $(2,2,4)$, $l(a_1, a_2, a_3) = (3,2,4)$.

We will be using these definitions both on the free modules F_i in our free resolution, as well as the polynomial ring S.

Proposition 3.9. *Suppose $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ are a free basis of homogeneous elements in F_i. Then,*

$$\ell(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m) \leq ((i+1)(n-1), \ldots, (i+1)(n-1)).$$

Proof. We proceed by induction on i. For our base case, we take $i = 0$. The degree-preserving map ϕ_0, by definition, sends each element $\mathbf{e}_i$ in our basis of F_0 to some monomial (up to a factor in $\mathbb{C}$ which doesn't affect degree) in S^{χ_k, C_n}. Then by Proposition 2.3, each $\mathbf{e}_i$, has degree at most $n-1$. In particular, each

$\mathrm{md}(\mathbf{e}_i) \leq (n-1, n-1, \dots, n-1)$, so the coordinate-wise maximum of the multi-degrees of all of the $\mathbf{e}_i$ is at most $(n-1, n-1, \dots, n-1)$. Thus $\ell(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m) \leq (n-1, n-1, \dots, n-1)$, proving our base case.

For the inductive step, suppose we've shown that, for a free basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m$ of F_i, we have that $\ell(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m) \leq ((i+1)(n-1), (i+1)(n-1), (i+1)(n-1), \dots, (i+1)(n-1))$. Consider a free basis $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_{m'}$ of F_{i+1}. By exactness, we know that $\phi_{i+1}(\mathbf{f}_j)$ lies in the kernel of ϕ_i for each j. By the fact that our resolution is minimal, these images are minimal generators of the kernel. Now, consider any one of these images, say $\phi_{i+1}(\mathbf{f}_j)$, which we write as $\sum c_{a,j}\mathbf{e}_a$ for $c_{a,j}$ monomials in S^{C_n}. Suppose that $\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \dots, \mathbf{e}_{a_k}$ are precisely the indices a with $c_{a,j} \neq 0$.

We will now proceed to compare $\mathrm{md}(\mathbf{f}_j)$ and $\ell(\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \dots, \mathbf{e}_{a_m})$. Observe that, since f_j is homogeneous and ϕ_{i+1} preserves multi-degree (with multiplication by elements of S^{C_n} corresponding to coordinate-wise addition of these multi-degrees), we have that $\mathrm{md}(\mathbf{f}_j) = \mathrm{md}(\mathbf{e}_{a_\ell}) + \mathrm{md}(c_{a_\ell,j})$. In particular, $\mathrm{md}(\mathbf{f}_j) \geq \mathrm{md}(\mathbf{e}_{a_\ell})$ for each ℓ, meaning that $\mathrm{md}(\mathbf{f}_j) \geq \ell(\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \dots, \mathbf{e}_{a_m})$. We will now show that $\mathrm{md}(\mathbf{f}_j) - \ell(\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \dots, \mathbf{e}_{a_m}) \leq (n-1, n-1, \dots, n-1)$, which implies that $\mathrm{md}(\mathbf{f}_j) - \ell(\mathbf{e}_1, \dots, \mathbf{e}_m) \leq (n-1, n-1, \dots, n-1)$.

Suppose for the sake of contradiction that this wasn't the case. Then, $\mathrm{md}(f_j) - \ell(\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \dots, \mathbf{e}_{a_m})$ has a coordinate that is at least n. Observe, however, that

$$\mathrm{md}(\mathbf{f}_j) - \ell(\mathbf{e}_{a_1}, \mathbf{e}_{a_2}, \dots, \mathbf{e}_{a_m}) \leq \mathrm{md}(\mathbf{f}_j) - \mathrm{md}(\mathbf{e}_a) = \mathrm{md}(c_{a,j})$$

for all a, meaning that $g(c_{1,j}, c_{2,j}, \dots, c_{l,j}) \geq \mathrm{md}(\mathbf{f}_j) - l(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m)$. Then $g(c_{1,j}, c_{2,j}, \dots, c_{l,j})$ has an entry that is at least n. Recalling that each $c_{a,j}$ is a monomial, we therefore get that $\gcd(c_{1,j}, c_{2,j}, \dots, c_{l,j})$ is divisible by y_b^n for some b, or that $c_{a,j}$ is divisible by y_b^n for each n. But then $\mathbf{f}_j$ cannot be part of a free basis since we could have simply divided $\phi_{i+1}(\mathbf{f}_j)$ by y_b^n, so we are done. □

Proposition 3.9 implies the following corollary.

Corollary 3.10. *If $\beta_{i,j} \neq 0$, then we require $j \leq (i+1)(n^2 - n)$.*

Proof. Consider a free basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k$ of F_i. By Proposition 3.9, we have $\mathrm{md}(\mathbf{e}_j) \leq ((i+1)(n-1), (i+1)(n-1), (i+1)(n-1), \dots, (i+1)(n-1))$. However, this means that the degree of $\mathbf{e}_j$ for each j is at most $n(i+1)(n-1) = (i+1)(n^2-n)$.

In particular, this means that, for $\beta_{i,j} \neq 0$, we need a basis element to have degree j, and so thus that $j \leq (i+1)(n^2-n)$, as desired. □

This also allows us to provide an upper bound to the rank of the free modules within the free resolution, which gives us the second and final part of Proposition 3.3.

Corollary 3.11. *The rank of F_i in the minimal free resolution for S^{χ_1, C_n} is at most $n^{(i+1)n}((i+1)!)^n$.*

Proof. We induct on i. For the base case $i = 0$, note that the elements of F_0 are indexed by the multi-degrees of the elements in S^{χ_k, C_n}. From Proposition 3.9 we know that for each basis element $\mathbf{e}_i \in F_0$, the inequality $\mathrm{md}(\mathbf{e}_i) \leq (n-1, n-1, \dots, n-1)$ holds. Furthermore, there is only at most one $\mathbf{e}_i$ per multi-degree (as their images are unique monomials in S^{χ_k, C_n}). Therefore, $\dim F_0 \leq n^n$, proving the base case.

Now, suppose that we've shown the statement for F_i. Let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k$ be a free basis for F_i. We consider the dimension of the free subspace of F_{i+1} with multi-degree $\mathbf{a}$, viewing this as a $\mathbb{C}$-vector space. Let this subspace have free basis $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_k$. We consider $\mathbf{f}_j$ for some j and write $\phi_{i+1}(\mathbf{f}_j) = \sum c_{a,j}\mathbf{e}_a$. We know that $\sum c_{a,j}\mathbf{e}_a$ is homogeneous, meaning that $c_{a,j}$ is either zero or is of the form $c'_{a,j} m_{\mathbf{a},a}$ for $c'_{a,j} \in \mathbb{C}^\times$ where $m_{\mathbf{a},a} = \mathbf{y}^{(\mathbf{a} - \mathrm{md}(\mathbf{e}_a))}$.

Recall that for a minimal free resolution, the images $\phi_{i+1}(\mathbf{f}_j)$ have to be part of a minimal generating set of $\ker \phi_i$. We then claim that the vectors $\mathbf{c}'_j = (c'_{1,j}, c'_{2,j}, \dots, c'_{l,j})$ have to be linearly independent. Indeed, suppose that $\sum b_j \mathbf{c}'_j = 0$ for some scalars b_j. Then, observe that

$$\sum_j b_j \sum_a c_{a,j}\mathbf{e}_a = \sum_a \sum_j b_j c'_{a,j} m_{\mathbf{a},a}\mathbf{e}_a = 0.$$

But given that the b_j are scalars, this means that, if one of them is nonzero, we can re-arrange this to get one of the $\sum c_{a,j}\mathbf{e}_a$ as a linear combination of the others, which contradicts minimality. Therefore, all the b_j are zero, meaning that these vectors are linearly independent.

But then we see that, from linear algebra, we need $j \leq k$. In other words, for each multi-degree component, the dimension of the multi-degree component is at most k, the rank of F_i.

We also know that by Proposition 3.9, the multi-degree of each of these elements is at most $((i+2)(n-1), (i+2)(n-1), (i+2)(n-1), \ldots, (i+2)(n-1))$. This means that we have at most $((i+2)(n-1)+1)^n \leq (i+2)^n n^n$ multidegrees among elements of F_{i+1}, so $\dim F_{i+1} \leq \dim F_i (i+2)^n n^n$. By the inductive hypothesis, $\dim F_i \leq n^{(i+1)n}((i+1)!)^n$, meaning that $\dim F_{i+1} \leq n^{(i+2)n}((i+2)!)^n$, which finishes the induction and proves the claim. $\square$

Neither of the bounds given in Proposition 3.3 are sharp. For instance, for $n = 4$, the rank of the free module F_i in the free resolution of S^{χ_1} is equal to $8 \cdot 3^i$, but the bounds given by the two corollaries are 2^{i+1} and $4^{4(i+1)}((i+1)!)^4$. This does not appear to be an isolated incident; see the data tables in the Appendix.

3.2 Degree Bounds on the Entries of ϕ_1

In an attempt to sharpen the bounds that we found in the previous subsection, we prove a stronger statement specifically about the map ϕ_1. For this subsection, we will be working for a generic isotypic component, rather than S^{χ_1, C_n} (and the other k relatively prime to n).

For this lemma, we suppose that we've specified a homogeneous free basis for each F_i, and that our ϕ_i are maps between these free modules. We first prove that the entries in the matrix ϕ_1 are bounded in degree, starting with a lower bound.

For this section, we will let our free basis of F_0 is $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$, with each of these $\mathbf{e}_i$ being sent to one of the monic indecomposable monomials of S^{χ_k, C_n}. From here, we have the following definitions.

Definition 6. Let $m_{i,k}$ be the image of $\mathbf{e}_i$ under ϕ_0, and suppose that our $\mathbf{e}_i$ are ordered by degree (so $m_{i,k}$ has degree at most as large as that of $m_{i+1,k}$ for each i).

Furthermore, let $g_{i,j} \in S$ be the greatest common divisor of $m_{i,k}$ and $m_{j,k}$, and suppose that this monomial lies in the $(k_{i,j})^{\text{th}}$ isotypic component of S.

Lemma 3.12. *The matrix for ϕ_1 has no entries of degree less than* 2.

Proof. Notice that it is enough to show that every element in the kernel of ϕ_0, when written as $\sum_{i=1}^{m} c_i \mathbf{e}_i$, with $c_i \in S^{C_n}$ and $\mathbf{e}_i$ the free basis for F_0, must have every c_i have degree at least 2. To see this, suppose for the sake of contradiction that some to c_i has degree either 1 or 0. Without loss of generality, this is c_1. Then, applying ϕ_0 yields the equation

$$c_1 m_{1,k} = -\sum_{i=1}^{m} c_i m_{i,k}.$$

Each of the $m_{i,k}$ is indecomposable, as the $\phi_0(\mathbf{e}_i)$ by construction form a minimal generating set of S^{χ_k, C_n}.

Now, the only terms of degree 1 or 0 in S^{C_n} are of the form c or cy_0, where $c \in \mathbb{C}$, so c_1 is one of these forms. In either case, taking the component with the same multi-degree as $c_1 m_{1,k}$, with $c_1 \neq 0$, notice that we must have that some $m_{i,k}$ that divides $y_0 m_{1,k}$ (as both sides will necessarily be monomials) and thus $m_{1,k}$, since $y_0 \in S^{C_n}$ implies that $m_{i,k}$ cannot be divisible by y_0. Furthermore, $m_{i,k}$ doesn't equal $m_{1,k}$, which contradicts indecomposability of $m_{1,k}$ (as their quotient will necessarily lie in S^{C_n}). Thus, each element in the kernel of ϕ_0, or the image of ϕ_1, is of the form $\sum_{i=1}^{m} c_i \mathbf{e}_i$, where the degree of each c_i is at least 2. $\square$

Our main goal is to now show the following proposition, which gives an upper bound to the degrees of the entries in ϕ_1.

Proposition 3.13. *For any positive integer $n \geq 3$ and nonzero residue $k \pmod n$, the matrix ϕ_1 corresponding to the first step of the free resolution of S^{χ_k, C_n} can be constructed so that every entry has degree at least 2 and at most $2n-2$, and every column contains an entry with degree at most n.*

To prove this proposition, we require several lemmas. The main objective is to start with a generating set and show that we can shrink it down to a smaller generating set, where we can finally arrive at the proposition above. Note the similarity with Lemma 3.6, although here we construct a *superset*, not a subset, of a minimal generating set.

In order to state the first lemma, we have the following definition.

Lemma 3.14. *The set*

$$M = \left\{ c\frac{m_{i,k}}{g_{i,j}}\mathbf{e}_j - c\frac{m_{j,k}}{g_{i,j}}\mathbf{e}_i \;\middle|\; 1 \leq i < j \leq m \text{ and } c \in S^{\chi_{n-k+k_{i,j}}, C_n} \text{ indecomposable} \right\}$$

is a generating set of the kernel of ϕ_0, and so of $\operatorname{im}\phi_1$.

Here, we take $S^{\chi_{n-k+k_{i,j}}, C_n}$ to be the isotypic component that is equivalent to $n-k+k_{i,j} \pmod n$.

Proof. We will prove the lemma by starting with every element in $\ker\phi_0$ in our generating set, and show that we may remove all of the elements other than those in M while still maintaining that this is a generating set.

Suppose that $\sum_{i=1}^{m} d_i\mathbf{e}_i$ lies in the kernel of ϕ_0. It's not hard to see that this lies in $\ker\phi_0$ if and only each homogeneous component lies in $\ker\phi_0$, so the set of homogeneous elements in $\ker\phi_0$ generates this submodule.

Now, say that $\sum_{i=1}^{m} d_i\mathbf{e}_i$ is such a homogeneous element; suppose that $i_1, i_2, \ldots, i_l$ are the i so $d_i \neq 0$. Express each of these d_i as $s'_i c_i$, where $s'_i \in \mathbb{C}$ and c_i is a monomial.

Then, by considering coefficients, we see that $\sum_{s=1}^{l} c'_{i_s} = 0$. But then, notice that

$$\sum_{i=1}^{m} d_i\mathbf{e}_i = \sum_{s=1}^{l}\left(\sum_{j=1}^{i} c'_{i_s}\right)(-c_{i_{s+1}}\mathbf{e}_{i_{s+1}} + c_{i_s}\mathbf{e}_{i_s}),$$

and that $(-c_{i_{s+1}}\mathbf{e}_{i_{s+1}} + c_{i_s}\mathbf{e}_{i_s})$ lies in the kernel as well. Thus, we see that the set of elements of this form can be taken to generate the kernel. Furthermore, to lie in the kernel we need $c_{i_{s+1}} m_{i_{s+1},k} = c_{i_s} m_{i_s,k}$.

Finally, suppose $c_i\mathbf{e}_i - c_j\mathbf{e}_j \in \ker\phi_0$. Notice that, from the above, we have that $c_i m_{i,k} = c_j m_{j,k}$, meaning that c_j is divisible by $\frac{m_{i,k}}{g_{i,j}}$ and c_i is divisible by $\frac{m_{j,k}}{g_{i,j}}$. Furthermore, we also have that $c_i = c\frac{m_{j,k}}{g_{i,j}}, c_j = c\frac{m_{i,k}}{g_{i,j}}$, for some c. Since c_i, c_j lie in S^{C_n}, we see that, considering the isotypic value, we need to have that $c \in S^{\chi_{n-k+k_{i,j}}, C_n}$. In particular, notice that if it isn't indecomposable, then our element $c_i\mathbf{e}_i - c_j\mathbf{e}_j$ can be written as a product of elements in $S_+^{C_n}$ times an element of the form $c\frac{m_{i,k}}{g_{i,j}}\mathbf{e}_j - c\frac{m_{j,k}}{g_{i,j}}\mathbf{e}_i$, where c is indecomposable. Thus, we see that the elements in M generate the kernel of ϕ_0, which is what we wanted. $\square$

We now sharpen the statement of the above lemma. In particular, our goal is to now show that the elements of M where both coefficients are of degree more than n can be removed.

Lemma 3.15. *The set*

$$M' = \left\{ c\frac{m_{i,k}}{g_{i,j}}\mathbf{e}_j - c\frac{m_{j,k}}{g_{i,j}}\mathbf{e}_i \;\middle|\; 1 \leq i < j \leq m,\; c \in S^{\chi_{n-k+k_{i,j}}, C_n} \text{ indecomposable, and } \deg\left(c\frac{m_{i,k}}{g_{i,j}}\right) \leq n \right\}$$

is a generating set of $\ker\phi_0$, and so of $\operatorname{im}\phi_1$.

Proof. Our goal here is to show that the elements whose coefficients only degrees more than n in M can be removed and for the set to still remain a minimal generating set. It suffices then to express each element in M as a linear combination of those in our set M'. Let $c\frac{m_{i,k}}{g_{i,j}}\mathbf{e}_j - c\frac{m_{j,k}}{g_{i,j}}\mathbf{e}_i$ be one such element. We have two cases.

Case 1: Suppose for the sake of contradiction that one such element couldn't be expressed as a linear combination of our elements in M, and that the degree of c is more than $\frac{n}{2}$. Notice that the degree of c is at most $n-1$. Thus, there is some maximal degree among those that can't be expressed; pick one such c that has this maximal degree.

By Theorem 2.4, we know that c can be expressed as

$$y_s^{\deg(c)-\sum_{j=1}^{n-1}\lambda(j)} y_{2s}^{\lambda_1} y_{3s}^{\lambda_2} \cdots y_{(n-1)s}^{\lambda_{n-2}},$$

with indices taken $\pmod n$ and $(\lambda_1, \lambda_2, \ldots, \lambda_{n-2})$ forming some partition of an integer $p < n - \deg(c)$, and s relatively prime to n. Notice that this lies in the $s(\deg(c)+p)$-isotypic component; by our above, we see that this is equivalent to $n - k + k_{i,j}$.

From here, observe that, since $\deg(c) > \frac{n}{2}$, we have that

$$\deg(c) - \sum_{j=1}^{n-1}\lambda(j) > (n - \deg(c)) - \sum_{j=1}^{n-1} j\lambda(j) = n - \deg(c) - p.$$

However, notice that $y_s^{n-\deg(c)-p}$ has isotypic component equivalent to $s(n-\deg(c)-p) \equiv n-(n-k+k_{i,j}) \equiv k - k_{i,j} \pmod n$. This means that, in particular, $g_{i,j}y_s^{n-\deg(c)-p} \in S^{\chi_k, C_n}$. Then, there is some element of the form $c'\mathbf{e}_q$ which maps to $g_{i,j}y_s^{n-\deg(c)-p}$ under the map ϕ_0, for some index q. Observe, now, that we may write

$$c_i\mathbf{e}_i - c_j\mathbf{e}_j = c_i\mathbf{e}_i - \left(\frac{m_{j,k}m_{i,k}c}{g_{i,j}^2 y_s^{n-\deg(c)-p}}\right)c'\mathbf{e}_q + \left(\frac{m_{j,k}m_{i,k}c}{g_{i,j}^2 y_s^{n-\deg(c)-p}}\right)c'\mathbf{e}_q - c_j\mathbf{e}_j,$$

which is a sum of two elements in the kernel (as each term is mapped to $\frac{m_{j,k}m_{i,k}c}{g_{i,j}}$).

We now consider the first pair of terms; the latter pair follows similarly. Notice that they can be written as

$$\frac{m_{j,k}c}{g_{i,j}}\mathbf{e}_i - (\frac{m_{j,k}m_{i,k}c}{g_{i,j}^2 y_s^{n-\deg(c)-p}})c'\mathbf{e}_q,$$

with a shared factor divisible by $\frac{m_{j,k}c}{g_{i,j}^2 y_s^{n-\deg(c)-p}}$. However, the degree of this term is

$$\deg(c) + \deg(m_{j,k}) - n + \deg(c) + p - 2\deg(g_{i,j}).$$

By assumption, we have that $\deg(c) + \deg(m_{j,k}) - \deg(g_{i,j}) > n$, meaning that

$$\deg(c) + \deg(m_{j,k}) - n + \deg(c) + p - 2\deg(g_{i,j}) > \deg(c) + p - \deg(g_{i,j}) \geq \deg(c) - \deg(g_{i,j}),$$

as $p \geq 0$.

By maximality, however, each of these can be expressed as a linear combination of elements in M', meaning that our original difference can be too, contradiction.

Case 2: Otherwise, if $\deg(c) \leq \frac{n}{2}$, it follows that both $\frac{m_{i,k}}{g_{i,j}}$ and $\frac{m_{j,k}}{g_{i,j}}$ have degree larger than $\frac{n}{2}$. We will use a similar argument; in this case, we suppose that we've picked the minimal value of the degree of c_j.

We translate our work in order to employ [7, Theorem 7] from Savchev and Chen's paper, and then work back. For a given monomial $m = y_{i_1}y_{i_2}\cdots y_{i_k}$, let $s(m)$ be the sum of the indices $i_1, i_2, \ldots, i_k$ taken $\pmod n$, with values in $\{0, 1, 2, \ldots, n-1\}$. From section 2, recall that we can then view our monomial $\frac{m_{i,k}}{g_{i,j}}$ as being a sequence of integers, coming from the indices of the variables.

Now, [7, Theorem 7] tells us that there is some index x among those in our monomial so that, multiplying each index by x in $\frac{m_{i,k}}{g_{i,j}}$ yields a monomial $m'_{j,k}$ so the set $\{s(m) : m \text{ divides } m'_{j,k}\}$ is a set of consecutive integers starting at 1 and ending at $l < n$.

Applying the inverse of that action, it follows that

$$\left\{s(m) : m \text{ divides } \frac{m_{j,k}}{g_{i,j}}\right\} = \{x, 2x, \dots, bx\}$$

for some x, b, where we take the products $\pmod n$. Notice that necessarily we need for x to be relatively prime to n, and furthermore that b is equal to $s(m'_{j,k})$. But then we see that $b \geq \frac{m_{j,k}}{g_{i,j}}$, as each index is at least 1 in $m'_{j,k}$, which in turn is larger than $n/2$ by assumption.

Now, recall that $c\frac{m_{i,k}}{g_{i,j}}, c\frac{m_{j,k}}{g_{i,j}}$ are both degree larger than n, meaning that they are not indecomposable by Proposition 2.3. This means that they can be written as a product of two elements of $S_+^{C_n}$, as these monomials lie in $S_+^{C_n}$. As such, write

$$c\frac{m_{i,k}}{g_{i,j}} = (c_1 m_1)(c_2 m_2),$$

where $c_1 c_2 = c$.

Here, we will refer to the residue of an integer a, denoted $r(a)$, to be the unique value $a' \in \{0, 1, 2, \dots, n-1\}$ so $a \equiv a' \pmod n$.

We now analyze $\frac{m_{j,k}}{g_{i,j}}$. Suppose that

$$\{r(s(m)) : m \text{ divides either } c_1 \text{ or } c_2\}$$

is disjoint from $\{r((n-1)x), r((n-2)x), \dots, r((n-b)x)\}$. Write $c_1 = y_{i_1} y_{i_2} y_{i_3} \cdots y_{i_{l_1}}$, and consider the residues $x^{-1}s(y_{i_1}), x^{-1}s(y_{i_1}y_{i_2})$, and so forth, where $x^{-1} \in \{1, 2, \dots, n-1\}$ is so $xx^{-1} \equiv 1 \pmod n$. Then, notice that this has to be an increasing sequence; otherwise, if it decreases when we add y_{i_j}, it follows that, since

$$r(x^{-1}s(y_{i_1} y_{i_2} y_{i_3} \cdots y_{i_{j-1}})) < n - b < n/2,$$

it follows that $r(x^{-1}i_j) > n/2$ (adding $x^{-1}i_j$ causes us to loop around back to 0), contradiction. We can do something similar with c_2.

Thus, our sequence is increasing for both c_1 and c_2. Furthermore, if we multiply each divisor of c_2 by c_1, and adjoin those after our sequence for c_1, we will still get another increasing sequence; but the maximal residue is $r(x^{-1}s(c_2)) < n - b < n/2$, and $r(x^{-1}s(c_1)) < n/2$.

Therefore, we see that we have an increasing sequence of length $\deg(c)$ that ends at $r(x^{-1}s(c))$. But notice that $r(x^{-1}s(c)) + r(x^{-1}s(\frac{m_{j,k}}{g_{i,j}})) = n$ by definition (as both can't be zero). Furthermore, we also know that $r(x^{-1}s(\frac{m_{j,k}}{g_{i,j}})) = b > \deg(\frac{m_{j,k}}{g_{i,j}})$, and that $\deg(c) < r(x^{-1}s(c))$. But then we have that $n < \deg(c) + \deg(\frac{m_{j,k}}{g_{i,j}}) < n$, contradiction.

Thus, some divisor c_3 of one of c_1 or c_2 (without loss of generality, c_1) satisfies

$$r(s(c_3)) \in \{r((n-1)x), r((n-2)x), \dots, r((n-b)x)\},$$

or that $s(c_3) \equiv yx \pmod n$ for some y so $n - b \leq y < n$. But we know that a divisor m_3 of $\frac{m_{j,k}}{g_{i,j}}$ exists so $s(m_3) \equiv (n-y)x \pmod n$, so $s(m_3 c_3) \equiv 0 \pmod n$, or that the product lies in S^{C_n}.

Now, consider the product $(\mathbf{c_3} m_3)(c_2 m_2)$. Notice that by construction this divides $\frac{cm_{i,k}m_{j,k}}{g_{i,j}}$; let

$$m = \frac{cm_{i,k}m_{j,k}}{g_{i,j}(c_3 m_3)(c_2 m_2)}.$$

Suppose this has preimage $c'\mathbf{e}_q$ under ϕ_0. Then, notice that

$$c_i\mathbf{e}_i - (c_3 m_3)(c_2 m_2)c'\mathbf{e}_q + (c_3 m_3)(c_2 m_2)c'\mathbf{e}_q - c_j\mathbf{e}_j$$

is again a sum of two elements in the kernel. Notice that c_j is divisible by (c_2m_2) and c_i is divisible by (c_3m_3), with both of these factors lying in S^{C_n}.

This means, however, that our element in the kernel is a linear combination of elements where the degree of the corresponding c is smaller. By our assumption of minimality, our element is a linear combination of elements in M', contradiction.

This covers both of our cases and proves the lemma. □

Once we have this lemma, the proof of Proposition 3.13 is relatively straightforward.

Proof. From Lemma 3.15, if we pick our minimal generating set to be a subset of M', we can take the columns of ϕ_1 to be so that every column has two entries, where one of them has degree at most n.

Furthermore, from Proposition 2.3, given an element $c\frac{m_{i,k}}{g_{i,j}}\mathbf{e}_j - c\frac{m_{j,k}}{g_{i,j}}\mathbf{e}_i$ in our minimal generating set, we know that the degrees of $c, m_{i,k}, m_{j,k}$ are all at most $n-1$, meaning that the degree of each of the coefficients of our minimal generators are at most $(n-1)+(n-1) = 2n-2$. This, in turn, means that all the entries of ϕ_1 (which are the coefficients above) are degree at most $2n-2$, which is what we want. □

3.3 Free Resolutions in the $n = 2$ and $n = 3$ cases

Here, we review the minimal free resolutions of S over $R = S^{C_n}$ when $n = 2$ and $n = 3$, including examining individually the R-resolutions of each direct summand $M = S^{\chi_k}$ in the decomposition

$$S = S^{C_n} \oplus S^{\chi_1} \oplus S^{\chi_2} \oplus \cdots \oplus S^{\chi_{n-1}}.$$

We also write down the multigraded Betti number generating functions (or *Poincaré series*)

$$\mathrm{Poin}_R(M; t, \mathbf{y}) := \sum_{i=0}^{\infty} \sum_{\alpha \in \mathbb{N}^{n-1}} \beta_{i,\alpha} t^i y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}} \tag{4}$$

since upon setting $t = -1$, this should specialize to the ratio of multigraded Hilbert series

$$[\mathrm{Poin}_R(M; t, \mathbf{y})]_{t=-1} = \frac{\mathrm{Hilb}(M, \mathbf{y})}{\mathrm{Hilb}(R, \mathbf{y}}$$

which are easily computable ahead of time, either via `Macaulay2` or a Molien series calculation.

3.3.1 The $n = 2$ case

This case is easy, since then $\mathbb{C}[y_0, y_1]^{C_2} = \mathbb{C}[y_0, y_1^2]$ is a polynomial algebra, and $\mathbb{C}[y_0, y_1]$ will be free as a module over $\mathbb{C}[y_0, y_1]^{C_2}$, with basis elements $\{1\}$ for S^{C_2}, and $\{y_1\}$ for S^{χ_1}. Hence both resolutions stop at the 0^{th} step, and there is not much more to say.:

$$0 \to R(-0) \to S^{C_2} \to 0,$$
$$0 \to R(-1) \to S^{\chi_1} \to 0.$$

3.3.2 The $n = 3$ case

Here We get lucky. In this case,

$$R = \mathbb{C}[y_0, y_1, y_2]^{C_3} = \mathbb{C}[y_0, y_1^3, y_2^3, y_1y_2] \cong \mathbb{C}[A, B, C, D]/(D^3 - BC)$$

is a *hypersurface ring*, that is, a polynomial ring modulo a single polynomial. Since $S = \mathbb{C}[y_0, y_1, y_2]$ is a maximal Cohen-Maculay module over R, meaning one whose depth is 3, one can apply Eisenbud's theorem on hyperfurface rings and *matrix factorizations*[1]

[1] See the book by Leuschke and Wiegand [4, §8.1].

Theorem 3.16 ([1]). *If $R = S/(f(x))$ is a hypersurface ring and M is an R-module, M's minimal free resolution over R takes the form*

$$\cdots \xrightarrow{\varphi_1} R^{\beta_0} \xrightarrow{\varphi_2} R^{\beta_0} \xrightarrow{\varphi_1} R^{\beta_0} \xrightarrow{\varphi_2} R^{\beta_0} \xrightarrow{\varphi_1} R^{\beta_0} \xrightarrow{\varphi_0} M \to 0,$$

where $\varphi_1\varphi_2 = \varphi_2\varphi_1 = f(x)I_{\beta_0}$.

In particular, note that this free resolution is 2-periodic. In fact, the same holds for each of the summands S^{χ_k} with $k = 0, 1, 2$ in the C_3-isotypic decomposition

$$S = S^{C_3} \oplus S^{\chi_1} \oplus S^{\chi_2},$$

and each can be resolved separately, and very explicitly. $R = S^{C_3}$ is a free module over itself of rank one, with 1 as its basis element. S^{χ_1} is generated as a module over $R = S^{C_3}$ by $\{y_1, y_2^2\}$, and its 2-periodic resolution looks like

$$\cdots \xrightarrow{\varphi_1} \begin{matrix} R(-13) \\ \oplus \\ R(-14) \end{matrix} \xrightarrow{\varphi_2} \begin{matrix} R(-10) \\ \oplus \\ R(-11) \end{matrix} \xrightarrow{\varphi_1} \begin{matrix} R(-7) \\ \oplus \\ R(-8) \end{matrix} \xrightarrow{\varphi_2} \begin{matrix} R(-4) \\ \oplus \\ R(-5) \end{matrix} \xrightarrow{\varphi_1} \begin{matrix} R(-1) \\ \oplus \\ R(-2) \end{matrix} \xrightarrow{\varphi_0} M \to 0$$

in which φ_1, φ_2 are reductions in $R = \mathbb{C}[A, B, C, D]/(D^3 - BC)$ of

$$\begin{bmatrix} -C & D^2 \\ D & -B \end{bmatrix} \text{ and } \begin{bmatrix} B & D^2 \\ D & C \end{bmatrix},$$

respectively. One can obtain the R-resolution for S^{χ_2} from this resolution of S^{χ_1}, by applying the nontrivial symmetry in $(\mathbb{Z}/3\mathbb{Z})^{\times}$, which swaps $y_1 \leftrightarrow y_2$, and fixes y_0. Explicitly, the free resolution for S^{χ_2} is

$$\cdots \xrightarrow{\varphi_1} \begin{matrix} R(-13) \\ \oplus \\ R(-14) \end{matrix} \xrightarrow{\varphi_2} \begin{matrix} R(-10) \\ \oplus \\ R(-11) \end{matrix} \xrightarrow{\varphi_1} \begin{matrix} R(-7) \\ \oplus \\ R(-8) \end{matrix} \xrightarrow{\varphi_2} \begin{matrix} R(-4) \\ \oplus \\ R(-5) \end{matrix} \xrightarrow{\varphi_1} \begin{matrix} R(-1) \\ \oplus \\ R(-2) \end{matrix} \xrightarrow{\varphi_0} M \to 0$$

where φ_1, φ_2 are reductions in $R = \mathbb{C}[A, B, C, D]/(D^3 - BC)$ of

$$\begin{bmatrix} -B & D^2 \\ D & -C \end{bmatrix} \text{ and } \begin{bmatrix} C & D^2 \\ D & B \end{bmatrix},$$

respectively. Thus in this case one has this expressions for the Poincaré series (4) of $M = S^{\chi_1}$:

$$\mathrm{Poin}(S^{\chi_1}; t, y_1, y_2) = \frac{y_1 + y_2^2 + t(y_1y_2^3 + y_1^3y_2^2)}{1 - t^2y_1^3y_2^3} \quad \overset{t=-1}{\rightsquigarrow} \quad \frac{y_1 + y_2^2 - (y_1y_2^3 + y_1^3y_2^2)}{1 - y_1^3y_2^3} = \frac{y_1 + y_2^2 + y_1^2y_2}{1 + y_1y_2 + y_1^2y_2^2}, \quad (5)$$

and the one for S^{χ_2} is obtained by swapping $y_1 \leftrightarrow y_2$. Both of the $\mathbb{N}$-graded Betti tables for S^{χ_1}, S^{χ_2} look like this, continuing infinitely downward:

```
o9 = total: 2 2 2
         0: . . .
         1: 1 . .
         2: 1 . .
         3: . 1 .
         4: . 1 .
         5: . . 1
         6: . . 1
```

3.4 The $n = 4$ case

In this subsection, we consider the minimal free resolution of S^{χ_k}, for $k = 1, 2, 3$. We will be able to almost explicitly describe the matrices involved in the minimal free resolution. Here, we have $S = \mathbb{C}[y_0, y_1, y_2, y_3]$.

Throughout, when we refer to the matrix of a mapping, this matrix will necessarily be for the minimal resolution; in particular, this means that the columns of the matrix minimally generate the image of the linear map. In addition, we will suppress the C_4 in the isotypic notation. As such, when we write S^{χ_k} in this subsection, we mean S^{χ_k, C_4}.

Our approach in this subsection will construct a minimal free resolution for S and then prove it is minimal.

Lemma 3.17. *Suppose that we have a diagonal block matrix given by* $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. *Then, if* $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}$ *are a minimal generating set for the kernel of* A, *and* $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_l}$ *are a minimal generating set for the kernel of* B, *then* $\{\left(\begin{smallmatrix} \mathbf{v_1} \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathbf{v_2} \\ 0 \end{smallmatrix}\right), \ldots \left(\begin{smallmatrix} \mathbf{v_k} \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ \mathbf{w_1} \end{smallmatrix}\right), \ldots \left(\begin{smallmatrix} 0 \\ \mathbf{w_l} \end{smallmatrix}\right)\}$ *minimally generate the kernel of the diagonal block matrix.*

Proof. We show that this is a generating set, and then that this is minimal.

First, suppose that $\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix}$ is in the kernel of this matrix. Then, it follows that $\begin{pmatrix} A\mathbf{v} \\ B\mathbf{w} \end{pmatrix} = 0$, or that $\mathbf{v} \in \ker A, \mathbf{w} \in \ker B$. In particular, this means that the set of vectors that we provided above generate the kernel of our block diagonal matrix.

Now, suppose that our set wasn't a set of minimal generators. Without loss of generality, say that $\begin{pmatrix} \mathbf{v_i} \\ 0 \end{pmatrix}$ is redundant. Then, we require that $\begin{pmatrix} \mathbf{v_i} \\ 0 \end{pmatrix} = \sum c_j \begin{pmatrix} \mathbf{v_j} \\ 0 \end{pmatrix} + \sum d_j \begin{pmatrix} 0 \\ \mathbf{w_j} \end{pmatrix}$. But then $\mathbf{v_i} = \sum c_j \mathbf{v_j}$, contradicting the fact that the $\mathbf{v_i}$ are minimal generators. $\square$

Notice that this lemma can be easily extended to several blocks, by using induction and repeatedly applying this lemma.

Lemma 3.18. *Let* R *be a polynomial graded ring over* $\mathbb{C}$. *Suppose that a minimal free resolution of an* R*-module* M *has block matrix form* $\phi_j = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right)$, *and that the the following conditions are satisfied:*

- $B \cdot \ker C \subset \text{im } A$.
- *A nonzero* $\mathbb{C}$*-linear combination of the columns of* A *always has a coordinate with an indecomposable monomial term.*

Then we can write ϕ_{j+1} *as* $\left(\begin{smallmatrix} \ker(A) & B' \\ 0 & \ker(C) \end{smallmatrix}\right)$ *with the following condition satisfied:*

$$B' \cdot \ker(\text{matrix } \ker C) \subset \ker A,$$

where by $\ker(A)$ *we mean a matrix whose columns form a homogeneous minimal generating set for the kernel of* A.

Remark. In our block form, if A is $i \times j$ and C is $k \times l$, then B is $i \times l$, so the first condition makes sense dimension-wise.

Proof. Suppose $\phi_j \left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) = 0$. Then we have $A\mathbf{v} + B\mathbf{w} = 0, C\mathbf{w} = 0$. Now, let $\mathbf{w_1}, \ldots, \mathbf{w_k}$ be a homogeneous minimal generating set for $\ker C$. Such a homogeneous minimal generating set exists since ϕ_j, and thus A, B, C, preserve multi-degree. Note that all entries consist of non-scalar monomials. If not, then some $\mathbf{w_i}$ has a scalar entry, meaning that by the first condition, there exists a $\mathbf{v_i}$ so that $A\mathbf{v_i} = -B\mathbf{w_i}$. But then $\begin{pmatrix} \mathbf{v_i} \\ \mathbf{w_i} \end{pmatrix}$ contains a scalar and so ϕ isn't part of a minimal generating set, contradiction. By the first condition on B, we can find $\mathbf{u_1}, \ldots, \mathbf{u_k}$ such that $A\mathbf{u_i} + B\mathbf{w_i} = 0$ for all $1 \le i \le k$.

We claim that we can choose $\mathbf{u_1}, \ldots, \mathbf{u_k}$ to each consist of monomial entries. Indeed, let $\mathbf{d}$ be the multi-degree of $\mathbf{w_i}$. Then, consider the $\mathbf{d}$ homogeneous component of $\mathbf{u_i}$, say $\mathbf{u_i'}$. As A preserves multi-degree, we see that $A(\mathbf{u}-\mathbf{u_i'})$ cannot have any $\mathbf{d}$-component, meaning that it equals zero as $B\mathbf{w_i}$ is also homogeneous with multi-degree $\mathbf{d}$. This implies that we can pick homogeneous elements, which thus implies that they must have monomial entries.

For a general $\left(\begin{smallmatrix}\mathbf{v}\\\mathbf{w}\end{smallmatrix}\right)$ with $\phi_j\left(\begin{smallmatrix}\mathbf{v}\\\mathbf{w}\end{smallmatrix}\right)=0$, we can write $\mathbf{w}$ as a linear combination of the $\mathbf{w_i}$; let $\mathbf{u}$ be the corresponding linear combination of the $\mathbf{u_i}$. We then have $\phi_j\left(\begin{smallmatrix}\mathbf{u}\\\mathbf{w}\end{smallmatrix}\right)=0$, meaning that $A\cdot(\mathbf{v}-\mathbf{u})=0$. Then if $\mathbf{x_1},\dots,\mathbf{x}_\ell$ form a minimal generating set for $\ker A$, we have that $\{\left(\begin{smallmatrix}\mathbf{x_1}\\0\end{smallmatrix}\right),\cdots,\left(\begin{smallmatrix}\mathbf{x}_\ell\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}\mathbf{u_1}\\\mathbf{w_1}\end{smallmatrix}\right),\cdots,\left(\begin{smallmatrix}\mathbf{u_k}\\\mathbf{w_k}\end{smallmatrix}\right)\}$ form a generating set for $\ker\phi_j$.

We claim that this generating set is minimal. Suppose not. Then since the $\{\mathbf{x_i}\}$ and the $\{\mathbf{w_j}\}$ form minimal generating sets, the only possibility is that an R-linear combination of the $\left(\begin{smallmatrix}\mathbf{u_j}\\\mathbf{w_j}\end{smallmatrix}\right)$ equals a $\mathbb{C}$-linear combination of the $\left(\begin{smallmatrix}\mathbf{x_i}\\0\end{smallmatrix}\right)$. Suppose we have homogeneous elements $m_1,\dots,m_k$ with $\sum m_j\mathbf{w_j}=0$. Then none of the m_i can be a scalar, as otherwise $\mathbf{w_i}$ would be generated by the other $\mathbf{w_j}$s. But then $\sum m_j\mathbf{u_j}$ cannot contain any indecomposable elements; by the assumption that ϕ_j is part of a minimal free resolution, B does not contain any scalars. Thus $\sum m_j\mathbf{u_j}$ cannot be equal to a $\mathbb{C}$-linear combination of the columns of A by the second condition.

Thus we can write ϕ_{j+1} as the concatenation of these columns. We claim that the condition is satisfied. An element in the kernel of the matrix $\left(\begin{smallmatrix}\mathbf{v_1}\mathbf{v_2}\cdots\mathbf{v_k}\end{smallmatrix}\right)$ can be written as a vector $\begin{pmatrix}m_1\\m_2\\\vdots\\m_k\end{pmatrix}$ with $\sum m_i\mathbf{v_i}=0$. Then

$$A\cdot B'\begin{pmatrix}m_1\\m_2\\\vdots\\m_k\end{pmatrix}=A\cdot\sum m_i\mathbf{u_i}=\sum m_iA\mathbf{u_i}=\sum m_i(-B\mathbf{v_i})=(-B)\sum m_i\mathbf{v_i}=0.$$

□

Remark. Notice that this matrix B' is defined by the equation $AB'=-B\ker C$, and furthermore it isn't difficult to see that all matrices B' that satisfy this property are allowed to be taken in the above lemma.

Corollary 3.19. *In the language of Lemma 3.18, suppose that $B=\underline{y}I$, $A=\phi_j$, and $C=\phi_{j-1}$. Then $B'=-\underline{y}I$, where the two identity matrices are not necessarily the same dimension.*

Proof. Since $A=\phi_j$ and $C=\phi_{j-1}$, we see that by construction $\ker C=A$. Now, the previous remark states that $AB'=-B\ker C$, from which the result follows. □

We are now ready to move toward the statement of the main theorem of this subsection. We first define some matrices, which we claim lie in the minimal free resolutions of S^{χ_1},S^{χ_2} and S^{χ_3}.

Definition 7. Let M be a matrix with entries in S^{C_4}. Then we define $s(M)$ to be the matrix M, except with y_1 replaced with y_3 everywhere, and y_3 replaced with y_1 everywhere.

For example, applying s to $\begin{pmatrix}y_1^4 & y_1y_3\\ y_2^2 & y_3^4\end{pmatrix}$ yields $\begin{pmatrix}y_3^4 & y_1y_3\\ y_2^2 & y_1^4\end{pmatrix}$.

We now recursively construct the matrices that will form our free resolutions.

Definition 8. We define series of matrices $\phi_j^1,\phi_j^2,\phi_j^3$ for $j\geq 1$ and Y_j for $j\geq 3$ with entries in S^{C_4} as follows:

1. Let
$$\phi_1^1=\begin{pmatrix}y_2y_3^2 & y_1y_2^2y_3 & y_1^3y_2y_3 & 0 & 0 & 0 & y_3^4 & y_1^3y_3^3\\ -y_1y_3 & -y_1^2y_2 & -y_1^4 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 & 0 & 0\\ 0 & 0 & 0 & -y_1^2y_2 & -y_2^2 & -y_2y_3^2 & -y_1y_3 & -y_1^4\end{pmatrix},$$
$$\phi_1^2=\begin{pmatrix}y_1^2y_2 & y_2^2 & y_2y_3^2 & 0 & 0 & 0 & y_1^2y_3^2 & y_3^4\\ -y_1^4 & -y_1^2y_2 & -y_1^2y_3^2 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 & 0 & 0\\ 0 & 0 & 0 & -y_1^2y_2 & -y_2^2 & -y_2y_3^2 & -y_1^4 & -y_1^2y_3^2\end{pmatrix},$$
and $\phi_1^3=s(\phi_1^1)$.

2. Let

$$\phi_2^1 = \begin{pmatrix} \phi_1^3 & 0 & 0 & \begin{pmatrix} y_1^5y_3 & y_1^3y_2y_3 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ 0 & \phi_1^2 & 0 & \begin{pmatrix} y_1^4 & y_1^2y_2 & y_1y_3 & 0 & 0 \\ 0 & 0 & 0 & y_1y_3 & 0 \\ 0 & 0 & 0 & 0 & y_1y_3 \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} y_1^4 & y_1^3y_2y_3 & y_1^3y_3^3 \\ -y_1y_3 & -y_2y_3^2 & -y_3^4 \end{pmatrix} & \begin{pmatrix} 0 & 0 & -y_1^2y_2 & y_2^2 & -y_2y_3^2 \\ -y_1^2y_2 & -y_2^2 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$\phi_2^2 = \begin{pmatrix} \phi_1^2 & 0 & 0 & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ y_1^4 & y_1^2y_2 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 \end{pmatrix} \\ 0 & \phi_1^2 & 0 & \begin{pmatrix} y_1^4 & 0 & 0 & 0 & 0 \\ 0 & y_1^4 & 0 & 0 & 0 \\ 0 & 0 & y_1^4 & y_1^2y_3^2 & y_1^2y_3^2 \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} y_1^2y_3^2 & y_2y_3^2 & y_3^4 \\ -y_1^4 & -y_1^2y_2 & y_1^2y_3^2 \end{pmatrix} & \begin{pmatrix} -y_1^2y_2 & -y_2^2 & -y_2y_3^2 & 0 & 0 \\ 0 & 0 & 0 & -y_2^2 & -y_2y_3^2 \end{pmatrix} \end{pmatrix},$$

and $\phi_2^3 = s(\phi_2^1)$.

3. We define

$$Y_3 = -\begin{pmatrix} y_2^2 & 0 & 0 & 0 & 0 & 0 & y_1^2y_2 & 0 \\ 0 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2^2 & 0 & 0 & 0 & 0 & y_1^2y_2 \end{pmatrix}$$

and

$$Y_3' = -\begin{pmatrix} y_2^2 & 0 & 0 & 0 & 0 & 0 & y_2y_3^2 & 0 \\ 0 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2^2 & 0 & 0 & 0 & 0 & y_2y_3^2 \end{pmatrix}.$$

Then let

$$\phi_3^1 = \begin{pmatrix} \phi_2^3 & 0 & 0 & 0 & 0 \\ 0 & \phi_2^2 & 0 & 0 & -y_1y_3I \\ 0 & 0 & \phi_1^1 & Y_3 & 0 \\ 0 & 0 & 0 & X_1 & X_2 \end{pmatrix},$$

where

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & -y_2y_3^2 & -y_2^2 & -y_1^2y_2 & -y_1y_3 & -y_3^4 \\ -y_1y_3 & -y_2y_3^2 & -y_3^4 & y_1^2y_3^2 & y_1^2y_2 & y_1^4 & 0 & 0 \\ y_1^2y_2 & y_1y_2^2y_3 & y_1y_2y_3^3 & 0 & 0 & 0 & y_1^4 & y_1^3y_3^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_1^2y_2 & y_1y_2^2y_3 & y_1y_2y_3^3 & 0 & 0 & 0 & y_1^4 & y_1^3y_3^3 \\ -y_1y_3 & -y_2y_3^2 & -y_3^4 & y_1^2y_3^2 & y_1^2y_2 & y_1^4 & 0 & 0 \\ 0 & 0 & 0 & -y_2y_3^2 & -y_2^2 & -y_1^2y_2 & -y_1y_3 & -y_3^4 \end{pmatrix},$$

and let

$$\phi_3^2 = \begin{pmatrix} \phi_2^2 & 0 & 0 & 0 & 0 \\ 0 & \phi_2^2 & 0 & -y_1^4I & 0 \\ 0 & 0 & \phi_1^2 & 0 & Y_3' \\ 0 & 0 & 0 & X_1' & X_2' \end{pmatrix},$$

where

$$X_1' = \begin{pmatrix} y_2y_3^2 & y_1y_2^2y_3 & y_1^3y_2y_3 & 0 & 0 & 0 & y_3^4 & y_1^3y_3^3 \\ -y_1y_3 & -y_1^2y_2 & -y_1^4 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 & 0 & 0 \\ 0 & 0 & 0 & -y_1^2y_2 & -y_2^2 & -y_2y_3^2 & -y_1y_3 & -y_1^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$X_2' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_2y_3^2 & y_1y_2^2y_3 & y_1^3y_2y_3 & 0 & 0 & 0 & y_3^4 & y_1^3y_3^3 \\ -y_1y_3 & -y_1^2y_2 & -y_1^4 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 & 0 & 0 \\ 0 & 0 & 0 & -y_1^2y_2 & -y_2^2 & -y_2y_3^2 & -y_1y_3 & -y_1^4 \end{pmatrix},$$

with $\phi_3^3 = s(\phi_3^1)$.

4. For $j \geq 4$, let Y_j be a matrix with only monomial entries such that $Y_{j-1}\phi_{j-2}^3 = -\phi_{j-3}^1 Y_j$, and let Y_j' be a matrix with only monomial entries such that $Y_{j-1}'\phi_{j-2}^2 = -\phi_{j-3}^2 Y_j'$. Then we define

$$\phi_j^1 = \begin{pmatrix} \phi_{j-1}^3 & 0 & 0 & 0 & 0 \\ 0 & \phi_{j-1}^2 & 0 & 0 & (-1)^j y_1y_3 I \\ 0 & 0 & \phi_{j-2}^1 & Y_j & 0 \\ 0 & 0 & 0 & \phi_{j-2}^3 & 0 \\ 0 & 0 & 0 & 0 & \phi_{j-2}^2 \end{pmatrix},$$

$$\phi_j^2 = \begin{pmatrix} \phi_{j-1}^2 & 0 & 0 & 0 & 0 \\ 0 & \phi_{j-1}^2 & 0 & (-1)^j y_1^4 I & 0 \\ 0 & 0 & \phi_{j-2}^2 & 0 & Y_j' \\ 0 & 0 & 0 & \phi_{j-2}^2 & 0 \\ 0 & 0 & 0 & 0 & \phi_{j-2}^2 \end{pmatrix}.$$

and $\phi_j^3 = s(\phi_j^1)$.

Remark. The existence of these Y_j follows from the fact that they satisfy the property in Lemma 3.18 with $A = \phi_{j-2}^1, C = \phi_{j-2}^3$. The fact that this holds for 3 can be verified manually (see the proof of Lemma 3.22, which we introduce later, in the appendix).

Now, given that this holds for Y_{j-1}, if $\mathbf{v} \in \ker \phi_{j-2}^3$, then $Y_{j-1}\phi_{j-2}^3 v = 0$, so $-\phi_{j-3}^1 Y_j \mathbf{v} = 0$. But this requires that $Y_j\mathbf{v} \in \ker \phi_{j-3}^1 = \operatorname{im} \phi_{j-2}^1$ by exactness. In fact, it is this property which allows us to argue that these matrices exist, using the argument of Lemma 3.18.

As such, notice that the recursion employed in defining the Y_j and Y_j' are similar to that given in the proof of Lemma 3.18.

Proposition 3.20. *A nonzero $\mathbb{C}$-linear combination of the columns of ϕ_i^j will always have a coordinate with an indecomposable monomial term.*

Proof. We use induction. Our base cases are $j = 1, 2, 3$; for $j = 1, 2$ this can be easily verified since each column has a coordinate where no other column has that indecomposable monomial in that coordinate.

For $j = 3$, notice that ϕ_3^k's matrix form have ϕ_2 and ϕ_1 matrices along diagonals, both of which can be easily seen to also have this property. The only concern we might have is with regards to X_1, X_2. But these can be resolved, as for the S^{χ_1} case these matrices take on this form:

$$\begin{pmatrix} 0 & 0 & 0 & \boxed{-y_2y_3^2} & \boxed{-y_2^2} & \boxed{-y_1^2y_2} & \boxed{-y_1y_3} & \boxed{-y_3^4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \boxed{-y_1y_3} & \boxed{-y_2y_3^2} & \boxed{-y_3^4} & y_1^2y_3^2 & y_1^2y_2 & y_1^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_1^2y_2 & y_1y_2^2y_3 & y_1y_2y_3^3 & 0 & 0 & 0 & y_1^4 & y_1^3y_3^3 & y_1^2y_2 & y_2^2 & \boxed{y_2y_3^2} & 0 & 0 & 0 & y_1^2y_3^2 & \boxed{y_3^4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{-y_1^4} & \boxed{-y_1^2y_2} & -y_1^2y_3^2 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{-y_1^2y_2} & \boxed{-y_2^2} & \boxed{-y_2y_3^2} & \boxed{-y_1^4} & -y_1^2y_3^2 \end{pmatrix},$$

and for S^{χ_2} the matrices take on this form:

$$\begin{pmatrix} \boxed{y_1^2y_2} & \boxed{y_2^2} & \boxed{y_2y_3^2} & 0 & 0 & 0 & y_1^2y_3^2 & \boxed{y_3^4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_1^4 & -y_1^2y_2 & -y_1^2y_3^2 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-y_1^2y_2} & \boxed{-y_2^2} & \boxed{-y_2y_3^2} & \boxed{-y_1^4} & -y_1^2y_3^2 & y_1^2y_2 & y_2^2 & y_2y_3^2 & 0 & 0 & 0 & y_1^2y_3^2 & \boxed{y_3^4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{-y_1^4} & \boxed{-y_1^2y_2} & -y_1^2y_3^2 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{-y_1^2y_2} & \boxed{-y_2^2} & \boxed{-y_2y_3^2} & \boxed{-y_1^4} & -y_1^2y_3^2 \end{pmatrix}.$$

Notice that in the first matrix, each of the boxed monomials is the first time that monomial appears in that row. In particular, if we have a linear combination of the columns in $\mathbb{C}$, then the right-most one is so that the corresponding boxed monomial is preserved.

For the second matrix, a similar logic applies, except for the 11th column; either in the linear combination the $y_2y_3^2$ is still there, or it is cancelled out by the 6th column. But if it is, then notice that this is the only column with a y_3^4, meaning again we have a row where one of the entries has a y_3^4 in it. This finishes the base case.

Now, suppose that we've shown this for all $j \leq m$, for each S^{χ_k}. Consider the matrix for ϕ_{m+1}^k for some S^{χ_k}. If we have some linear combination $\sum c_a \mathbf{v_a}$, where the $\mathbf{v_a}$ are columns of ϕ_{m+1}^k and the c_a are complex numbers (so at least one is nonzero), consider the rightmost block of columns where one of the columns is nonzero. Then, the only columns with a nonzero coordinate in that row block are those in that block. The existence of an indecomposable monomial then follows from the inductive hypothesis. $\square$

Our main theorem is the following:

Theorem 3.21. *For $k = 1, 2, 3$, the sequence of matrices ϕ_j^k form the maps of a minimal free resolution of S^{χ_k} as an S^{C_4}-module:*

$$\cdots \xrightarrow{\phi_3^k} F_2^k \xrightarrow{\phi_2^k} F_1^k \xrightarrow{\phi_1^k} F_0^k \to S^{\chi_k} \to 0.$$

Remark. The data in the appendix for $n = 4$ motivates the above recursive form, rather than the periodic behavior of both the Betti numbers and the matrices in the $n = 3$ case.

Proof. We proceed by induction. Our base cases are $m = 1, 2, 3, 4$. The fact that the maps $\phi_1^k, \phi_2^k, \phi_3^k, \phi_4^k$ from Definition 8 form maps for a minimal free resolution is the subject of Lemma 3.22.

From here, suppose that we know that the ϕ_m form the maps for our minimal free resolution for $m \leq j$, where $j \geq 4$. We now consider our map ϕ_{j+1} and show that this is a valid map for our minimal free resolution, using the above lemmas. We can view our matrix as having three blocks; by Lemma 3.17, we can treat each block separately.

The first block, ϕ_{j-1}^{4-i}, clearly gives possible next matrix ϕ_j^{4-i}.

The second block, with diagonal blocks ϕ_{j-2}^i and ϕ_{j-2}^{4-i} and off-diagonal block one of $Y_j, Y_j', s(Y_j)$ (depending on whether i is $1, 2$, or 3, satisfies the hypotheses of 3.18 by 3.20 and the remark after the definition). Then by the definition of Y_{j+1}, Lemma 3.18 tells us that the next matrix for this block can be written as it is in ϕ_{j+1}^i: diagonal blocks ϕ_{j-1}^i and ϕ_{j-1}^{4-i}, and off-diagonal block one of $Y_{j+1}, Y_{j+1}', s(Y_j)$ (depending on whether i is $1, 2$, or 3).

The third block is similar to above, except the off-diagonal block is either $(-1)^j y_1y_3I$ or $(-1)^j y_1^4 I$. The fact that this block transforms appropriately according to Lemma 3.18 follows from Corollary 3.19, so we are done. $\square$

In the proof of Theorem 3.21, we have yet to complete the base case. The base case is the following lemma, whose proof we defer to the appendix.

Lemma 3.22. *Theorem 3.21 holds when $n = 1, 2, 3, 4$.*

As a result of this theorem, we deduce the following, which will allow us to describe the multi-graded betti table of our minimal free resolution.

Corollary 3.23. *Within the free resolution for S^{χ_1}, for $n \geq 4$, we have that*

$$F_n = F_{(n-1),3}(-(1,1,1)) \oplus F_{(n-1),2}(-(0,1,3)) \oplus F_{(n-2),1}(-(4,0,4)) \oplus F_{(n-2),3}(-(3,1,3)) \oplus F_{(n-2),2}(-(1,1,4)),$$

where $F_{n,i}(-(a,b,c))$ is F_n in the free resolution of S^{χ_i}, with degrees shifted up (a,b,c).

Similarly, we have that for S^{χ_2},

$$F_n = F_{(n-1),2}(-(2,1,0)) \oplus F_{(n-1),2}(-(0,1,2)) \oplus F_{(n-2),2}(-(4,0,4)) \oplus F_{(n-2),2}(-(4,1,2)) \oplus F_{(n-2),2}(-(2,1,4)).$$

Remark. Because all of the F_i are free modules, the main content of this corollary is the distribution of degree shifts, and how the multi-degrees of homogeneous basis elements in F_n relate to that of the previous free modules.

Proof. It suffices to prove this for $n = 4$, for then Theorem 3.21 can be inductively applied for all larger n. All of the data referenced in this proof can be found in the proof of Lemma 3.22 in the appendix.

We can begin with starting out at $n = 2$ with the block forms, using $\phi_2 : F_2 \to F_1$. Notice that the first three blocks are already present; we see that in the first block the first term using the `Macaulay2` code has multi-degree shifted from S^{χ_3} by $(1,1,1)$, the second block has multi-degree shifted by $(0,1,3)$, in the case for S^{χ_1}, and for S^{χ_2} these degree shifts are $(2,1,0)$ and $(0,1,2)$.

Now, looking at ϕ_3, we know that blocks X_1 and the combined X_2, X_3 eventually separate into blocks, as argued in Lemma 3.22, when going from $n = 3$ to $n = 4$. We just need to show that the degree shifts match when we analyze the $n = 3$ blocks. This means just looking at the degrees in F_3, both for S^{χ_1} and S^{χ_2}. Recall that the degrees of the columns in the map ϕ_2 are precisely the degrees of basis elements in F_3, which correspond to rows of ϕ_3.

For the map ϕ_2^1, one can check that the set of degrees among third block of columns have the same degrees as S^{χ_1} (and so F_0), but shifted by $(4,0,4)$. Similarly, the degrees of the fourth block of columns (the fifth, fourth, and third-to-last) are those of S^{χ_3} but shifted by $(3,1,3)$, and the fifth block of rows (the last three) have degrees like that of S^{χ_2}, but shifted by $(1,1,4)$. This gives us the set of degree shifts given in S^{χ_1}'s free resolution, since these degree shifts are maintained when we look at the free basis of F_4 (the columns of ϕ_3^1) due to their block form.

Looking now at ϕ_3^2, again we see the third block of rows have multi-degrees like that for S^{χ_2} but shifted by $(4,0,4)$, since the columns for the ϕ_1 map for S^{χ_2} have multi-degrees

$$(4,1,0), (2,2,0), (2,1,2), (2,1,2), (0,2,2), (0,1,4), (4,0,2), (2,0,4).$$

Similarly, the multi-degrees of the fourth block of rows (the fifth, fourth, and third-to-last rows) are that of S^{χ_2} but shifted in degree by $(4,1,2)$, and those of the fifth block of rows (the last three rows in ϕ_3) also resemble that of S^{χ_2}, shifted in degree by $(2,1,4)$. This gives us our desired, since then moving to F_4 all of these degree shifts apply to the direct sum components as per the above. $\square$

We can rephrase this corollary in terms of the following multi-graded Betti number recurrences for the first, third, and second isotypic components respectively, where $n \geq 4$:

$$\begin{aligned}\beta_{n,1,(a_1,a_2,a_3)} = \beta_{n-1,3,(a_1-1,a_2-1,a_3-1)} &+ \beta_{n-1,2,(a_1,a_2-1,a_3-3)} \\ &+ \beta_{n-2,1,(a_1-4,a_2,a_3-4)} + \beta_{n-2,3,(a_1-3,a_2-1,a_3-3)} + \beta_{n-2,2,(a_1-1,a_2-1,a_3-4)},\end{aligned} \tag{6}$$

$$\begin{aligned}\beta_{n,3,(a_1,a_2,a_3)} = \beta_{n-1,1,(a_1-1,a_2-1,a_3-1)} &+ \beta_{n-1,2,(a_1-3,a_2-1,a_3)} \\ &+ \beta_{n-2,3,(a_1-4,a_2,a_3-4)} + \beta_{n-2,1,(a_1-3,a_2-1,a_3-3)} + \beta_{n-2,2,(a_1-4,a_2-1,a_3-1)},\end{aligned} \tag{7}$$

and

$$\begin{aligned}\beta_{n,2,(a_1,a_2,a_3)} = \beta_{n-1,2,(a_1-2,a_2-1,a_3)} &+ \beta_{n-1,2,(a_1,a_2-1,a_3-2)} \\ &+ \beta_{n-2,2,(a_1-4,a_2,a_3-4)} + \beta_{n-2,2,(a_1-4,a_2-1,a_3-2)} + \beta_{n-2,2,(a_1-2,a_2-1,a_3-4)},\end{aligned} \tag{8}$$

where $\beta_{n,i,(a_1,a_2,a_3)}$ is the rank of the (a_1, a_2, a_3) multigraded component (notice that $a_0 = 0$ always) in F_n, for the S^{χ_i} component.

Using generating functions, this multigraded data allows us to obtain a recurrence for the ungraded Betti numbers for each isotypic component, which we phrase as the following corollary.

Corollary 3.24. *With the free resolution of S^{χ_k} where $k = 1, 2, 3$, we have the following recurrence for $i \geq 2$ and $(i, j) \neq (2, 4)$:*

$$\beta_{i,j} = 2\beta_{i-1,j-3} + \beta_{i-1,j-4}.$$

Proof. We construct the generating function

$$\sum_{n=0}^{\infty} t^n \sum_{a_1,a_2,a_3} \beta_{n,i,(a_1,a_2,a_3)} y_1^{a_1} y_2^{a_2} y_3^{a_3}$$

for each isotypic component.

Starting with S^{χ_2}, after plugging in the multigraded initial data for $n = 0, 1, 2, 3$, we obtain the rational function

$$F = \frac{y_2 + y_3^2 + y_1^2(1 - ty_2y_3^2)}{1 - t(y_2y_3^2 + y_1^2y_2 + y_1^2y_3^2)}.$$

Specializing $y \coloneqq y_1 = y_2 = y_3$ to remove the multigrading, one sees the desired recurrence in the denominator of the above fraction: we have $(1 - t(2y^3 + y^4))F = y + 2y^2 - ty^5$. Also, specializing $t = -1$ gives

$$[F]_{t=-1} = \frac{y_2 + y_3^2 + y_1^2 + y_1^2y_2y_3^2}{1 + y_2y_3^2 + y_1^2y_2 + y_1^2y_3^2} = \frac{\mathrm{Hilb}(S^{\chi_2}, \mathbf{y})}{\mathrm{Hilb}(R, \mathbf{y})}.$$

Since the Betti numbers of S^{χ_1} and S^{χ_2} depend on each other, we solve for both of them together. In partcular, letting G and H be the multigraded generating function for S^{χ_1} and S^{χ_2} resolutions respectively, we may first define $\Delta = G - H$ and $\Sigma = G + H$, and solve for each of them, as Δ and Σ satisfy equations that only depend on themselves and F, which we already know. Solving gives Not sure if we want this to be in another form?

$$G = \frac{\Delta + \Sigma}{2} = \frac{y_1 + ty_1y_3^4 + y_3(y_2 + y_3^2) + ty_1^4y_3(y_2 + y_3^2) - ty_1^3y_2(1 + ty_3^4)}{(1 + ty_1^2y_3^2)(1 - t(y_1^2y_2 + y_1^2y_3^2 + y_2y_3^2))}.$$

One has a similar expression

$$H = \frac{\Sigma - \Delta}{2} = \frac{y_3 + ty_3y_1^4 + y_1(y_2 + y_1^2) + ty_3^4y_1(y_2 + y_1^2) - ty_3^3y_2(1 + ty_1^4)}{(1 + ty_3^2y_1^2)(1 - t(y_1^2y_2 + y_1^2y_3^2 + y_2y_3^2))},$$

which could also be obtained from that of G by swapping $y_1 \leftrightarrow y_3$.

Taking G, for example, and specializing to $y \coloneqq y_1 = y_2 = y_3$ and simplifying gives

$$G = \frac{y(1 + y + y^2 - ty^3)}{1 - t(2y^3 + y^4)},$$

which again yields the desired recurrence; the same holds for H.

Also, specializing $t = -1$, after some cancellation (note the change in denominator), yields

$$\begin{aligned}[G]_{t=-1} &= \frac{y_1 + y_1y_2y_3^2 + y_1^3(y_2 + y_3^2) + y_3(y_2 + y_3^2) + y_1^2(y_3 + y_2y_3^3)}{(1 + y_1y_3)(1 + y_2y_3^2 + y_1^2y_3^2 + y_1^2y_2)} \\ &= \frac{\mathrm{Hilb}(S^{\chi_1}, \mathbf{y})}{\mathrm{Hilb}(R, \mathbf{y})},\end{aligned}$$

as well as

$$\begin{aligned}[H]_{t=-1} &= \frac{y_3 + y_3y_2y_1^2 + y_3^3(y_2 + y_1^2) + y_1(y_2 + y_1^2) + y_3^2(y_1 + y_2y_1^3)}{(1 + y_1y_3)(1 + y_2y_3^2 + y_1^2y_3^2 + y_1^2y_2)} \\ &= \frac{\mathrm{Hilb}(S^{\chi_3}, \mathbf{y})}{\mathrm{Hilb}(R, \mathbf{y})},\end{aligned}$$

where again we take the substitution $y_1 \leftrightarrow y_3$. $\square$

Remark. The same recurrence relationship holds if we replace S^{χ_1} with S, since we know that $S = S^{\chi_0} \oplus S^{\chi_1} \oplus S^{\chi_2} \oplus S^{\chi_3}$, and free resolutions of direct sums of modules are direct sums of the free resolutions.

Note that F, G, H, all have denominator with t-degree at most 2, and comparing with (5), the same holds for $n = 3$. This is a consequence of the fact that Eisenbud's resolution for $n = 3$ is 2-periodic, and for $n = 4$, the recursive description of the maps ϕ_i only relied on ϕ_{i-1} and ϕ_{i-2}. This raises the following question.

Guess 3.25. *For higher $n = 5, 6, 7, \ldots$, is $\mathrm{Poin}_R(S^{\chi_k}; t, \mathbf{y})$ always a rational function having denominator with t-degree at most 2? Is it because one eventually has for $i >> 0$ a recursive descriptions of the maps ϕ_i in terms of ϕ_{i-1} and ϕ_{i-2}?*

4 The Garsia-Stanton Method

The goal here is to find an explicit basis for $R = S^G = \mathbb{C}[x_1, \ldots, x_n]^G$ as a module over $S^{\mathfrak{S}_n}$ for a permutation group $G \subset \mathfrak{S}_n$. Garsia and Stanton's paper [2] suggests a general methodology for doing this, given any permutation group G. This methodology involves *partitioning* the faces of the quotient complex Δ_n/G, which we will define.

After finding such a partitioning, one then needs to check invertibility of the square $\{0, 1\}$ incidence matrix describing which restriction faces $R(F_i)$ lie in which facets F_j. Here, these restriction faces $R(F_i)$ are the lower bounds of each of the parts in the partitioning, in a sense which is defined below as well.

Unfortunately, there are relatively few families of permutation groups G for which this has been carried out. Reiner and White [6] proposed such a partitioning when $G = C_n$, for n a prime. We show that their construction provides a valid partitioning.

To do this, we first review some definitions.

Definition 9. Let Δ_n be the order complex of the Boolean algebra on $\{1, \ldots, n\}$. That is, Δ_n is the simplicial complex whose faces are given by **flags** $\mathcal{F}$ of subsets of $\{1, \ldots, n\}$. These flags are chains of the form

$$(\emptyset \subset) F_1 \subset F_2 \subset \ldots \subset F_k (\subset \{1, 2, \ldots, n\}).$$

Note that $0 \leq k \leq n$; if $k = n$, then $\mathcal{F}$ is a **maximal flag**. We define the action of an element g in subgroup $G \subset \mathfrak{S}_n$ on this complex to send the flag

$$(\emptyset \subset) F_1 \subset F_2 \subset \ldots \subset F_k (\subset \{1, 2, \ldots, n\})$$

to

$$(\emptyset \subset) gF_1 \subset gF_2 \subset \ldots \subset gF_k (\subset \{1, 2, \ldots, n\}),$$

where $g(\{x_1, x_2, \ldots, x_\ell\}) = \{gx_1, gx_2, \ldots, gx_\ell\}$. Then, Δ_n/G is the quotient complex, whose faces are orbits of the action of G.

We say that elements $\mathcal{F}_1 \leq \mathcal{F}_2$ in Δ_n/G if $\mathcal{F}_2$ contains each subset in the flag $g\mathcal{F}_1$ for some $g \in G$. Then, we can define the interval $[\mathcal{F}_1, \mathcal{F}_2]$ as the set of flags $\mathcal{F}$ so $\mathcal{F}_1 \leq \mathcal{F} \leq \mathcal{F}_2$.

Given this quotient complex Δ_n/G, we can associate to a permutation a maximal face of this quotient complex as follows.

Definition 10. Given a permutation w, we can assign it the orbit of the flag given by

$$(\emptyset \subset)\{w(1)\} \subset \{w(1), w(2)\} \subset \cdots \subset \{w(1), w(2), \ldots, w(n-1)\} (\subset \{w(1), w(2), \ldots, w(n)\} = \{1, 2, \ldots, n\}).$$

Per the notation of [6], we will write w to also denote the orbit of the maximal flag associated to w (which lies in Δ_n/G).

Assigned to this permutation as well is its $\mathrm{Des}(w)$, the set $\{i \in \{1, 2, \ldots, n-1\} : w(i) > w(i+1)\}$ of descents of w. Given this set, we define $\mathrm{maj}(w)$ to be $\sum_{x \in \mathrm{Des}(w)} x$.

From here, let A_n be the set of $w \in \mathfrak{S}_n$ such that $\mathrm{maj}(w^{-1}) \equiv 0 \pmod{n}$.

Finally, we will write $w|_{\mathrm{Des}(w)}$ to be the orbit of the following subflag of w, namely

$$(\emptyset \subset)\{w(1), w(2), \ldots, w(d_1)\} \subset \{w(1), \ldots, w(d_2)\} \subset \cdots \subset \{w(1), \ldots, w(d_m)\}(\subset \{1, 2, \ldots, n\}),$$

where $d_1, d_2, \cdots, d_m$ are the descents of w.

For instance, let $w = 13254$. Then, $\mathrm{Des}(w) = \{2, 4\}$, meaning that $w|_{\mathrm{Des}(w)} = \{1, 3\} \subset \{1, 2, 3, 5\}$, and $\mathrm{maj}(w) = 6$.

To simplify notation (and following the notation of [6]), we will often drop the braces for the set: as such we can write the above flag as $13 \subset 1235$. Another element in its orbit is $24 \subset 1234$.

We are now ready to state the theorem. Here, $G = C_n$ for n prime.

Theorem 4.1. *The formula* $\Delta_n/C_n = \bigsqcup_{w \in A_n} [w|_{Des(w)}, w]$ *given as Question 6.1 in [6] is indeed a partitioning.*

To make things precise in the proof, let c generate C_n and $\mathcal{F}$ be a flag in Δ_n; then define $c^m\mathcal{F}$ to be the flag formed by incrementing all entries of $\mathcal{F}$ by $m \pmod n$. We can view the quotient complex Δ_n/C_n then as the complex Δ_n modulo the relation $\mathcal{F} = c\mathcal{F}$ for any flag $\mathcal{F}$.

We first present the idea of the proof of Theorem 4.1 with an illustrative example.

Example. Let $n = 5$ and our flag be $\mathcal{F} = (1 \subset 134)$. We wish to find $w \in A_n$ such that $w|_{\mathrm{Des}(w)} \subset c^m\mathcal{F} \subset w$, for some m. First suppose $m = 0$. Write $w = abcde$ in one-line notation; then our condition is equivalent to the following criteria:

- $a = 1$ and $\{a, b, c\} = \{1, 3, 4\}$;
- The descents of w are a subset of $\{1, 3\}$, so $b < c$ and $d < e$.

In this case, we conclude $a = 1, abc = 134$, and $abcde = 13425$. Doing a similar process for $m = 1, 2, 3, 4$, we get a list of five candidate flags to check:

$\mathcal{F}$	$a.bc.de$	$\mathrm{maj}(w^{-1}) \bmod n$
$(1 \subset 134)$	1.34.25	2
$(2 \subset 245)$	2.45.13	4
$(3 \subset 351)$	3.15.24	1
$(4 \subset 412)$	4.12.35	3
$(5 \subset 523)$	5.23.14	0

Note that exactly one of them gives a statistic of 0 mod n, so exactly one of the candidate flags belongs to A_n. Hence, there is exactly one w such that $[w|_{\mathrm{Des}(w)}, w]$ contains $\mathcal{F}$.

In the example, notice that the third column forms an arithmetic sequence $2, 4, 1, 3, 0$ mod 5. This is not a coincidence. To formalize this idea, we will need a few definitions.

Definition 11. For a flag $\mathcal{F} = (F_1 \subset F_2 \subset \cdots \subset F_r)$, define $k(\mathcal{F})$ to be the number of cyclic inverse descents of the flag, i.e. the number of $1 \leq i \leq n$ such that $i + 1$ belongs to a strictly earlier F_j than i (where we take $n + 1$ to mean 1). Notice that $k(\mathcal{F})$ only depends on the equivalence class of $\mathcal{F}$ mod C_n.

Also, define the permutation $f(\mathcal{F})$ as follows: first, sort each F_i in increasing order. Then, concatenate $F_1, F_2 \setminus F_1, \cdots, F_r \setminus F_{r-1}, \{1, 2, \ldots, n\} \setminus F_r$ in that order to get the one-line notation of $f(\mathcal{F})$. Observe that $\mathcal{F}$ need not be maximal: for instance if $\mathcal{F} = \{1\} \subset \{1, 3, 4\}$, then $f(\mathcal{F})$ is the permutation 13425.

Now, we are ready to state our key lemma, which holds, like in [6], even when n is not prime:

Lemma 4.2. *We have* $\mathrm{maj}(f(c\mathcal{F})^{-1}) \equiv \mathrm{maj}(f(\mathcal{F})^{-1}) + k(\mathcal{F}) \pmod n$.

This lemma generalizes the main claim in Proposition 6.5 from [6] if we take $\mathcal{F}$ to be a maximal flag.

Proof. From the proof of Proposition 6.5 of [6], we know that $\mathrm{maj}(f(\mathcal{F})^{-1})$ equals the sum of the values of the cyclic inverse descent set of $f(\mathcal{F})$ (which we abbreviate as CDS) The CDS of $f(\mathcal{F})$ might differ from the CDS of $\mathcal{F}$, but only by n (namely, when n and 1 belong to the same F_j in the flag). Hence $\mathrm{maj}(f(\mathcal{F})^{-1})$ is equivalent to the sum of values in the CDS of $\mathcal{F}$. Since the values of the CDS increase by 1 mod n when passing from $\mathcal{F}$ to $c\mathcal{F}$, we see that $\mathrm{maj}(f(c\mathcal{F})^{-1}) - \mathrm{maj}(f(\mathcal{F})^{-1}) \equiv k(\mathcal{F}) \pmod n$, as desired. $\square$

Now we are ready to prove the main theorem.

Proof of Theorem 4.1. It suffices to show that for any flag $\mathcal{F}$, there exists a unique $w \in A_n$ such that $w|_{\mathrm{Des}(w)} \subset c^m\mathcal{F} \subset w$. Write $\mathcal{F} = F_1 \subset F_2 \subset \cdots \subset F_r$. Notice that the argument in the Example can be run for each m. Done more generally, the condition that $c^m\mathcal{F} \subset w$ implies that $w(\{1, 2, \ldots, |F_i|\}) = c^m F_i$ for $i = 1, 2, \ldots, r$, and the condition $w|_{\mathrm{Des}(w)} \subset c^m\mathcal{F}$ implies that w is increasing on $\{|F_{i-1}|+1, |F_{i-1}|+2, \ldots, |F_i|\}$ (where $F_0 = \emptyset$). But this uniquely specifies w for each m; in fact, this requires that $w = f(c^m\mathcal{F})$.

We now want to show that there exists a unique $w \in A_n$ for which $w = f(c^m\mathcal{F})$ for some m. By repeating Lemma 4.2, we have that

$$\mathrm{maj}(f(c^m\mathcal{F})^{-1}) \equiv \mathrm{maj}(f(\mathcal{F})^{-1}) + mk(\mathcal{F}) \pmod n.$$

However, the fact $1 \le k(\mathcal{F}) \le n-1$ implies that there exists a unique m such that $\mathrm{maj}(f(c^m\mathcal{F})^{-1}) \equiv 0 \pmod n$; $k(\mathcal{F})$ is invertible $\pmod n$ as n is prime. Hence, the choice for w is unique as well. $\square$

5 Appendix

5.1 Proof of Lemma 3.22

Recall the statement of the lemma: Theorem 3.21 holds when $j = 1, 2, 3, 4$.

Throughout this proof, we will refer to the matrices given in definition 8. We will drop the superscript when the isotypic component we're referring to isn't relevant (or has been mentioned previously).

A useful fact for us here, from [5, Theorem 2.12], is that minimal generating sets of a finitely generated graded module over a graded Noetherian ring R have the same size.

Proof. This proof consists of going through each value of j one at a time, beginning with the map ϕ_1. We work with all the isotypic components at the same time. The main goal is to show that we can obtain the matrices referenced in Theorem 3.21 in our minimal free resolution.

The map $\phi_1 : F_1 \to F_0$. These matrices can be explicitly computed using `Macaulay2`. The resulting matrices, after applying invertible column operations, are the following, first for S^{χ_1} :

$$\phi_1^1 = \begin{pmatrix} y_2y_3^2 & y_1y_2^2y_3 & y_1^3y_2y_3 & 0 & 0 & 0 & y_3^4 & y_1^3y_3^3 \\ -y_1y_3 & -y_1^2y_2 & -y_1^4 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 & 0 & 0 \\ 0 & 0 & 0 & -y_1^2y_2 & -y_2^2 & -y_2y_3^2 & -y_1y_3 & -y_1^4 \end{pmatrix},$$

with the first row corresponding to the basis element $\mathbf{e}_1 \in F_0$ (which maps to y_1 under ϕ_0^1), the second row corresponding to $\mathbf{e}_2$ (which maps to y_2y_3), and the third row corresponding to $\mathbf{e}_3$ (which maps to y_3^3). Similarly, we see that for S^{χ_2} this is

$$\phi_1^2 = \begin{pmatrix} y_1^2y_2 & y_2^2 & y_2y_3^2 & 0 & 0 & 0 & y_1^2y_3^2 & y_3^4 \\ -y_1^4 & -y_1^2y_2 & -y_1^2y_3^2 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 & 0 & 0 \\ 0 & 0 & 0 & -y_1^2y_2 & -y_2^2 & -y_2y_3^2 & -y_1^4 & -y_1^2y_3^2 \end{pmatrix},$$

with the first row corresponding to the basis element which maps to y_3^2 under ϕ_0^2, the second row corresponding to the basis element which maps to y_2 and the third row corresponding to the basis element which maps to y_1^2. Finally, for S^{χ_3} we have

$$\phi_1^3 = \begin{pmatrix} y_1^2y_2 & y_1y_2^2y_3 & y_1y_2y_3^3 & 0 & 0 & 0 & y_1^4 & y_1^3y_3^3 \\ -y_1y_3 & -y_2y_3^2 & -y_3^4 & y_1^2y_3^2 & y_1^2y_2 & y_1^4 & 0 & 0 \\ 0 & 0 & 0 & -y_2y_3^2 & -y_2^2 & -y_1^2y_2 & -y_1y_3 & -y_3^4 \end{pmatrix},$$

with the first row corresponding to the basis element which maps to y_3 under ϕ_0^3, the second row corresponding to the basis element which maps to y_1y_2, and the third row corresponding to the basis element which maps to y_1^3. There are precisely the matrices that we wanted in 3.21.

The map $\phi_2 : F_2 \to F_1$. Again, we can compute the matrices for S^{χ_1}, S^{χ_2} and S^{χ_3}. For the sake of space, notice that we can get between the S^{χ_1} and S^{χ_3} components by our map s, which swaps y_1 and y_3. Thus, it suffices to prove that the ϕ_2 as defined in Definition 8 is the matrix of the next map in the minimal free resolution for the S^{χ_1}, S^{χ_2} isotypics, as the s map will allow us to conclude this for S^{χ_3}.

For S^{χ_1}, we can employ a similar logic as in the inductive argument. We claim that we can use the map given in Definition 8. We need to verify that these columns are elements of the kernel of ϕ_1^1, and that they are minimal generators for the module they generate.

To see this, we again rely on `Macaulay2`. Applying column operations to the original `Macaulay2` output, we now arrive at the following form for the matrix, which is the form given in Definition 8:

$$\begin{pmatrix} \phi_1^3 & 0 & 0 & \begin{pmatrix} y_1^5y_3 & y_1^3y_2y_3 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ 0 & \phi_1^2 & 0 & \begin{pmatrix} y_1^4 & y_1^2y_2 & y_1y_3 & 0 & 0 \\ 0 & 0 & 0 & y_1y_3 & 0 \\ 0 & 0 & 0 & 0 & y_1y_3 \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} y_1^4 & y_1^3y_2y_3 & y_1^3y_3^3 \\ -y_1y_3 & -y_2y_3^2 & -y_3^4 \end{pmatrix} & \begin{pmatrix} 0 & 0 & -y_1^2y_2 & y_2^2 & -y_2y_3^2 \\ -y_1^2y_2 & -y_2^2 & 0 & 0 & 0 \end{pmatrix} \end{pmatrix}.$$

Similarly, for S^{χ_2}, the second part of the free resolution, using a similar logic, has the matrix given in the Definition 8, which we can again see this with `Macaulay2`. Again, following the procedure above, we end up with the following matrix:

$$\phi_2^2 = \begin{pmatrix} \phi_1^2 & 0 & 0 & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ y_1^4 & y_1^2y_2 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 \end{pmatrix} \\ 0 & \phi_1^2 & 0 & \begin{pmatrix} y_1^4 & 0 & 0 & 0 & 0 \\ 0 & y_1^4 & 0 & 0 & 0 \\ 0 & 0 & y_1^4 & y_1^2y_3^2 & y_1^2y_3^2 \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} y_1^2y_3^2 & y_2y_3^2 & y_3^4 \\ -y_1^4 & -y_1^2y_2 & y_1^2y_3^2 \end{pmatrix} & \begin{pmatrix} -y_1^2y_2 & -y_2^2 & -y_2y_3^2 & 0 & 0 \\ 0 & 0 & 0 & -y_2^2 & -y_2y_3^2 \end{pmatrix} \end{pmatrix}.$$

We can verify that each of the columns is homogeneous as well by explicit computation, keeping in mind the multi-degrees of the columns in the ϕ_1 (which are assigned so ϕ_1 fixes degree). Finally, for minimality, from [5, Theorem 2.12], all minimal generating sets have the same size, and from the data in the Appendix this size for $\ker \phi_1$ is 24 for both S^{χ_1} and S^{χ_2}. Thus, as we have 24 columns that generate the kernel of ϕ_1, they are a minimal generating set.

The map $\phi_3 : F_3 \to F_2$.

Unlike the previous two cases, we do not use `Macaulay2` and attempt to apply column operations, as this would be too large. Rather, we analyze the matrices more algebraically to show that these matrices are the next step in the minimal free resolution.

For S^{χ_1}, recall that we want to consider the following matrix:

$$\phi_3^1 = \begin{pmatrix} \phi_2^3 & 0 & 0 & 0 & 0 \\ 0 & \phi_2^2 & 0 & 0 & -y_1y_3I \\ 0 & 0 & \phi_1^1 & Y_3 & 0 \\ 0 & 0 & 0 & X_1 & X_2 \end{pmatrix},$$

where

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & -y_2y_3^2 & -y_2^2 & -y_1^2y_2 & -y_1y_3 & -y_3^4 \\ -y_1y_3 & -y_2y_3^2 & -y_3^4 & y_1^2y_3^2 & y_1^2y_2 & y_1^4 & 0 & 0 \\ y_1^2y_2 & y_1y_2^2y_3 & y_1y_2y_3^3 & 0 & 0 & 0 & y_1^4 & y_1^3y_3^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_1^2y_2 & y_1y_2^2y_3 & y_1y_2y_3^3 & 0 & 0 & 0 & y_1^4 & y_1^3y_3^3 \\ -y_1y_3 & -y_2y_3^2 & -y_3^4 & y_1^2y_3^2 & y_1^2y_2 & y_1^4 & 0 & 0 \\ 0 & 0 & 0 & -y_2y_3^2 & -y_2^2 & -y_1^2y_2 & -y_1y_3 & -y_3^4 \end{pmatrix},$$

and

$$Y_3 = -\begin{pmatrix} y_2^2 & 0 & 0 & 0 & 0 & 0 & y_1^2y_2 & 0 \\ 0 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2^2 & 0 & 0 & 0 & 0 & y_1^2y_2 \end{pmatrix}.$$

Similarly, for S^{χ_2}, we have the matrix

$$\phi_3^2 = \begin{pmatrix} \phi_2^2 & 0 & 0 & 0 & 0 \\ 0 & \phi_2^2 & 0 & -y_1^4 I & 0 \\ 0 & 0 & \phi_1^2 & 0 & Y_3' \\ 0 & 0 & 0 & X_1' & X_2' \end{pmatrix},$$

where

$$X_1' = \begin{pmatrix} y_2y_3^2 & y_1y_2^2y_3 & y_1^3y_2y_3 & 0 & 0 & 0 & y_3^4 & y_1^3y_3^3 \\ -y_1y_3 & -y_1^2y_2 & -y_1^4 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 & 0 & 0 \\ 0 & 0 & 0 & -y_1^2y_2 & -y_2^2 & -y_2y_3^2 & -y_1y_3 & -y_1^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$X_2' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_2y_3^2 & y_1y_2^2y_3 & y_1^3y_2y_3 & 0 & 0 & 0 & y_3^4 & y_1^3y_3^3 \\ -y_1y_3 & -y_1^2y_2 & -y_1^4 & y_1^2y_3^2 & y_2y_3^2 & y_3^4 & 0 & 0 \\ 0 & 0 & 0 & -y_1^2y_2 & -y_2^2 & -y_2y_3^2 & -y_1y_3 & -y_1^4 \end{pmatrix},$$

and

$$Y_3' = -\begin{pmatrix} y_2^2 & 0 & 0 & 0 & 0 & 0 & y_2y_3^2 & 0 \\ 0 & y_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_2^2 & 0 & 0 & 0 & 0 & y_2y_3^2 \end{pmatrix}.$$

We can verify that each column of ϕ_3^1, for instance, lives in the kernel of ϕ_2^1, and similarly for ϕ_3^2 and ϕ_2^2. Notice that each block of rows in ϕ_3^1 corresponds to a block of columns in ϕ_2^1, which helps with the explicit verification.

In order for our ϕ_3^k maps to be part of our minimal free resolution, they need to minimal generate the kernel of ϕ_2^k. It is enough to show that they generate the kernel. Once we've shown this, the minimality follows from the fact that, from `Macaulay2`, we know that the rank of F_3 is 72. This means that a minimal set of generators for the kernel will have 72 columns. But notice that we have $24 + 24 + 8 + 8 + 8 = 72$ columns in our matrix, which generate the kernel, and as we noted before, all minimal generating sets have the same size by [5, Theorem 2.12].

We now show that the columns of ϕ_3^1 generate the kernel of ϕ_2^1; the S^{χ_2} case follows basically the same argument (although c_{45}, c_{44} will be replaced with c_{41}, c_{42}, as we'll define later). First, suppose that we have a vector that lies in the kernel of ϕ_2^1.

If such a vector has zeroes in the last five entries, it follows that it can be written as a block vector form $\mathbf{c} = \begin{pmatrix} \mathbf{c_1} & \mathbf{c_2} & \mathbf{c_3} & \mathbf{0} \end{pmatrix}^T$, with $\mathbf{c_1}$ getting sent to zero by the first block of ϕ_2^1, $\mathbf{c_2}$ by the second block, and $\mathbf{c_3}$ by the third block. But notice that the columns of each of the three diagonal blocks, by our previous cases, generate the kernels for those respective blocks within ϕ_2^1. For the third block, notice that the span of the columns of the seventeenth, eighteenth, and nineteenth columns in ϕ_2^1 generate a S^{C_n} module isomorphic to S^{χ_1}, by sending $p \in S^{\chi_1}$ to $\begin{pmatrix} py_3^2 \\ -py_1^2 \end{pmatrix}$. Then, notice that we may think of this block matrix as being a ϕ_0^1 map, whose kernel is minimally generated by the columns of ϕ_1^1 map. In this case, we see that our vector can be generated by the columns of ϕ_3^1.

Otherwise, we suppose the last five coordinates of the vector aren't zero. It suffices to show that each *homogeneous* $\mathbf{c} \in \ker \phi_2^1$ can be expressed as a linear combination of the columns in ϕ_3^1. Our main approach from here will be to subtract off linear combinations of columns in ϕ_3^1 and show we eventually get another vector that we already know is a linear combination of columns in ϕ_3^1. In particular, we will eventually reduce to the case where $\mathbf{c_4} = \mathbf{0}$.

Now, if X is submatrix of the last five columns of ϕ_2^1, and $\mathbf{c} = \begin{pmatrix} \mathbf{c_1} & \mathbf{c_2} & \mathbf{c_3} & \mathbf{c_4} \end{pmatrix}^T$ is a block vector, it follows that $X\mathbf{c_4}$ is a linear combination of the first 19 columns. In particular, we need

$$X\mathbf{c_4} = \begin{pmatrix} -\phi_1^3 \mathbf{c_1} \\ -\phi_1^2 \mathbf{c_2} \\ -\begin{pmatrix} y_1^4 & y_1^3 y_2 y_3 & y_1^3 y_3^3 \\ -y_1 y_3 & -y_2 y_3^2 & -y_3^4 \end{pmatrix} \mathbf{c_3} \end{pmatrix}.$$

But this forces $\phi_1^3 \mathbf{c_1} = \mathbf{0}$, since we see that the only possible nonzero coordinate in the first three rows of $X\mathbf{c_4}$ is the third row, but $\phi_1^3 \mathbf{c_1}$ cannot have just one nonzero coordinate.

Thus, we see that $\mathbf{c_1}$ lies in the kernel of ϕ_1^3, which means it is a linear combination of columns in ϕ_2^3. But then we may subtract off a linear combination of the corresponding columns in the first block in ϕ_1^3 to get the vector $\mathbf{c}' = \begin{pmatrix} \mathbf{0} & \mathbf{c_2} & \mathbf{c_3} & \mathbf{c_4} \end{pmatrix}^T$, which also lies in the kernel of ϕ_1^2.

It suffices to now consider vectors of this form. Suppose that $\mathbf{c_4}$ has coordinates $\begin{pmatrix} c_{41} & c_{42} & c_{43} & c_{44} & c_{45} \end{pmatrix}^T$. We will first show that we can assume that $c_{45} = 0$.

Indeed, suppose that $c_{45} \neq 0$. Then, observe that it cannot be the only nonzero coordinate in $\mathbf{c_4}$, similar to how we saw above with our argument for $\mathbf{c_1}$. But then, looking at the seventh coordinate of $\phi_1^2 \mathbf{c}' = 0$ and noting that $\mathbf{c}'$ is still homogeneous (meaning that each coordinate $\phi_1^2 \mathbf{c}'$ will also be homogeneous), we require that c_{45} is divisible by one of $y_1^4, y_1^2 y_2, y_1^2 y_3^2, y_2 y_3^2, y_2^2$, depending on which of the other coordinates is nonzero. We can suppose that it is divisible by $y_1^2 y_2$; the other four follow by a similar logic.

But then if $c_{45} = c'_{45} y_1^2 y_2$, then

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{c_2} \\ \mathbf{c_3} \\ \begin{pmatrix} c_{41} \\ c_{42} \\ c_{43} \\ c_{44} \\ c_{45} \end{pmatrix} \end{pmatrix} + c'_{45} \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{v} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_1^4 \\ -y_1^2 y_2 \end{pmatrix} \end{pmatrix}$$

has last coordinate zero, with this second vector coming from the third-to-last column of ϕ_3^1. Thus, we may assume $c_{45} = 0$.

Similarly, we will show that we can assume $c_{44} = 0$. Again, suppose that $c_{44} \neq 0$. Then, for the seventh coordinate of $\phi_2^1 c'$ to be zero, again by homogenity we require c_{44} to be divisible by one of $y_1^4, y_1^3 y_2 y_3, y_1^2 y_3^2, y_1^2 y_2$. But again, like in the c_{45} case, each of these appear as entries in the row of X_2 corresponding to the c_{44} variable, meaning we may assume that $c_{44} = 0$.

As a result, similarly to $\mathbf{c_1}$, we can assume that $\mathbf{c_2} = \mathbf{0}$ at this point. Indeed, notice that $\phi_2^1 c'$ is equal to $\phi_1^2 \mathbf{c_2} + \begin{pmatrix} y_1^4 c_{41} + y_1^2 y_2 c_{42} + y_1 y_3 c_{43} \\ 0 \\ 0 \end{pmatrix}$. But to be zero, as $\phi_1^2 \mathbf{c_2}$ cannot have just one nonzero coordinate, we require $\phi_1^2\mathbf{c_2} = 0$, meaning that $\mathbf{c_2}$ is a linear combination of the columns of ϕ_2^2. Subtracting off the corresponding linear combination of columns in the second block from ϕ_3^1 allows then to assume that $\mathbf{c_2} = 0$.

From here, we will show that $\mathbf{c_4}$ can be taken to be zero. Indeed, notice that this equation also forces

$$y_1^4c_{41} + y_1^2y_2c_{42} + y_1y_3c_{43} = 0.$$

But again, if $c_{41} \neq 0$, it must be divisible by one of $y_2y_3^2, y_1^2y_2, y_1y_3, y_3^4, y_2^2$. There then exists a multiple of a column of ϕ_3^1 that we can subtract off so the resulting vector is $\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{c_3} \\ \begin{pmatrix} 0 \\ c_{42}' \\ c_{43}' \\ 0 \\ 0 \end{pmatrix} \end{pmatrix}$. It is not hard to then see that this is a multiple of yet another column of X_1.

To summarize, there exists a linear combination of columns in ϕ_3^1 that we can subtract from

$$\mathbf{c}' = \begin{pmatrix} \mathbf{c_1} & \mathbf{c_2} & \mathbf{c_3} & \mathbf{c_4} \end{pmatrix}^T \in \ker \phi_2^1$$

so the resulting vector has the form $c = \begin{pmatrix} \mathbf{c_1}' & \mathbf{c_2}' & \mathbf{c_3}' & \mathbf{0} \end{pmatrix}^T$, taking us back to the $\mathbf{c_4} = \mathbf{0}$ case, which we've already discussed. We can then conclude that the columns of ϕ_3^1 generate the kernel of ϕ_2^1, and thus minimally generate the kernel.

To finish, we want our minimal generating set to be a homogeneous minimal generating set in our multi-graded free resolution. In order to prove this, we first note that each column of ϕ_3^1 has at most 3 nonzero entries. Now, consider some column v of ϕ_3^1, which we can write as $d_1\mathbf{e}_1 + d_2\mathbf{e}_2 + d_3\mathbf{e}_3$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are basis elements of F_2 and $d_1, d_2, d_3 \in S^{C_n}$. We know that v lies in the kernel of ϕ_2^1; but as ϕ_2^1 preserves multi-degrees by construction, we require that $d_1\phi_2^1(\mathbf{e}_1) + d_2\phi_2^1(\mathbf{e}_2) + d_3\phi_2^1(\mathbf{e}_3) = 0$, where each term is homogeneous. We want to now show that all terms have the same degree.

First, notice that $\phi_2^1(\mathbf{e}_i)$ are zero, and this equation holds if we fix any multi-graded component. But then either all terms where $d_i \neq 0$ are the same multi-degree, or some multi-degree only has one nonzero term, which is impossible. Thus, all the terms $d_i\mathbf{e}_i$ where $d_i \neq 0$ are the same multi-degree, or that v is homogeneous.

Thus, we see that we may take the form for the ϕ_3 maps for the minimal free resolutions of S^{χ_1} and S^{χ_2}. We are now ready to begin the base case of the inductive portion in Theorem 3.21, with the $j = 4$ case.

The map $\phi_4 : F_4 \to F_3$. For S^{χ_1} and S^{χ_2} we claim that we can take matrices given in Theorem 3.21. To do this, we show that the columns of our proposed ϕ_4 matrix form a minimal generating set for the kernel of ϕ_3 for each of S^{χ_1}, S^{χ_2}. Once we do this, we will have completed the base case and thus proven the theorem.

The argument that this generates the kernel, like in Theorem 3.21, can be seen to come from Lemma 3.18. In order to apply this lemma, we need to verify that Y_3, Y_3' satisfies the properties given in Lemma 3.18, so that we can apply it to show that ϕ_4^1 and ϕ_4^2 are the maps from $F_4 \to F_3$ in the minimal free resolution. For instance, X_1 is the matrix ϕ_1^3, but with the rows flipped. But then notice that the columns ϕ_2^3 still minimally generate the kernel of X_1, so we check with each column of $\phi_2^3 = s(\phi_2^3)$. For instance, using the nineteenth column of ϕ_2^3, we check that

$$Y_3 \begin{pmatrix} y_3^5y_1 \\ 0 \\ 0 \\ y_3^4 \\ 0 \\ 0 \\ 0 \\ -y_3^2y_2 \end{pmatrix} = \begin{pmatrix} y_1y_2^2y_3^5 \\ -y_1^2y_2^2y_3^2 \end{pmatrix} = y_1y_2^2y_3 \begin{pmatrix} y_3^4 \\ -y_1y_3 \end{pmatrix},$$

the seventh column of ϕ_1^1. Thus, we may use Lemma 3.18 to show that the columns of ϕ_4 form a homogeneous generating set for the kernel of ϕ_3.

In order to obtain the matrices ϕ_4 in Definition 8 from Lemma 3.18, we need to show that the kernel of the matrix $\begin{pmatrix} X_1 & X_2 \end{pmatrix}$ is generated by the columns of $\begin{pmatrix} \phi_2^3 & 0 \\ 0 & \phi_2^2 \end{pmatrix}$. Indeed, if $\begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \end{pmatrix}$ (both being 8-dimensional vectors) lies in the kernel of $\begin{pmatrix} X_1 & X_2 \end{pmatrix}$, then $X_2\mathbf{v_2}$ cannot have the only nonzero coordinate be the third row and still have $X_1\mathbf{v_1} + X_2\mathbf{v_2} = 0$, meaning that $\mathbf{v_2}$ is in $\ker \phi_1^2$ and so $\mathbf{v_1} \in \ker \phi_1^3$.

For minimality, notice that we can track the sizes of our matrices. We saw that, for each isotypics, ϕ_1 was 3×8 and ϕ_2 was 8×24, so ϕ_3 was 24×72 and thus ϕ_4 is 72×216. However, from `Macaulay2` computations in the appendix, we know that F_4 in the minimal resolution has rank 216, meaning that our generators are in fact a minimal set (again, by [5, Theorem 2.12], all minimal generating sets have the same size). It follows that our proposed ϕ_4 is indeed a matrix that we can take for the next step in our minimal free resolution.

This finishes the base case, and so we've proven the lemma. $\square$

5.2 Free Resolution Data

Here, we present some data on the free resolutions of $n = 4, 5, 6$. Some of the data was used to justify minimality in the proof of Lemma 3.22; as such we include it here for completeness. As for the $n = 5, 6$ case one can take a look at the shape the nonzero entries form in the Betti table as a way to motivate Conjecture 5.5. We also hope this data will be useful in trying to obtain some concrete statements about the minimal free resolutions for $n = 5, 6$.

5.2.1 The case $n = 4$

Here is the Betti table for S as a S^{C_4}-module; row j, column i represents the value of $\beta_{i,i+j}$.

```
              0  1  2   3   4    5    6     7
o13 = total: 10 24 72 216 648 1944 5832 17496
          0:  1  .  .   .   .    .    .     .
          1:  3  .  .   .   .    .    .     .
          2:  4  .  .   .   .    .    .     .
          3:  2  4  .   .   .    .    .     .
          4:  . 10  .   .   .    .    .     .
          5:  .  8  8   .   .    .    .     .
          6:  .  2 24   .   .    .    .     .
          7:  .  . 26  16   .    .    .     .
          8:  .  . 12  56   .    .    .     .
          9:  .  .  2  76  32    .    .     .
         10:  .  .  .  50 128    .    .     .
         11:  .  .  .  16 208   64    .     .
         12:  .  .  .   2 176  288    .     .
         13:  .  .  .   .  82  544  128     .
         14:  .  .  .   .  20  560  640     .
         15:  .  .  .   .   2  340 1376   256
         16:  .  .  .   .   .  122 1664  1408
         17:  .  .  .   .   .   24 1240  3392
         18:  .  .  .   .   .    2  584  4704
         19:  .  .  .   .   .    .  170  4144
         20:  .  .  .   .   .    .   28  2408
         21:  .  .  .   .   .    .    2   924
         22:  .  .  .   .   .    .    .   226
         23:  .  .  .   .   .    .    .    32
         24:  .  .  .   .   .    .    .     2
```

Rather than considering all of S at once, one can simplify a bit by considering resolutions over ~~$R = S^{C_4}$~~ of the separate C_4-isotypic summands here:

$$S = S^{C_4} \oplus S^{\chi_1} \oplus S^{\chi_2} \oplus S^{\chi_3}$$

Again, recall due to symmetry that the only isotypic components we are worried about are χ_1 and χ_2.

For the χ_1 isotypic component, we can use `Macaulay2` code to compute the resolution, yielding the following Betti table for S^{χ_1} as a module over S^{C_n}:

```
               0  1  2   3   4    5    6
o10 = total:   3  8 24  72 216  648 1944
          1:   1  .  .   .   .    .    .
          2:   1  .  .   .   .    .    .
          3:   1  1  .   .   .    .    .
          4:   .  3  .   .   .    .    .
          5:   .  3  2   .   .    .    .
          6:   .  1  7   .   .    .    .
          7:   .  .  9   4   .    .    .
          8:   .  .  5  16   .    .    .
          9:   .  .  1  25   8    .    .
         10:   .  .  .  19  36    .    .
         11:   .  .  .   7  66   16    .
         12:   .  .  .   1  63   80    .
         13:   .  .  .   .  33  168   32
         14:   .  .  .   .   9  192  176
         15:   .  .  .   .   1  129  416
         16:   .  .  .   .   .   51  552
         17:   .  .  .   .   .   11  450
         18:   .  .  .   .   .    1  231
         19:   .  .  .   .   .    .   73
         20:   .  .  .   .   .    .   13
         21:   .  .  .   .   .    .    1
```

Also, using `Macaulay2` computation, we were able to find the Hilbert series of the module $M = S^{C_4,\chi_1}$ is equal to the following rational function: $\frac{t+t^2+t^3}{(1-t^3)(1-t^2)^2(1-t)} = \frac{t}{(1-t)^4(1+t)^2}$. But at the same time, the graded ring S^{C_4} has the Hilbert series $\frac{1+2t^3+t^4}{(1-t^4)(1-t^2)^2(1-t)}$.

Therefore, it follows that the series $\sum_{i,j}(-1)^i \beta_{i,j} t^j$ is equal to their quotient, namely $\frac{t(1+t^2)(1+t)}{1+2t^3+t^4} = \frac{t+t^2+t^3+t^4}{1+2t^3+t^4}$. One can make a similar statement if we instead consider the multi-graded case, for computing the series $\sum_{i,j}(-1)^i \beta_{i,\mathbf{a}} \mathbf{t}^{\mathbf{a}}$, where we understand $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} t_2^{a_2} t_3^{a_3}$.

We find that for the module $M = S^{\chi_1}$ we have the following rational function for the Hilbert series:

$$\mathrm{Hilb}(M; t_1, t_2, t_3, t_4) = \frac{t_1 + t_2 t_3 + t_1^2 t_3 + t_3^3 + t_1^3 t_2 + t_1 t_2 t_3^2 + t_1^3 t_3^2 + t_1^2 t_2 t_3^3}{(1-t_3^4)(1-t_1^4)(1-t_2)^2(1-t_0)}. \tag{9}$$

For the invariant ring, we see that the Hilbert series is equal to

$$\mathrm{Hilb}(S^{C_4}; t_1, t_2, t_3, t_4) = \frac{1 + t_1 t_3 + t_1^2 t_2 + t_2 t_3^2 + t_1^2 t_3^2 + t_1^3 t_2 t_3 + t_1 t_2 t_3^3 + t_1^3 t_3^3}{(1-t_3^4)(1-t_1^4)(1-t_2)^2(1-t_0)}, \tag{10}$$

meaning that our series with the Betti coefficients is now

$$\sum_{i,\mathbf{a}}(-1)^i\beta_{i,\mathbf{a}}\mathbf{t}^{\mathbf{a}} = \frac{t_1 + t_2t_3 + t_1^2t_3 + t_3^3 + t_1^3t_2 + t_1t_2t_3^2 + t_1^3t_3^2 + t_1^2t_2t_3^3}{1 + t_1t_3 + t_1^2t_2 + t_2t_3^2 + t_1^2t_3^2 + t_1^3t_2t_3 + t_1t_2t_3^3 + t_1^3t_3^3}; \tag{11}$$

we can see that setting all the t_i equal to t yields the same Hilbert series.

Similarly, for the χ_2 component we get the following generating function:

$$\sum_{i,\mathbf{a}}(-1)^i\beta_{i,\mathbf{a}}\mathbf{t}^{\mathbf{a}} = \frac{t_2 + t_1^2 + t_3^2 + t_1t_2t_3 + t_1^3t_3 + t_1t_3^3 + t_1^2t_2t_3^2 + t_1^3t_2t_3^3}{1 + t_1t_3 + t_1^2t_2 + t_2t_3^2 + t_1^2t_3^2 + t_1^3t_2t_3 + t_1t_2t_3^3 + t_1^3t_3^3}. \tag{12}$$

For completeness, here is the table for the χ_2-isotypic component. Notice that the same dimensions are present, and that we have the same recurrence relationship here from Corollary 3.24.

```
            0  1  2   3   4    5    6
    total:  3  8 24  72 216  648 1944
        1:  1  .  .   .   .    .    .
        2:  2  .  .   .   .    .    .
        3:  .  2  .   .   .    .    .
        4:  .  4  .   .   .    .    .
        5:  .  2  4   .   .    .    .
        6:  .  . 10   .   .    .    .
        7:  .  .  8   8   .    .    .
        8:  .  .  2  24   .    .    .
        9:  .  .  .  26  16    .    .
       10:  .  .  .  12  56    .    .
       11:  .  .  .   2  76   32    .
       12:  .  .  .   .  50  128    .
       13:  .  .  .   .  16  208   64
       14:  .  .  .   .   2  176  288
       15:  .  .  .   .   .   82  544
       16:  .  .  .   .   .   20  560
       17:  .  .  .   .   .    2  340
       18:  .  .  .   .   .    .  122
       19:  .  .  .   .   .    .   24
       20:  .  .  .   .   .    .    2
```

5.2.2 The case $n = 5$

This is the betti table for S^{χ_1, C_5}. Notice that the action of $g \in (\mathbb{Z}/5\mathbb{Z})^\times$ that sends y_i to y_{gi} is an isomorphism between the isotypic components, meaning that it is sufficient for us to just consider this isotypic component.

```
             0  1   2    3     4
o22 = total: 6 54 534 5286 52326
          1: 1  .   .    .     .
          2: 2  .   .    .     .
          3: 2  2   .    .     .
          4: 1  8   .    .     .
          5: . 15   6    .     .
          6: . 16  32    .     .
          7: . 10  82   18     .
          8: .  3 130  120     .
```

```
     9: .  . 137  390    54
    10: .  .  96  806   432
    11: .  .  42 1162  1698
    12: .  .   9 1210  4306
    13: .  .   .  911  7798
    14: .  .   .  480 10566
    15: .  .   .  162 10922
    16: .  .   .   27  8618
    17: .  .   .    .  5097
    18: .  .   .    .  2160
    19: .  .   .    .   594
    20: .  .   .    .    81
```

We conjecture the following recurrence:

Conjecture 5.1. *Let $j \geq 2$. Then, we have $\beta_{i+1,j} = 3\beta_{i,j-3} + 4\beta_{i,j-4} + 3\beta_{i,j-5} - \beta_{i-1,j-8}$.*

Using the data from the table above, we can compute that the Poincare series is given by

$$Z = \frac{y + 2y^2 + 2y^3 + y^4 - t(y^4 + 2y^5 + 2y^6 + y^7)}{1 - (3ty^3 + 4ty^4 + 3ty^5 - t^2y^8)}.$$

If we plug in $t = -1$, we end up with, after simplifying,

$$[Z]_{t=-1} = \frac{y - y^2 + y^3}{1 - 3y + 5y^2 - 3y^3 + y^4} = \frac{\mathrm{Hilb}(S^{\chi_1}, \mathbf{y})}{\mathrm{Hilb}(R, \mathbf{y})},$$

where the second equality has been verified by Macaulay2. It is not yet clear what the right guess should be for the multigraded denominator is in the Poincare series.

5.2.3 The case $n = 6$

Here are the betti numbers for S^{χ_1, C_6} :

```
                0   1    2     3
o19 = total: 8 102 1390 18950
           1: 1   .    .     .
           2: 2   .    .     .
           3: 3   2    .     .
           4: 1  12    .     .
           5: 1  26    9     .
           6: .  27   67     .
           7: .  19  192    42
           8: .  12  299   361
           9: .   3  311  1285
          10: .   1  254  2617
          11: .   .  157  3618
          12: .   .   69  3815
          13: .   .   26  3209
          14: .   .    5  2137
          15: .   .    1  1150
          16: .   .    .   501
          17: .   .    .   163
          18: .   .    .    44
          19: .   .    .     7
          20: .   .    .     1
```

and for S^{χ_2,C_6}, we have:

```
               0  1    2     3      4
o9 = total: 7 86 1170 15950 217450
          1: 1  .    .     .      .
          2: 3  .    .     .      .
          3: 2  3    .     .      .
          4: 1 16    .     .      .
          5: . 26   14     .      .
          6: . 22   90     .      .
          7: . 13  216    65      .
          8: .  5  285   496      .
          9: .  1  260  1535    302
         10: .  .  178  2743   2666
         11: .  .   88  3384  10036
         12: .  .   31  3179  22467
         13: .  .    7  2333  35006
         14: .  .    1  1340  41611
         15: .  .    .   604  39481
         16: .  .    .   208  30434
         17: .  .    .    53  19253
         18: .  .    .     9  10017
         19: .  .    .     1   4248
         20: .  .    .     .   1450
         21: .  .    .     .    388
         22: .  .    .     .     79
         23: .  .    .     .     11
         24: .  .    .     .      1
```

5.3 Conjectures on the Asymptotic Behavior of $\beta_{i,j}$

Here, we include some conjectures on the behavior of $\beta_{i,j}$, of the same asymptotic style as we considered in subsections 3.1 and 3.2 formulated based off of the data for the minimal free resolutions that we included above, as well as our work with the minimal free resolution when $n = 4$.

Based off of the initial data, both from the Betti table and the construction for the $n = 4$ free resolution, one might be tempted to conjecture the following.

Guess 5.2. *For $1 \leq i \leq n-1$ there exist matrices $\phi_j^{(i)}$ satisfying the following properties:*

(a) *These $\phi_j^{(i)}$ are the matrices, for appropriate bases, corresponding to the maps in the minimal free resolution of S^{χ_i,C_n} as a S^{C_n}-module.*

(b) *Every nontrivial $\mathbb{C}$-linear combination of the columns in $\phi_j^{(i)}$ will always have a coordinate with an indecomposable monomial term.*

(c) *For $j \geq 2$, $\phi_j^{(i)}$ is block upper-triangular, having the form $\phi_j^{(i)} = \begin{pmatrix} D_{i,j} & X_{i,j} \\ 0 & Y_{i,j} \end{pmatrix}$.*

(d) *For $j \geq 2$, $D_{i,j}$ is a block-diagonal matrix consisting of $\mathrm{Hilb}(S/(S_+^{C_n}), 1) - 1$ copies of $\phi_{j-1}^{(n-i)}$.*

(e) *For $j \geq 2$, $X_{i,j} \cdot \ker Y_{i,j} \subset \operatorname{im} D_{i,j}$.*

This also suggests the following weaker statement.

Guess 5.3. *Let* $\cdots \to F_1 \to F_0 \to S^{\chi_i, C_n} \to 0$ *be the minimal free resolution of* S^{χ_i, C_n}*. There are bases for* F_1 *and* F_0 *such that the matrix of* ϕ_1 *has two nonzero coordinates in each column, and at least one of the nonzero coordinates is an indecomposable monomial in* $S_+^{C_n}$*.*

Note that every column having an indecomposable monomial for ϕ_1 is weaker than condition (b) in 5.2, as each column is indeed a $\mathbb{C}-$linear combination.

Unfortunately, that the above guess is false, and so part (b) of the first guess is also not correct.

Proof. We set $n = 150$ for our counterexample, and work with S^{χ_1}. Let $\mathbf{e}_i$ be the basis element of F_1 in the free resolution that gets sent to the monomial $y_{45}y_{54}y_{52}$ and $\mathbf{e}_j$ gets sent to the monomial $y_{45}y_{70}y_{36}$ by the map ϕ_0. We claim that the element $y_{70}y_{30}y_{50}y_{36}y_{66}y_{48}\mathbf{e}_i - y_{54}y_{30}y_{66}y_{52}y_{50}y_{48}\mathbf{e}_j$ cannot be expressed as a linear combination of other elements in the kernel (other than multiples of itself). This means that if it is included in a generating set, any minimal generating set resulting from this (by taking a subset) must contain the above element (or some $\mathbb{C}$ multiple of it).

To see this, we can first restrict just to the homogeneous component with multi-degree with only ones on $30, 36, 45, 48, 50, 52, 54, 66, 70$, and zeroes on the rest, since we know that our maps can be constructed so that they preserve degree. Suppose that we have $y_{70}y_{30}y_{50}y_{36}y_{66}y_{48}\mathbf{e}_i - y_{54}y_{30}y_{66}y_{52}y_{50}y_{48}\mathbf{e}_j = \sum c_i\mathbf{k}_i$, where $c_i \in S^{C_{150}}$ and $\mathbf{k}_i$ lie in the kernel of the map ϕ_0. We know that degree is preserved; thus, we see that the multi-degree of $\mathbf{k}_i$ has to be so that the only nonzero components are among those in the set $\{30, 36, 45, 48, 50, 52, 54, 66, 70\}$. However, we see that, checking with Sage, the only elements in $S^{\chi_1, C_{150}}$ that are indecomposable (not divisible by any element in $S_+^{C_{150}}$) are $y_{45}y_{54}y_{52}$ and $y_{45}y_{70}y_{36}$. Therefore, we see that each $\mathbf{k}_i$ must be of the form $a_i\mathbf{e}_i - b_i\mathbf{e}_j$, where $a_i, b_i \in S^{\chi_1, C_{150}}$. But notice that a_i has to divide $y_{70}y_{30}y_{50}y_{36}y_{66}y_{48}$ and b_i has to divide $y_{54}y_{30}y_{66}y_{52}y_{50}y_{48}$.

But then, within $S = \mathbb{C}[y_0, y_1, \ldots, y_{n-1}]$, we see that the greatest common divisor of a_i, b_i is $y_{48}y_{30}y_{50}y_{66}$. But this can be seen to not be divisible by any element in $S_+^{C_{150}}$, meaning that, within $\sum c_i\mathbf{k}_i$, we know that c_ia_i is a $\mathbb{C}$-multiple of $y_{70}y_{30}y_{50}y_{36}y_{66}y_{48}$ and c_ib_i is a $\mathbb{C}$-multiple of $y_{54}y_{30}y_{66}y_{52}y_{50}y_{48}$ means that c_i is in $\mathbb{C}$.

Therefore, we see that each of the $\mathbf{k}_i$ is just a multiple of our original kernel element. In particular, this means that if our above element appears in any generating set, then any minimal generating set that is given as a subset contains this kernel element. But at the same time, notice that each of the coefficients given, $y_{70}y_{30}y_{50}y_{36}y_{66}y_{48}$ and $y_{54}y_{30}y_{66}y_{52}y_{50}y_{48}$, aren't indecomposable, with $(y_{70}y_{30}y_{50})(y_{36}y_{66}y_{48})$ and $(y_{54}y_{30}y_{66})(y_{52}y_{50}y_{48})$. Thus, the conjecture above is false. $\square$

It is still possible, however, that the other parts of the first conjecture are true, that we may decompose our matrices in that form. However, given the usefulness of condition (b) to our construction of the $n = 4$ case, it's possible that a weaker version of this condition holds true instead, which will serve a similar purpose. In particular, we posit the following new conjecture:

Conjecture 5.4. *For the minimal free resolution of* S^{χ_k, C_n}*, the matrix* ϕ_j *is such that every column contains a monomial of the form* m_1m_2*, where* m_1, m_2 *are indecomposable elements in* $S^{\chi_l}, S^{\chi_{n-l}}$ *for some integer* l*.*

Furthermore, we conjecture that the following is true, which can be formulated based off of the initial `Macaulay2` data in the last subsection:

Conjecture 5.5. *For the minimal free resolution of* S^{χ_k, C_n}*, we have* $\beta_{i,j} = 0$ *unless* j *is in the interval* $[3i + 1, ni + n - 1]$.

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References

[1] David Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 35–64. MR 570778

[2] A. M. Garsia and D. Stanton, *Group actions of Stanley-Reisner rings and invariants of permutation groups*, Adv. in Math. **51** (1984), no. 2, 107–201. MR 736732

[3] John C. Harris and David L. Wehlau, *Non-negative integer linear congruences*, Indag. Math. (N.S.) **17** (2006), no. 1, 37–44. MR 2337163

[4] Graham J. Leuschke and Roger Wiegand, *Cohen-Macaulay representations*, Mathematical Surveys and Monographs, vol. 181, American Mathematical Society, Providence, RI, 2012. MR 2919145

[5] Irena Peeva, *Graded syzygies*, Algebra and Applications, vol. 14, Springer-Verlag London, Ltd., London, 2011. MR 2560561

[6] V. Reiner and D. White, *Some notes on polya's theorem, kostka numbers and the rsk correspondence*, 2012; available at `http://www-users.math.umn.edu/~reiner/Papers/UnfinishedPolya.pdf`.

[7] Svetoslav Savchev and Fang Chen, *Long zero-free sequences in finite cyclic groups*, Discrete Math. **307** (2007), no. 22, 2671–2679. MR 2351696

[8] Richard P. Stanley, *Invariants of finite groups and their applications to combinatorics*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), no. 3, 475–511. MR 526968

[9] Pingzhi Yuan, *On the index of minimal zero-sum sequences over finite cyclic groups*, Journal of Combinatorial Theory, Series A **114** (2007), no. 8, 1545–1551.