Friezes and Dissections

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Led by Esther Banaian
UMN REU 2021

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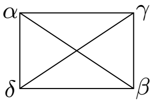
Frieze on a polygon

Definition

Let P be an n-gon with vertices $V = \{0, 1, \ldots, n-1\}$ and let R be an integral domain. A *frieze on P* is a map $f : V \times V \to R$ assigning every arc to a weight where for $\alpha, \beta \in V$

1. $f(\alpha, \beta) = 0 \iff \alpha = \beta$
2. $f(\alpha - 1, \alpha) = 1$
3. $f(\alpha, \beta) = f(\beta, \alpha)$
4. If $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ are crossing diagonals of P, then we have the Ptolemy relation
   $f(\alpha, \beta)f(\gamma, \delta) = f(\alpha, \gamma)f(\beta, \delta) + f(\alpha, \gamma)f(\gamma, \beta)$.



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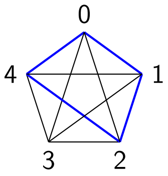
Example:



$$f(0,0) = f(1,1) = \cdots = f(4,4) = 0$$
$$f(0,1) = f(1,2) = f(2,3) = f(3,4) = 1$$

E.g. if $f(0,2) = 1$ and $f(2,4) = 2$

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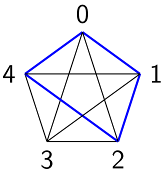

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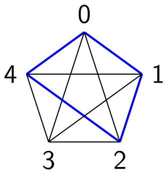

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$$f(1,4)\cdot f(0,2)=f(0,1)\cdot f(2,4)+f(0,4)\cdot f(1,2)$$
$$\implies f(1,4)=1$$

Frieze pattern

Definition

A *frieze pattern of width* $n \in \mathbb{Z}_{\geq 0}$ has $n+4$ horizontally infinite rows of elements of an integral domain. Every diamond

$$\begin{matrix} & a & \\ b & & c \\ & d & \end{matrix}$$

must satisfy the *diamond relation* $ad - bc = 1$.

0		0		0		0		0		0
	1		1		1		1		1	
1		3		1		2		2		1
	2		2		1		3		1	
1		1		1		1		1		1
	0		0		0		0		0	

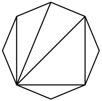
Useful fact

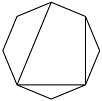
$\{$frieze patterns of width $n\} \longleftrightarrow \{$friezes on an $(n+3)$-gon$\}$.

Conway and Coxeter's integer friezes

Conway—Coxeter studied finite integral frieze patterns and how they come from triangulating a polygon.

Example of triangulation and dissection:





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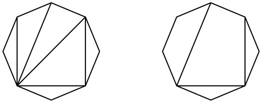


Holm—Jorgensen showed that

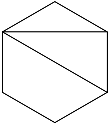
$$\left\{\begin{array}{c}\text{dissections of a polygon } P \\ \text{into sub-gons } P_1, \ldots, P_s \\ \text{where } P_i \text{ is a } p_i - \text{gon}\end{array}\right\} \hookrightarrow \left\{\begin{array}{c}\text{friezes on } P \text{ with} \\ \text{values in } \mathcal{O}_K\end{array}\right\}.$$

Here $\mathcal{O}_K$ is the ring of algebraic integers of the field $K = \mathbb{Q}(\lambda_{p_1}, \ldots, \lambda_{p_s})$ and $\lambda_p = 2\cos\left(\frac{\pi}{p}\right)$.

Dissection into frieze on polygon

Example of frieze from dissection

Holm—Jorgensen provide a way of making a dissection of a polygon into a frieze pattern (equivalently a frieze on a polygon).



$$\lambda_p = 2\cos\left(\frac{\pi}{p}\right)$$
$$\lambda_3 = 1$$
$$\lambda_4 = \sqrt{2}$$

0		0		0		0		0		0
	1		1		1		1		1	
1		2		$1+\sqrt{2}$		$\sqrt{2}$		$\sqrt{2}$		$2+\sqrt{2}$
	1		$1+2\sqrt{2}$		$1+\sqrt{2}$		1		$1+2\sqrt{2}$	
$\sqrt{2}$		$\sqrt{2}$		$2+\sqrt{2}$		1		2		$1+\sqrt{2}$
	1		1		1		1		1	
0		0		0		0		0		0

Studying friezes in $\mathbb{Z}[\sqrt{2}]$

We are interested in studying friezes over $\mathbb{Z}[\lambda_3, \lambda_4] = \mathbb{Z}[\sqrt{2}]$.

Here if a frieze came from dissection, the sub-gons would have to be triangles and quadrilaterals.

Motivating questions

- Holm—Jorgensen showed that there is an injection from dissections of a polygon to friezes on it. What is the image of this map for friezes over $\mathbb{Z}[\sqrt{2}]$?
- Conway—Coxeter showed that every frieze over $\mathbb{Z}_{>0}$ is unitary. How can we characterize unitary friezes over $\mathbb{Z}[\sqrt{2}]$?

Image of dissection to frieze map

Conjecture

The set of friezes where every arc's weight is ≥ 1 is equal to the set of friezes from dissections.

We proved the conjecture for quadrilaterals and have reason to believe that it is true for pentagons.

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However there is a counterexample for hexagons:

0		0		0		0		0	
	1		1		1		1		1
$1+\sqrt{2}$		$\sqrt{2}$		$3-\sqrt{2}$		$1+\sqrt{2}$		$\sqrt{2}$	
	$1+\sqrt{2}$		$-3+3\sqrt{2}$		$2\sqrt{2}$		$1+\sqrt{2}$		$-3+3\sqrt{2}$
$1+\sqrt{2}$		$\sqrt{2}$		$3-\sqrt{2}$		$1+\sqrt{2}$		$\sqrt{2}$	
	1		1		1		1		1
0		0		0		0		0	

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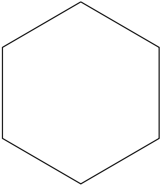

Empty dissection
But $\lambda_6 \notin \mathbb{Z}[\sqrt{2}]$

Types of friezes

Definition: (Frieze from dissection/ Holm—Jorgensen friezes over $\mathbb{Z}[\sqrt{2}]$)

In these friezes, arcs weighted 1 form a dissection into triangles and quadrilaterals.

Definition: ($\mathbb{Z}[\sqrt{2}]_{\geq 1}$ friezes)

A frieze where every arc's weight is ≥ 1.

Definition: (Unitary frieze)

A frieze on a polygon is *unitary* if there exists a triangulation of the polygon such that each arc's weight is a unit. In $\mathbb{Z}[\sqrt{2}]$ these are $(\pm 1 \pm \sqrt{2})^n$.

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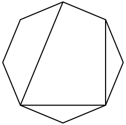

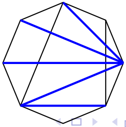

Types of friezes

Definition: (Frieze from dissection/ Holm—Jorgensen friezes)

In these friezes, arcs weighted 1 form a dissection into triangles and quadrilaterals.

Definition: ($\mathbb{Z}[\sqrt{1}]_{\geq 1}$ friezes)

A frieze where every arc's weight is ≥ 1.

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A frieze on a polygon is *unitary* if there exists a triangulation of the polygon such that each arc's weight is a unit. In $\mathbb{Z}[\sqrt{2}]$ these are $(\pm 1 \pm \sqrt{2})^n$.

A

frieze on a polygon is a $\mathbb{Z}_{\geq 0}[\sqrt{2}]$ frieze if every arc's weight is of the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Z}_{\geq 0}$.

Relations between the types of friezes

Proposition

We have the following relations between the four types of friezes:

1. $\{\text{friezes from dissections}\} \subsetneq \{\mathbb{Z}_{\geq 0}\left[\sqrt{2}\right] \text{ friezes}\} \subsetneq \{\mathbb{Z}\left[\sqrt{2}\right]_{\geq 1} \text{ friezes}\}$
2. $\{\text{unitary friezes}\}$ is incomparable with $\{\text{friezes from dissections}\}$, $\{\mathbb{Z}_{\geq 0}\left[\sqrt{2}\right] \text{ friezes}\}$ and $\{\mathbb{Z}\left[\sqrt{2}\right]_{\geq 1} \text{ friezes}\}$.

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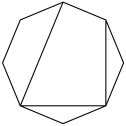
Intersection between Unitary Friezes and Friezes from Dissection

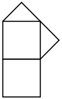
We want to characterize unitary friezes and in particular $\{\text{unitary friezes}\} \cap \{\text{friezes from dissection}\}$.

We wrote some code on Sage:

- Generate all dissections of an n-gon into triangles and quadrilaterals
- Given a dissection, produce the corresponding frieze, find all unitary arcs and determine whether they form a triangulation

Example of simplified pictures:



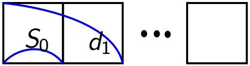


Families of Unitary and Non-Unitary Friezes

Proposition: a Straight Line of Squares

Dissecting a polygon into a straight line of squares produces a non-unitary frieze.
In particular, the arcs from the dissection are the only arcs with unit weights in this type of dissections.

Proof sketch:



i	0	1	2	3	4	$\cdots$
d_i	$\sqrt{2}$	3	$5\sqrt{2}$	17	$29\sqrt{2}$	$\cdots$
s_i	1	$2\sqrt{2}$	7	$12\sqrt{2}$	41	$\cdots$

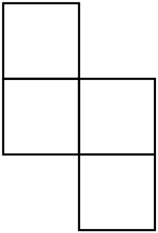
Families of Unitary and Non-Unitary Friezes

Corollary:

$(2n + 2)$-gons (where $n \geq 1$) always have a dissection leading to a non-unitary frieze.

Proposition: Any Arrangement of Squares

The family of dissections into any arrangement of squares is non-unitary.



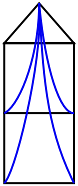
Families of Unitary and Non-Unitary Friezes

Proposition: Towers

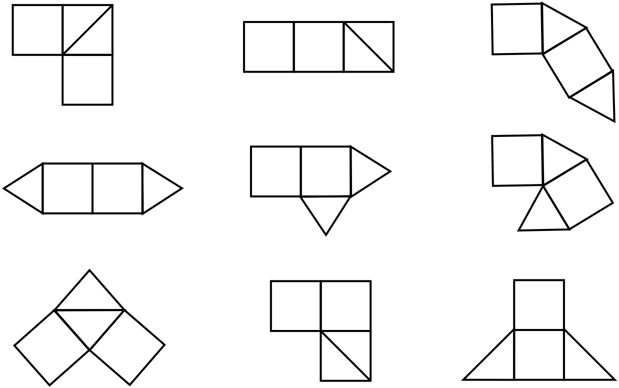
Consider the family of dissections into "towers" i.e. $n \geq 0$ straight squares with a triangle on top. This gives a unitary frieze on $(2n+3)$-gons (that isn't into all triangles).

Proof sketch:

$$\ell_k = d_{k-1} + s_{k-1} = (1+\sqrt{2})^{F_{k+1}}.$$



Examples of Dissections of Octagons



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The Second Big Conjecture

The Second Big Conjecture

A frieze from a dissection is unitary if and only if the dissection is a gluing of towers.

$\Leftarrow$ If a dissection is a gluing of towers, then the frieze from the dissection is unitary.

Proof sketch:

1. Each tower can be triangulated by its tower arcs.
2. All tower arcs are units.
3. Gluing together towers gives a unitary triangulation.

The First Approach

$\Rightarrow$ If a frieze from a dissection is unitary, then the dissection is a gluing of towers.

First approach: Trying to show a stronger statement.

The Stronger Conjecture

Only tower arcs are unitary.

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$\Rightarrow$ If a frieze from a dissection is unitary, then the dissection is a gluing of towers.

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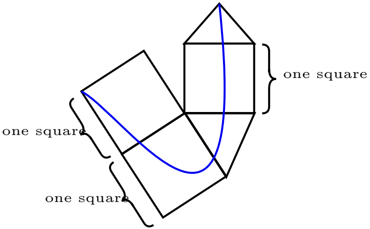
The Stronger Conjecture

Only tower arcs are unitary.

NOT TRUE!

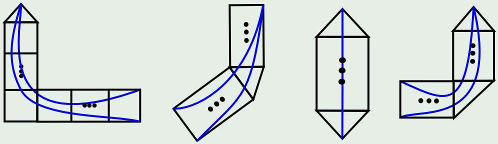
Counter-example: The blue arc has weight $5\sqrt{2} + 7$.



The First Approach

Decompose any arc into shorter pieces.
Express these shorter pieces in terms of d_i's, s_i's, and ℓ_i's and show that they are not units.

Tower + One Turn, Tower + Triangle, Tower + Square, Two Towers



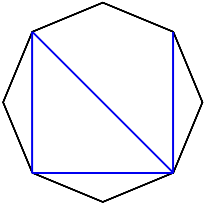
Proof Sketch for One Tower + One Turn

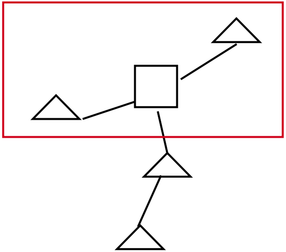
Assume the tower on the left has $n+1$ squares and m squares glued to its right. Then
$ad = d_n d_m - s_n d_{m-1} + d_m s_n - d_n d_{m-1} = (d_m - d_{m-1})(d_n + s_n)$ and
$ae = s_n s_m - s_{m-1} d_n + d_n s_m - s_n s_{m-1} = (s_m - s_{m-1})(s_n + d_n)$.

The Second Approach

The second approach

1. shifts from determining whether an arc is unitary or not to whether the arc could be a part of a unitary triangulation
2. is only sufficient to show that the second big conjecture holds for paths



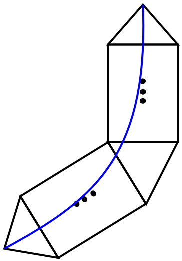


The Second Approach

Definition: k-arc

An arc is a k-arc if it passes through k triangles from its beginning to its end.
In particular, an arc passes a $\frac{1}{2}$ triangle if the triangle is at the beginning or the end;
An arc passes a 1 triangle if the triangle is in the middle.

Example: the green arc is a 2-arc



The Second Approach

Lemma

The only $\frac{1}{2}$-arcs that could exist in a triangulation of unit arcs are tower arcs.

Proposition

For a dissection into a path of triangles and squares, if there exists a k-arc in a triangulation of units with $k > 1$, then there must be a $1 \leq \ell < k$ arc in such a triangulation.

The Second Approach

An overview of the second approach:

1. Show that the proposition holds when k is maximum.

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3. By induction, the existence of k in the triangulation of units, then there must be some 1-arc in such a triangulation.

The Second Approach

An overview of the second approach:

1. Show that the proposition holds when k is maximum.
2. Then show that the proposition holds for $1 \leq \ell < k$.
3. By induction, the existence of k in the triangulation of units, then there must be some 1-arc in such a triangulation.
4. But we can show that all the 1-arcs are not units using the shorter pieces.

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3. By induction, the existence of k in the triangulation of units, then there must be some 1-arc in such a triangulation.
4. But we can show that all the 1-arcs are not units using the shorter pieces.
5. By the contrapositive, it is not possible to have a k-arc in the triangulation of units.

The Second Approach

We have shown that the proposition holds when k is maximum.

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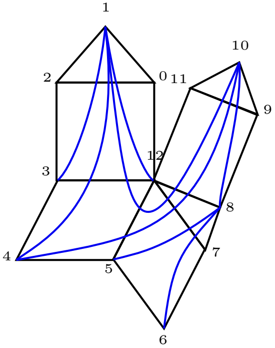
To show that the proposition holds for $1 \leq \ell < k$, we can repeat the argument for the case when k is maximum most of the times, except for the family of paths that has a unitary triangulation without a 1-arc.

We have identified a couple of such families of paths.
For example, the arcs $(1, 10)$ and $(4, 10)$ are units and non-tower-arcs.



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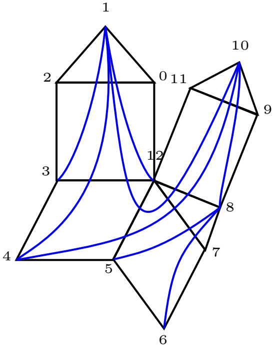

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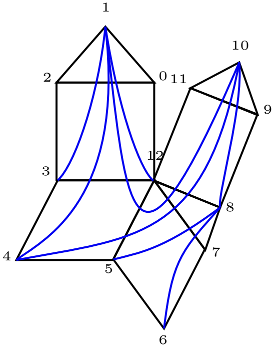

1. We hope to show that all such families are gluings of towers.
2. Once this is shown, steps 3,4, and 5 immediately follow, and the proof is complete.

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Future Directions

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3. How many dissections into triangles and squares are there up to symmetry?
4. Intersections and finiteness of the other types of friezes
5. Move beyond polygons/type A and into punctured polygons/type D

Acknowledgement

We would like to thank our mentor and TAs, Esther, Kayla, and Libby, for their guidance and support, Vic for organizing this REU, and Trevor for helping us with Sage.