Whittaker coefficients and crystals

Aidan Kelley and Siki Wang
joint with Swapnil Garg and Frank Lu
Mentor: Prof. Ben Brubaker, TAs: Emily Tibor, Kayla Wright, Meagan Kenney

August 4, 2021
Root system \( \Phi \)
Root system $\Phi$

- Vectors in $\mathbb{R}^{n+1}$, like $\alpha_1 = (1, -1, 0, 0, \ldots)$. 

- Contain simple roots, positive roots, etc.

- Matrix parameterization. E.g., for $A_2$:

\[
\begin{pmatrix}
\ast & \alpha_1 \\
\ast & \alpha_1 + \alpha_2 \\
\ast & \ast
\end{pmatrix}
\]

- Dynkin diagram for the associated Weyl group. E.g, Aidan Kelley and Siki Wang joint with Swapnil Garg and Frank Lu Mentor: Prof. Ben Brubaker, TAs: Emily Tibor, Kayla Wright, Meagan Kenney

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\]

- **Dynkin diagram** for the associated Weyl group. E.g,

\[
A_5: \quad \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5
\]
Dirichlet L-series

\[ L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^s. \]

E.g., \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \ldots \) (Riemann zeta-function)
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Dirichlet L-series

- $\mathcal{L}(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$.
- E.g., $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + ...$ (Riemann zeta-function)
Combinatorics of Dirichlet Series

There is a way to associate a multiple Dirichlet series to a Dynkin diagram. Example: Dynkin diagram for the Weyl group of $A_5$.

Associating each simple root $\alpha_i \in \Phi^+$ with a complex variable $s_i$, we get the corresponding multiple Dirichlet series

$$\sum_{d_1,\ldots,d_5=1}^\infty (d_1 d_2) (d_2 d_3) (d_3 d_4) (d_4 d_5) d_1 s_1 d_2 s_2 d_3 s_3 d_4 s_4 d_5 s_5$$
Combinatorics of Dirichlet Series

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**Example**

Dynkin diagram for the Weyl group of $A_5$.

Associating each simple root $\alpha_i \in \Phi^+$ with a complex variable $s_i$, we get the corresponding multiple Dirichlet series

$$\sum_{d_1, \ldots, d_5=1}^{\infty} \frac{d_1}{d_2} \left( \frac{d_2}{d_3} \right) \left( \frac{d_3}{d_4} \right) \left( \frac{d_4}{d_5} \right) \frac{s_1^{d_1}}{d_1} \frac{s_2^{d_2}}{d_2} \frac{s_3^{d_3}}{d_3} \frac{s_4^{d_4}}{d_4} \frac{s_5^{d_5}}{d_5}$$
We can compute the Whittaker coefficient for a maximal parabolic Eisenstein series (subgroups of $GL_n$):

$$W_{f_1, f_2, s}(1) \sum_{d_j \in \mathcal{O}/\mathcal{O} \times s} H(d_1, \ldots, d_N)$$

We want to compute the Whittaker coefficient for $A_5$. Why?

Conjecture of Bump, 1996: A multiple Dirichlet series (Chinta) coincide with the H-part (exponential sums) of the Whittaker coefficient.
(Brubaker-Friedberg) We can compute the Whittaker coefficient for a maximal parabolic Eisenstein series (subgroups of $GL_n$):

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**Conjecture of Bump, 1996**: A multiple Dirichlet series (Chinta) coincide with the $H$-part (exponential sums) of the Whittaker coefficient.
Chinta Series

A multiple Dirichlet series related to $A_5$:

$$\sum_{I} \chi_{I_1} \chi_{I_2}(\hat{I}_3) \chi_{I_4}(\hat{I}_3) \chi_{I_5}(\hat{I}_5) \prod_{i=1}^{5} |I_i| S_i \cdot g(I_1, \ldots, I_5)$$

With a change of variable, we get

$$g(I_1, \ldots, I_5) = H(x, y, z, w, v)$$

a polynomial of 366 terms:

$$1 - vw - xy + vwxy - wz + vwz + pv^2 w^2 z - \ldots + p^7 v^4 w^7 x^4 y^7 z^8.$$
(Chinta) A multiple Dirichlet series related to $A_5$:

$$\sum_{I} \frac{\chi_{I_2}(\hat{I}_1) \chi_{I_2}(\hat{I}_3) \chi_{I_4}(\hat{I}_3) \chi_{I_5}(\hat{I}_5)}{|I_1|^{S_1}|I_2|^{S_2}...|I_5|^{S_5}} \cdot g(I_1, ..., I_5)$$
(Chinta) A multiple Dirichlet series related to $A_5$:

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We suspect that the Chinta series comes from the Whittaker coefficient. Reason: Both have nice functional equations that generate a group isomorphic to the Weyl group of $A_5$. 

Aidan Kelley and Siki Wang joint with Swapnil

Whittaker coefficients and crystals

August 4, 2021
Our Goal (REU Problem 4)

1. Compute Whittaker coefficients using data from $A_5$.
2. Understand the support of $H(d_1, \ldots, d_N)$. (Does it form a polytope in the Euclidean space? It is infinite?)

Questions we ask:

- How do we simplify $H(d_1, \ldots, d_N)$ and when is it nonzero?
- How does the polynomial from Whittaker compare with the Chinta series?
Our Strategy for removing roots

Recall, a maximal parabolic corresponds to the choice of removing a simple root.

Heuristic:

\[ \sum d_i = 1 \]

For computation, removing \( \alpha_2 \) and \( \alpha_4 \) could give us a nicer polynomial to compare.
Our Strategy for removing roots

- Recall, a maximal parabolic corresponds to the choice of removing a simple root.
Our Strategy for removing roots

- Recall, a maximal parabolic corresponds to the choice of removing a simple root.
- Heuristic:

\[
\sum_{d_i=1}^{\infty} \frac{d_1}{d_2} \frac{d_2}{d_3} \frac{d_3}{d_4} \frac{d_4}{d_5} = \sum_{d_2,d_4=1}^{\infty} \frac{\mathcal{L}(s_1, \chi d_2) \mathcal{L}(s_3, \chi d_2d_4) \mathcal{L}(s_5, \chi d_4)}{d_2^s \, d_4^s}\]

For computation, removing \(\alpha_2\) and \(\alpha_4\) could give us a nicer polynomial to compare.
(Brubaker-Friedberg) Theorem 4.1:

\[ \mathcal{W}_{f_1, f_2, s(1)} \sum_{d_j \in \mathfrak{o}_s / \mathfrak{o}_s^x} H(d_1, \ldots, d_N) \delta_P^{s+1/2}(\mathfrak{D}) \Psi(\mathfrak{D}) \zeta_\mathfrak{D} c^\psi_{f_1, f_2}(\mathfrak{D}) \]

Some results:
Computing the Whittaker coefficient

(Brubaker-Friedberg) Theorem 4.1:

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Some results:

- \[ \delta_p^{s+1/2} (\mathfrak{D}) = (d_1d_2d_3d_4d_5d_6d_7d_8)^{-3s-3/2} \]
Computing the Whittaker coefficient

(Brubaker-Friedberg) Theorem 4.1:

$$W_{f_1, f_2, s(1)} \sum_{d_j \in o_s / o_s^\times} H(d_1, \ldots, d_N) \delta_P^{s+1/2}(\mathcal{D}) \Psi(\mathcal{D}) \zeta_\mathcal{D} c_{f_1, f_2}^\psi(\mathcal{D})$$

Some results:

- $$\delta_P^{s+1/2}(\mathcal{D}) = (d_1 d_2 d_3 d_4 d_5 d_6 d_7 d_8)^{-3s-3/2}$$
- $$H(d_1, \ldots, d_N) := \sum_{c_i \mod D_j} \prod_{k=1}^N \left( \frac{c_k}{d_k} \right) e^{2\pi i \sum_j v_j}$$: Gauss sums calculated from removing $$\alpha_2$$ (will be explained in detail)
Computing the Whittaker coefficient

(Brubaker-Friedberg) Theorem 4.1:

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- $$\zeta_D = (d_4 d_3 d_2 d_1, d_5)_S (d_4 d_3 d_2, d_6)_S (d_4 d_3, d_7)_S (d_4, d_8)_S$$.
Computing the Whittaker coefficient

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- \[ c_{f_1,f_2}^\psi(\mathfrak{D}) : \text{The inductive step for further removing roots from } A_1 \times A_3. \]
Root System $A_n$

- A root system $\Phi \subset \mathbb{R}^{n+1}$ is a finite collection of vectors (“roots”) under some axioms.
- There is a method of enumerating the positive roots.

### Example

Below is one possible enumeration of roots for the $A_3$ case:

\[
\begin{bmatrix}
* & \beta_3 & \beta_2 & \beta_1 \\
* & \beta_5 & \beta_4 \\
* & \beta_6 & *
\end{bmatrix}
\]
Removing the second root from $A_5$

- We want to split up $A_5$ into $A_1 \times A_3$ and “what’s left”

![Figure: The Dynkin diagram corresponding to removing the second node](image)

Figure: The Dynkin diagram corresponding to removing the second node
Removing the second root from $A_5$

- In the below diagram, the asterisks represent the $A_1$ and $A_3$ root systems.
- We can rig the enumeration to do the $A_1 \times A_3$ roots first and $\gamma_1, \gamma_2, \ldots, \gamma_8$ last.

\[
\begin{bmatrix}
* & * & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\
* & * & \gamma_8 & \gamma_7 & \gamma_6 & \gamma_5 \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
\end{bmatrix}
\]

- We’ll compute the asterisk $A_1 \times A_3$ part inductively.
Definition

\[ g_t(m, d) = \sum_{c \mod d} \left( \frac{c}{d} \right)^t e^{2\pi i \frac{mc}{d}} \]
Gauss Sums – A Prototype for the Exponential Sum

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**Example**

\[ g_1(1, p^2) = \sum_{c \mod p^2} \left( \frac{c}{p^2} \right) e^{2\pi i \frac{c}{p^2}} \]
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Reindex to \( c = x + py \) with \( x, y \mod p \).

\[ g_1(1, p^2) = \sum_{x, y \mod p} \left( \frac{x}{p^2} \right) e^{2\pi i \frac{x}{p^2} + \frac{y}{p}} = \sum_{x \mod p} \left( \frac{x}{p^2} \right) e^{2\pi i \frac{x}{p^2}} \sum_{y \mod p} e^{2\pi i \frac{y}{p}} \]

\[ = \sum_{x \mod p} \left( \frac{x}{p^2} \right) e^{2\pi i \frac{x}{p^2}} \cdot 0 = 0 \]
Gauss Sums – A Prototype for the Exponential Sum

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\[ g_t(m, d) = \sum_{c \mod d} \left( \frac{c}{d} \right)^t e^{2\pi i \frac{mc}{d}} \]

**Example**

\[ g_1(p, p) = \sum_{c \mod p} \left( \frac{c}{p} \right) e^{2\pi i \frac{cp}{p}} \]
Gauss Sums – A Prototype for the Exponential Sum

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\[ g_t(m, d) = \sum_{c \mod d} \left( \frac{c}{d} \right)^t e^{2\pi i \frac{mc}{d}} \]

Example

\[ g_1(p, p) = \sum_{c \mod p} \left( \frac{c}{p} \right) e^{2\pi i \frac{cp}{p}} \]

For \( c \mod p \), \( (c, p) = 1 \), half of \( c \) are squares and half are not, so

\[ g_1(p, p) = \sum_{c \mod p} \left( \frac{c}{p} \right) = 0. \]
Defining the Exponential Sum

- We associate an exponential sum to removing a certain root from a root system

**Definition (Brubaker-Friedberg)**

For $\mathbf{d} = (d_1, d_2, \ldots, d_N)$ with $d_i = p^{l_i}$ for some prime

$$H(\mathbf{d}) = \sum_{c_i \mod D_i} \exp \left( 2\pi i \left( \sum_i v_i \right) \right) \prod_{k=1}^{N} \left( \frac{c_k}{d_k} \right)$$
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We define $v_j = \frac{c_N}{d_N}$ when $j$ is the removed root and is otherwise

$$\sum_{(k,k') \in S_j} (-1)^{i+i'} \eta_{i,i',k,-k'}(b_k d_k^{-1})^i (c_{k'} d_{k'}^{-1})^{i'} \prod_{l \geq k} (d_l^{-1})^{\langle \alpha_j, \gamma_l \rangle} \prod_{k' < l < k} (d_l^{-1})^{i' \langle \gamma'_k, \gamma'_l \rangle}$$
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We define \( v_j = \frac{c_N}{d_N} \) when \( j \) is the removed root and is otherwise

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\sum_{(k,k') \in S_j} (-1)^{i+i'} \eta_{i,i',k,-k'} (b_k d_k^{-1})^{i} (c_k' d_k'^{-1})^{i'} \prod_{l \geq k} (d_l^{-1})^{\langle \alpha_j, \gamma_l^{\vee} \rangle} \prod_{k' < l < k} (d_l^{-1})^{i' \langle \gamma_{k'}, \gamma_l^{\vee} \rangle}
\]

\( D_j \) are defined in terms of \( d_j \)s as follows:

\[
D_j = d_j \prod_{k>j} d_k^{\langle \gamma_j, \gamma_k \rangle}
\]
Removing the second root from $A_5$

- We compute $H(d)$ in the $A_5$ case with second node of the Dynkin diagram removed.
Removing the second root from $A_5$

- We compute $H(d)$ in the $A_5$ case with second node of the Dynkin diagram removed.

$$H(d) = \sum_{c_i \mod D_i} \exp\left(2\pi i \left( -\frac{b_5 c_1 d_6 d_7 d_8}{d_1 d_2 d_3 d_4} - \frac{b_6 c_2 d_7 d_8}{d_2 d_3 d_4} - \frac{b_7 c_3 d_8}{d_3 d_4} - \frac{b_8 c_4}{d_4} \right) \right) \prod_{k=1}^{8} \left( \frac{c_k}{d_k} \right),$$

where loosely we define $b_i \equiv c_i^{-1} \mod d_i$.
Removing the second root from $A_5$

\[ H(d) = \sum_{c_i \mod D_i} \exp \left( 2\pi i \left( - \frac{b_5 c_1 d_6 d_7 d_8}{d_1 d_2 d_3 d_4} - \frac{b_6 c_2 d_7 d_8}{d_2 d_3 d_4} - \frac{b_7 c_3 d_8}{d_3 d_4} - \frac{b_8 c_4}{d_4} \right) \right) \]
\[ + \frac{c_8}{d_8} + \frac{b_4 c_3 d_8}{d_3 d_7} + \frac{b_8 c_7}{d_7} + \frac{b_3 c_2 d_7}{d_2 d_6} + \frac{b_7 c_6}{d_6} + \frac{b_2 c_1 d_6}{d_1 d_5} + \frac{b_6 c_5}{d_5} \right) \right) \prod_{k=1}^{8} \left( \frac{c_k}{d_k} \right), \]

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**Proposition (S. Garg-K.-F. Lu-W.)**

Put the $d_j$s in a matrix corresponding to the position of $\gamma_j$. Then,

\[ D_j = d_j \times d_k \text{s below } d_j \text{ in the same column} \]
\[ \times d_k \text{s to the left of } d_j \text{ in the same row} \]

Recall the original definition: \[ D_j = d_j \prod_{k>j} d_k^{\langle \gamma_j, \gamma_k \rangle} \]
Removing the second root from $A_5$

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Recall the original definition:

$$D_j = d_j \prod_{k > j} d_k^{\langle \gamma_j, \gamma_k \rangle}$$

**Example**

Here, the matrix is

$$
\begin{array}{cccc}
d_4 & d_3 & d_2 & d_1 \\
d_8 & d_7 & d_6 & d_5 \\
\end{array}
$$

We then have

$$D_3 = d_3d_7d_4, \quad D_4 = d_4d_8$$
Removing the second root from $A_5$

$$H(d) = \sum_{c_i \mod D_i} \exp \left( 2\pi i \left( -\frac{b_5 c_1 d_6 d_7 d_8}{d_1 d_2 d_3 d_4} - \frac{b_6 c_2 d_7 d_8}{d_2 d_3 d_4} - \frac{b_7 c_3 d_8}{d_3 d_4} - \frac{b_8 c_4}{d_4} \right) \right)$$

$$+ \frac{c_8}{d_8} + \frac{b_4 c_3 d_8}{d_3 d_7} + \frac{b_8 c_7}{d_7} + \frac{b_3 c_2 d_7}{d_2 d_6} + \frac{b_7 c_6}{d_6} + \frac{b_2 c_1 d_6}{d_1 d_5} + \frac{b_6 c_5}{d_5} ) \right) \prod_{k=1}^{8} \left( \frac{c_k}{d_k} \right),$$

where loosely we define $b_i \equiv c_i^{-1} \mod d_i$

**Proposition (GKLW)**

*Each term in the exponent other than $\frac{c_8}{d_8}$ is of the form*

$$\pm \frac{b_i c_j D_i}{D_j}$$

Recall the original definition of a term:

$$(-1)^{i+i'} \eta_{i,i',k,-k'} (b_k d_k^{-1})^i (c_{k'} d_{k'}^{-1})^{i'} \prod_{l \geq k} (d_l^{-1})^{\langle \alpha_j, \gamma_l \rangle} \prod_{k' < l < k} (d_l^{-1})^{i' \langle \gamma_{k'}, \gamma_l \rangle}$$
Removing the second root from $A_5$

\[
H(d) = \sum_{c_i \mod D_i} \exp \left(2\pi i \left( -\frac{b_5 c_1 d_6 d_7 d_8}{d_1 d_2 d_3 d_4} - \frac{b_6 c_2 d_7 d_8}{d_2 d_3 d_4} - \frac{b_7 c_3 d_8}{d_3 d_4} - \frac{b_8 c_4}{d_4} \right) \right)
\]

\[
+ \frac{c_8}{d_8} \cdot \frac{b_4 c_3 d_8}{d_3 d_7} + \frac{b_8 c_7}{d_7} + \frac{b_3 c_2 d_7}{d_2 d_6} + \frac{b_7 c_6}{d_6} + \frac{b_2 c_1 d_6}{d_1 d_5} + \frac{b_6 c_5}{d_5} \right) \right) \prod_{k=1}^{8} \left( \frac{c_k}{d_k} \right),
\]

where loosely we define $b_i \equiv c_i^{-1} \mod d_i$

**Proposition (GKLW)**

*Each term in the exponent other than $\frac{c_8}{d_8}$ is of the form*

\[
\pm \frac{b_i c_j D_i}{D_j}
\]

- We can check this for $b_4, c_3$ with $D_3 = d_3 d_7 d_4, D_4 = d_4 d_8$. 
Removing the second root from \( A_5 \)

\[
H(d) = \sum_{c_i \mod D_i} \exp \left( 2\pi i \left( -\frac{b_5 c_1 d_6 d_7 d_8}{d_1 d_2 d_3 d_4} - \frac{b_6 c_2 d_7 d_8}{d_2 d_3 d_4} - \frac{b_7 c_3 d_8}{d_3 d_4} - \frac{b_8 c_4}{d_4} \right) + \frac{c_8}{d_8} + \frac{b_4 c_3 d_8}{d_3 d_7} + \frac{b_8 c_7}{d_7} + \frac{b_7 c_6}{d_2 d_6} + \frac{b_6 c_5}{d_6} + \frac{b_2 c_1 d_6}{d_1 d_5} + \frac{b_6 c_5}{d_5} \right) \prod_{k=1}^{8} \left( \frac{c_k}{d_k} \right),
\]

where loosely we define \( b_i \equiv c_i^{-1} \mod d_i \)

- To better understand the sum, we draw a “dependency graph”

![Dependency Graph]

**Figure:** There is a \( b_i c_j \) term in the sum \( \iff \) there is an edge \( i \rightarrow j \) in the graph. We circle 8 to remember the \( \frac{c_8}{d_8} \) term.
Removing the second root from $A_5$

$$H(d) = \sum_{c_i \mod D_i} \exp \left( 2\pi i \left( -\frac{b_5 c_1 d_6 d_7 d_8}{d_1 d_2 d_3 d_4} - \frac{b_6 c_2 d_7 d_8}{d_2 d_3 d_4} - \frac{b_7 c_3 d_8}{d_3 d_4} - \frac{b_8 c_4}{d_4} \right) \right)$$

$$+ \frac{c_8}{d_8} + \frac{b_4 c_3 d_8}{d_3 d_7} + \frac{b_8 c_7}{d_7} + \frac{b_3 c_2 d_7}{d_2 d_6} + \frac{b_7 c_6}{d_6} + \frac{b_2 c_1 d_6}{d_1 d_5} + \frac{b_6 c_5}{d_5} \right) \right) \prod_{k=1}^{8} \left( \frac{c_k}{d_k} \right),$$

where loosely we define $b_i \equiv c_i^{-1} \mod d_i$

- To better understand the sum, we draw a “dependency graph”

![Dependency Graph](image)

**Figure:** There is a $b_i c_j$ term in the sum $\iff$ there is an edge $i \rightarrow j$ in the graph. We circle 8 to remember the $\frac{c_8}{d_8}$ term.
Removing the second root from $A_5$

Figure: There is a $b_i c_j$ term in the sum $\iff$ there is an edge $i \rightarrow j$ in the graph. We circle 8 to remember the $\frac{c_8}{d_8}$ term

- We can follow paths to compute what the other edges are in terms of the $a_j$s.

Example

Since $b_i = c_i^{-1}$, we have

$$b_4 c_3 = b_4 c_7 b_7 c_3 = a_2 a_3$$
Removing the second root from $A_5$

$$H(d) = \sum_{a_i} \exp\left(2\pi i \left(-\frac{a_7 d_6 d_7 d_8}{d_1 d_2 d_3 d_4} - \frac{a_5 d_7 d_8}{d_2 d_3 d_4} - \frac{a_3 d_8}{d_3 d_4} - \frac{a_1}{d_4}\right) + \frac{a_0}{d_8} + \frac{a_2 a_3 d_8}{d_3 d_7} + \frac{a_1 a_2}{d_7} + \frac{a_4 a_5 d_7}{d_2 d_6} + \frac{a_3 a_4}{d_6} + \frac{a_6 a_7 d_6}{d_1 d_5} + \frac{a_5 a_6}{d_5}\right) \prod_{k=1}^{8} \left(\frac{a_k}{\ldots}\right),$$

**Figure:** There is a $b_i c_j$ term in the sum $\iff$ there is an edge $i \to j$ in the graph. We circle 8 to remember the $\frac{c_8}{d_8}$ term.
Removing the second root from $A_5$

$$H(d) = \sum_{a_i} \exp\left(2\pi i \left( -\frac{a_7 d_6 d_7 d_8}{d_1 d_2 d_3 d_4} - \frac{a_5 d_7 d_8}{d_2 d_3 d_4} - \frac{a_3 d_8}{d_3 d_4} - \frac{a_1}{d_4} \right. \right.$$  

$$+ \frac{a_0}{d_8} + \frac{a_2 a_3 d_8}{d_3 d_7} + \frac{a_1 a_2}{d_7} + \frac{a_4 a_5 d_7}{d_2 d_6} + \frac{a_3 a_4}{d_6} + \frac{a_6 a_7 d_6}{d_1 d_5} + \frac{a_5 a_6}{d_5} \left. \right) \right) \prod_{k=1}^{8} \left( \frac{a_k}{\ldots} \right),$$

\begin{figure}
\begin{tikzpicture}
\node (a0) at (0,0) {$a_0$};
\node (a1) at (1,0) {$a_1$};
\node (a2) at (2,0) {$a_2$};
\node (a3) at (3,0) {$a_3$};
\node (a4) at (4,0) {$a_4$};
\node (a5) at (5,0) {$a_5$};
\node (a6) at (6,0) {$a_6$};
\node (a7) at (7,0) {$a_7$};
\draw (a0) -- (a1);
\draw (a1) -- (a2);
\draw (a2) -- (a3);
\draw (a3) -- (a4);
\draw (a4) -- (a5);
\draw (a5) -- (a6);
\draw (a6) -- (a7);
\end{tikzpicture}
\end{figure}

**Figure:** A visualization of the dependencies in the re-indexed sum
Progress Summary

- We compute a Dirichlet Series from a Dynkin Diagram
- We show how to interpret relevant quantities in terms of the geometry of the $\gamma_j$s
- We model the exponential sum as a graph and use it to facilitate re-indexing to “nicer” coordinates
  this gives us...
- An understanding of where the $H(d, t)$’s are supported: Finite cases (most exponents $\leq 1$) and a few infinite cases.
Future Directions

- Change of variables from the Whittaker coefficient to the Chinta polynomial
- Understand the 15 zeta functions that got pulled out from the Chinta series, and how it coincide with the normalizing zeta factor of the Whittaker function
- Another description of the same polynomial is through ”string data” defined in Littelmann. We bounded a polytope but it currently has 12624 vertices...
Acknowledgements

Thanks Ben for patiently explaining this problem to us again and again (until one of us finally starts to understand).

Thanks Kayla, Emily and Megan for all the TA sessions, answering endless discord Q&As' and spending 2.5 hours yesterday to help us polish the talk.

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The End!
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