PROBLEM 4: ROOT SYSTEMS, HIGHEST-WEIGHT POLYTOPES, AND EXPONENTIAL SUMS

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Abstract. In this paper, we consider the Dirichlet series that comes from the process of obtaining the Whittaker coefficients for the maximal parabolic Eisenstein series, outlined in from Brubaker and Friedberg's paper [BF15] in the case of the Dynkin diagram of A_5. Specifically, we take the maximal parabolic corresponding to the removal of the second simple root from A_5. We compute the support of the exponential sum corresponding to this removal of the second simple root from A_5, in order to see how the Dirichlet series to another Dirichlet series associated to A_5 outlined in Chinta's paper [Chi05].

1. Introduction

Brubaker and Friedberg describe in their paper [BF15] a method to compute the Whittaker coefficients of a parabolic Eisenstein series for metaplectic covers of a split reductive group G. The resulting coefficient turns out to be a Dirichlet series over several variables, with each variable corresponding to the removal of a given root. This whole process results in a maximal chain of nested parabolics between (and including) the Borel subgroup B and the whole G, with the removal of each root corresponding to a step in the chain. In particular, the first removal step takes us from G to a maximal parabolic P.

In particular, their main result, [BF15, Theorem 4.1], states that for a specific character ψ we have the following coefficient:

$$\mathcal{W}_{f_1,f_2,s}(1) \sum_{\substack{d_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times, d_j \neq 0 \\ j=1,2,\ldots,N}} H(d_1, d_2, \ldots, d_N)\delta_P^{s+1/2}(\mathfrak{D})\Psi(\mathfrak{D})\zeta_{\mathfrak{D}} c^{\psi}_{f_1,f_2}(\mathfrak{D}).$$

Of particular interest in studying this formula is the explicit computation of the exponential sum H, as this will provide a lot of information about which terms in the Dirichlet series have nonzero coefficient.

One interesting aspect of this exponential sum H is the support of H, which has been found to be related to the representation theory of G. In particular, certain inequalities that define the support of H appear related to those arising out of the combinatorics of these representations. For a more explicit description of the combinatorics, see Littelmann's paper [Lit98]. In particular, [Lit98]'s combinatorial machinery generates a polytope and various inequalities which Brubaker and Friedberg relate in [BF15] to the support of the function H in their explicit example of $G = GL_4(\mathbb{C})$, where the maximal parabolic has Levi subgroup $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ Additionally, an earlier paper by Brubaker, Bump, and Friedberg, [BBF11], goes through a specific case of $G = GL_n(\mathbb{C})$, with the chain of parabolics with respective Levi subgroups $GL_1(\mathbb{C}) \times GL_{n-1}(\mathbb{C}) \supset GL_1(\mathbb{C}) \times GL_1(\mathbb{C}) \times GL_{n-2}(\mathbb{C}) \supset$

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$\cdots \supset B$, observe as well how this combinatorial data is related to the evaluation of this exponential sum H.

A second process for generating a Dirichlet series, this time coming from a Dynkin diagram, is described by Chinta in [Chi05]. In particular, Chinta describes a Dirichlet series associated to the Dynkin diagram of A_5 and analyzes nice analytic properties of this function. However, from the outset this process isn't immediately related to the Dirichlet series outlined by [BF15]. A major goal of this project was to see if the series produced by Chinta and the series produced by Brubaker and Friedberg, in the case when we have $G = GL_6(\mathbb{C})$ and the associated Dynkin diagram A_5, are closely related (possibly equal up to a change of variables). Motivating this search are the nice analytic properties of both the series Chinta constructs and of Whittaker functions. Additionally, both of these constructions yield functional equations generating Weyl groups that are isomorphic to A_5, further suggesting a connection.

As such, in this paper, we go through the process outlined in [BF15] for the example where $G = GL_6(\mathbb{C})$, with the hope of analyzing the connection that this particular series has to both the work of [Lit98] and [Chi05]. While [BBF11] provides an inductive method for computing such a series for A_n, with parabolics given by removing the left-most root in the Dynkin diagram, the resulting form is difficult to compare with the series given in [Chi05]. In particular, the form of the series in [Chi05] suggests that a different choice of parabolics and running the process in [BF15] with this new choice of parabolics will yield a series that is easier to compare to Chinta's.

Our approach for the report is as follows. We begin in section 2 by introducing some notation and setting some conventions. We will also review some properties of Gauss sums and root systems, and explain some of the key parts of the process given [BF15], before working through the example done in [BF15] for $GL_4(\mathbb{C})$ in more detail.

Following this, in section 3 we provide some simplifications and describe general methods to computing the exponential sum $H(\mathbf{d}, \mathbf{t})$ corresponding to this process of removing the second root. These methods will also be useful in future study of this process in A_n. Section 4 then goes through some more specific computations of $H(\mathbf{d}, \mathbf{t})$ in the case of A_5 to determine the support of this function. From here, we go back to the whole Dirichlet series in section 5 and explicitly describe each of the parts of the formula from [BF15, Theorem 4.1]. In particular, we relate what form they take in our example.

Sections 6 and 7 are dedicated to describing some of the progress we've made in more explicit computations with Sage, with section 6 focused on analyzing the connection between the polytopes described in Littelmann's paper [Lit98] and section 7 more oriented on explicit values of $H(\mathbf{d}, \mathbf{t})$. Finally, in section 8, we outline future directions that we could take.

2. Preliminary Concepts and Definitions

This section introduces key concepts about Gauss sums and root systems. After introducing these concepts, we go through the example from [BF15] for their computation of the exponential sum H in the case where $G = GL_4(\mathbb{C})$ corresponding to A_3. This kind of computation will be our starting point for the work that we do in the rest of the report.

2.1. Gauss Sums.

Given integers m, t, d, where $d > 1$, define the Gauss sum as

$$g_t(m,d) = \sum_{c \bmod d} \left(\frac{c}{d}\right)_2^t e^{2\pi i \frac{mc}{d}}.$$

Of particular interest are the evaluations of $g_t(m,d)$ when d is a prime power, and m is relatively prime to d. In this case, if $d = p^k$ for some $k > 0$, again for integers m, t we can also define the related function

$$j_t(m,d) = \sum_{c \bmod d} \left(\frac{c}{p}\right)_2^t e^{2\pi i \frac{mc}{d}}.$$

We show a two important examples of Gauss sum examples that showcase techniques that will be important later in computing the full exponential sum $H(\mathbf{d},)$.

Example 2.1. In this example, we show how "summing over all roots of unity" can make a Gauss sum vanish. Here, consider the example

$$g_1(1,p^2) = \sum_{c \bmod p^2} \left(\frac{c}{p^2}\right)_2 e^{2\pi i \frac{c}{p^2}}$$

To solve this, we re-index to $c = x + py$ with $x, y \bmod p$ and notice that $\left(\frac{c}{p^2}\right)_2$ only depends on $c \bmod p$. Then, we get

$$\begin{aligned}
g_1(1,p^2) &= \sum_{x,y \bmod p} \left(\frac{x}{p^2}\right)_2 e^{2\pi i \frac{x}{p^2} + \frac{y}{p}} \\
&= \sum_{x \bmod p} \left(\frac{x}{p^2}\right)_2 e^{2\pi i \frac{x}{p^2}} \sum_{y \bmod p} e^{2\pi i \frac{y}{p}} \\
&= \sum_{x \bmod p} \left(\frac{x}{p^2}\right)_2 e^{2\pi i \frac{x}{p^2}} \cdot 0 = 0
\end{aligned}$$

,

where in the penultimate the sum over y vanishes because the sum of all roots of unity is 0.

Example 2.2. Next, we present an example where the sum vanishes due to symmetry in the quadratic residue symbol. We evaluate

$$g_1(p,p) = \sum_{c \bmod p} \left(\frac{c}{p}\right)_2 e^{2\pi i \frac{cp}{p}}$$

Here, the exponent of e is $2\pi i$ times an integer, so $e^{(\cdots)} = 1$. Then, we have

$$g_1(p,p) = \sum_{c \bmod p} \left(\frac{c}{p}\right)_2 = 0$$

because for $c \bmod p$, $\left(\frac{0}{p}\right)_2 = 0$ and for non-zero c, half have $\left(\frac{c}{p}\right)_2 = 1$ and half have $\left(\frac{c}{p}\right)_2 = -1$.

Certain values of Gauss sums for specific values of m and d and our related function j are well-known. For the sake of completeness, we list without proof a few of the more important ones we will use.

Proposition 2.3. *For a not divisible by p, we have*

$$g_t(ap^k, p^\ell) = \left(\frac{a}{p}\right)_2^{-\ell t} \cdot \begin{cases} 0 & \ell - k \geq 2 \\ p^{\ell-1} g_{\ell t}(1,p) & \ell - k = 1 \\ 0 & \ell - k \leq 0, \ell t \text{ odd} \\ p^{\ell-1}(p-1) & \ell - k \leq 0, \ell t \text{ even} \end{cases}$$

and

$$j_t(ap^k, p^\ell) = \left(\frac{a}{p}\right)_2^{-t} \cdot \begin{cases} 0 & \ell - k \geq 2 \\ p^{\ell-1} j_t(1,p) & \ell - k = 1 \\ 0 & \ell - k \leq 0, t \text{ odd} \\ p^{\ell-1}(p-1) & \ell - k \leq 0, t \text{ even.} \end{cases}$$

Proposition 2.4. *For an odd prime p, we have*

$$j_t(1,p) = g_t(1,p) = \begin{cases} -1 & t \text{ even} \\ \sqrt{p} & p \equiv 1 \bmod 4, t \text{ odd} \\ i\sqrt{p} & p \equiv 3 \bmod 4, t \text{ odd.} \end{cases}$$

Proposition 2.5. *For $k \geq l$ and for any a, in particular, $p \mid a$ is allowed, we have*

$$j_t(ap^k, p^l) = \begin{cases} 0 & t \text{ odd} \\ p^{l-1}(p-1) & t \text{ even.} \end{cases}$$

Proposition 2.6. *For $k < l$ and for any a, in particular, $p \mid a$ is allowed, we have*

$$j_t(ap^k, p^l) = p^k j_t(a, p^{l-k}).$$

Proposition 2.7. *We have the following special case: $j_t(x,1) = 1$.*

These propositions will turn out to be useful when we try to evaluate the exponential sum H later in the report.

2.2. **Root Systems.** We now go over some concepts about the combinatorics of root systems. For a more detailed overview, see [BB05], upon which some of the definitions are based.

Definition 1 (See p.4 from [BB05])**.** A **Weyl group** is a finite group W, generated by elements $S = \{s_1, s_2, \ldots, s_n\}$ so that

(W1) $s_i^2 = e$ for $i \in \{1, 2, \ldots, n\}$, and

(W2) $(s_i s_j)^{m_{ij}}$ for $i, j \in \{1, 2, \ldots, n\}$ where $i \neq j$,

and $m_{ij} \in \{2, 3, 4, 6\}$ for each $i, j \in \{1, 2, \ldots, n\}$ where $i \neq j$.

Weyl groups are a special subset of groups called **Coxeter group**. We will not need the definition of this whole class in generality. In fact, throughout our report we will typically consider the Weyl group S_n, the symmetric group on n letters.

Associated to this Weyl group is the root system A_{n-1}. We proceed to a definition of a root system.

Definition 2 (Definition on p. 10, [BB05])**.** A **root system** is a finite set $\Phi \subset \mathbb{R}^d \backslash \{0\}$ for some positive integer d, if for all $\alpha, \beta \in \Phi$, the following conditions hold:

(R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$,

(R2) Given a root $\alpha \in \Phi$, let σ_α, be the reflection about the hyperplane perpendicular to α. Then, $\sigma_\alpha(\Phi) = \Phi$.

(R3) $\sigma_\alpha(\beta) - \beta = m\alpha$, where $m \in \mathbb{Z}$.

The group generated by the σ_α is called the **Weyl group** of Φ. The naming is not coincidental; one can verify that this group satisifies axioms (W1) and (W2).

One way we can express our reflection explicitly is using the following formula:

$$\sigma_\beta(\alpha) = \alpha - \frac{2\langle \alpha_j, \beta \rangle}{\langle \beta, \beta \rangle}\beta,$$

where $\langle \bullet, \bullet \rangle$ is an inner product invariant under the action of the Weyl group.

Definition 3. Fix some choice of hyperplane through the origin, which can be defined by $\langle x, v \rangle = 0$ for some nonzero vector v. A **positive root** is a root $\alpha \in \Phi$ so that $\langle \alpha, v \rangle > 0$, and a **negative root** is a root where $\langle \alpha, v \rangle < 0$. We call the set of positive roots Φ^+ and the set of negative roots Φ^-. A **simple root** is a positive root that cannot be expressed a nonnegative linear combination of other positive roots.

The notion of positive roots lets us define a poset on the positive roots Φ^+. Then, say that $\beta > \beta'$ if $\beta - \beta' \in \Phi^+$. Say that β covers β' if $\beta > \beta'$ and there exists no $\delta \in \Phi^+$ such that $\beta > \delta > \beta'$.

In our case, we will be working exclusively with the type A root system which can be described as

$$\{e_i - e_j : i, j \in \{1, 2, \ldots, n\}, i \neq j\} \subset \mathbb{R}^n,$$

where e_i is the i^{th} standard basis vector. Then, using the hyperplane generated by the vector $v = (n, n-1, n-2, \ldots, 1)$, we have that the positive roots here are those where $i < j$ in the above set, and the simple roots are those where $j = i + 1$.

Then, the Weyl group associated to this root system is S_n. In this case, the simple roots have corresponding reflections which correspond to the simple reflections s_i, which generate S_n. For notational purposes, we will denote $\alpha_i := e_i - e_{i+1}$ as our simple roots and $\beta_{i,j} := e_i - e_j$.

2.3. **Reductive Groups.** We now consider how this work with root systems relates to a reductive group G (or, more precisely, the split metaplectic cover of the reductive group G). For us, we will always have $G = GL_n(\mathbb{C})$ for some n (and usually $n = 6$). This group is associated with the root system of type A_{n-1}.

We associate a maximal parabolic to a simple root as follows, along the lines of [BF15, §5.1]. For our group $GL_n(\mathbb{C})$ and a maximal parabolic P, which is the subgroup of block upper triangular matrices with two blocks of size m and $n-m$ for some $1 \le m \le n-1$, we let M be the Levi subgroup of P. In our case, this subgroup M will be isomorphic to $GL_m(\mathbb{C}) \times GL_{n-m}(\mathbb{C})$. Notice that we can associate a Weyl group to M as well, which will be $A_{m-1} \times A_{n-m-1} \subset A_{n-1}$. In this case, we associate the parabolic P with removing the simple root α_m. Indeed, the Weyl group for M is the subgroup of S_n generated by all the simple reflections except s_m, and we can think of the root system of $A_{m-1} \times A_{n-m-1} \subset A_{n-1}$ as the subset of our original root system.

Visually, we can think of P, M as the following respective matrix forms

$$\begin{pmatrix} [GL_m(\mathbb{C})] & * \\ 0 & \begin{bmatrix} \\ GL_{n-m}(\mathbb{C}) \\ \\ \end{bmatrix} \end{pmatrix}, \quad \begin{pmatrix} [GL_m(\mathbb{C})] & 0 \\ 0 & \begin{bmatrix} \\ GL_{n-m}(\mathbb{C}) \\ \\ \end{bmatrix} \end{pmatrix},$$

with the removed root "located" at the corner between the two blocks in M, like so:

$$\begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & \textcircled{0} & 0 & 0 & 0 \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{pmatrix}.$$

Remark. Later, when we analyze the entire Dirichlet series, we will be repeating this process with our Levi subgroup M until we eventually arrive at the Borel subgroup B, which for us is the set of upper triangular matrices. How this is done will be more precisely outlined when we consider the rest of the Dirichlet series.

From here, we enumerate the positive roots γ_i in the associated root system, following the method outlined by [BF15]. Let w_M be the longest word of the Weyl group of M. Then, if w_0 is the longest word of the Weyl group of G, we can decompose $w_0 = w_M w^P$. Write the reduced decomposition of w^P as $s_{i_{t+1}} s_{i_{t+2}} \cdots s_{i_m}$. Then, define $\gamma_1 = w_M(\alpha_{i_{t+1}}), \gamma_2 = w_M s_{i_{t+1}}(\alpha_{i_{t+2}}), \ldots, \gamma_N = w_M s_{i_{t+1}} s_{i_{t+2}} \cdots s_{i_{m-1}}(\alpha_{i_m})$. We will use this ordering of the roots when we review the definition of the exponential sum $H(\mathbf{d};\mathbf{t})$ given in [BF15].

2.4. **The Definition of H.** We now parse the exponential sum H that is given in [BF15], associated with a maximal parabolic subgroup $P \subset G$. See the next section for an example of these definitions.

In our case, we are particularly interested in certain characters $\psi_{\mathbf{t}}$ associated to a vector $\mathbf{t}$, as defined in the beginning of Section 6 of [BF15] by

$$\psi_{\mathbf{t}}(w_0 e_{-\alpha_j}(x) w_0^{-1}) = \psi(t_j x).$$

As such, as our exponential sum H depends on the character we choose, which [BF15] emphasizes in [BF15, Equation (30)], defining the sum as

$$H(\mathbf{d};\mathbf{t}) = \sum_{c_j \pmod{D_j}} \psi(\sum_j t_j v_j) \prod_{k=1}^{N} \left(\frac{c_k}{d_k}\right)_2^{q_k}.$$

For our computations, we treat [BF15, Proposition 5.7] as definition of the D_j, given by

$$D_j = d_j \prod_{l=j+1}^{N} d_l^{\langle \gamma_j, \gamma_l^\vee \rangle},$$

where the d_i are the coordinates of the vector $\mathbf{d}$ and γ_j are our positive roots with their enumeration. Additionally, for our report, we take $\psi(x) = e^{2\pi i x}$ and $q_k = 1$ for each k.

As for the v_j, they are defined in equation (26) of [BF15] as

$$v_j = \sum_{(k,k') \in S_j} \Big[(-1)^{i+i'} \eta_{i,i',\gamma_k,-\gamma_{k'}} (b_k d_k^{-1})^i (c_{k'} d_{k'}^{-1})^{i'} \prod_{l \geq k} (d_l^{-1})^{\langle \alpha_j, \gamma_l \rangle} \prod_{k' < l < k} (d_l^{-1})^{i' \langle \gamma'_k, \gamma_l \rangle} \Big],$$

summing over i, i', k, k' so that $i\gamma_k - i'\gamma_{k'} = -\alpha_j$.

The $\eta_{i,i',\gamma_k,-\gamma_{k'}}$ are constants that can be defined with the following equation, equation (21) in [BF15] (see [Ste16] for a more detailed exposition):

$$e_\alpha(s) e_\beta(t) e_\alpha(s)^{-1} = e_\beta(t) [\prod_{\substack{i,j \in \mathbb{Z}^+ \\ i\alpha + j\beta = \gamma \in \Phi}} e_\gamma(\eta_{i,j;\alpha,\beta} s^i t^j)].$$

In our report, as we are taking reductive group $G = GL_n(\mathbb{C})$, this map $e_\alpha(t)$ can be given by

$$e_\alpha(t) = I + tE_{i,j},$$

where $\alpha = e_i - e_j$ and $E_{i,j}$ is the matrix with 1 in i,jth entry and 0s elsewhere. Note that $E_{ij} E_{lk} = \delta_{jl} E_{ik}$.

In our particular example, we will not need to refer to this whole definition, because the case of $G = GL_n(\mathbb{C})$ turns out to make these η coefficients rather simple to compute. We will see this later when we consider A_n in general.

We now proceed to our example computation of $H(\mathbf{d}; \mathbf{t})$ when $n = 4$, with the H associated to removing the second root.

2.5. **Example: The $GL_4(\mathbb{C})$ example from** [BF15]**.** Here we will go through the example included in [BF15, §7] in more detail, to make it clear what the general process for computing the value of the exponential sum H looks like.

From Brubaker and Friedberg's paper [BF15], if we're given a maximal parabolic P corresponding to removing the second root, we can decompose our longest word w_0 (which in this case is the longest word of A_3) in the word $w_M w^P$, where w_M is the longest word in the subgroup obtained by removing a root. In this case, we are removing α_2, so the Weyl group of M is generated only by the reflections corresponding to α_1 and α_3. Recall that simple roots correspond to simple reflections (see the discussion under Definition 3), and so these reflections are namely s_1, s_3.

But then the longest word in the Levi subgroup being the permutation $w_M = s_1 s_3$, which means that we end up with a decomposition by $w_0 = s_1 s_3 s_2 s_1 s_3 s_2$.

From here, we can obtain the ordering of positive roots corresponding to this decomposition, as discussed at the end of subsection 2.3. In this case, this ordering corresponds to the following ordering of roots: $\gamma_1 = w_M(\alpha_2), \gamma_2 = w_M s_2(\alpha_1), \gamma_3 = w_M s_2 s_1(\alpha_3)$, and

$\gamma_4 = w_M s_2 s_1 s_3(\alpha_2)$. Applying these words, we obtain the following enumeration of the positive roots:

$$\begin{aligned}
\gamma_1 &= \alpha_1 + \alpha_2 + \alpha_3, \\
\gamma_2 &= \alpha_2 + \alpha_3, \\
\gamma_3 &= \alpha_1 + \alpha_2, \\
\gamma_4 &= \alpha_2.
\end{aligned}$$

In coordinates, we can express the simple roots as $\alpha_1 = (1,-1,0,0), \alpha_2 = (0,1,-1,0)$ and $\alpha_3 = (0,0,1,-1)$. From here, we can coordinatize the positive roots

$$\begin{aligned}
\gamma_1 &= \alpha_1 + \alpha_2 + \alpha_3 = (1,0,0,-1), \\
\gamma_2 &= \alpha_2 + \alpha_3 = (0,1,0,-1), \\
\gamma_3 &= \alpha_1 + \alpha_2 = (1,0,-1,0, \\
\gamma_4 &= \alpha_2 = (0,1,-1,0).
\end{aligned}$$

We can also picture these roots as corresponding the positions in a matrix, as shown below:

$$\begin{pmatrix} 1 & 0 & \gamma_3 & \gamma_1 \\ 0 & 1 & \gamma_4 & \gamma_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We now can compute the D's, recalling the formula: $D_j = d_j \prod_{l=j+1}^{N} d_l^{\langle \gamma_j, \gamma_l \rangle}$, so we have

$$D_1 = d_1 d_2^{\langle \gamma_1, \gamma_2 \rangle} d_3^{\langle \gamma_1, \gamma_3 \rangle} d_4^{\langle \gamma_1, \gamma_4 \rangle} = d_1 d_2 d_3,$$

$$D_2 = d_2 d_3^{\langle \gamma_2, \gamma_3 \rangle} d_4^{\langle \gamma_2, \gamma_4 \rangle} = d_2 d_4,$$

$$D_3 = d_3 d_4^{\langle \gamma_3, \gamma_4 \rangle} = d_3 d_4, \quad D_4 = d_4.$$

Next, the v's are computed by applying (26) of [BF15]:

$$v_j = \sum_{(k,k') \in S_j} \left[(-1)^{i+i'} \eta_{i,i',k,-k'} (b_k d_k^{-1})^i (c_{k'} d_{k'}^{-1})^{i'} \prod_{l \geq k} (d_l^{-1})^{\langle \alpha_j, \gamma_l \rangle} \prod_{k' < l < k} (d_l^{-1})^{i' \langle \gamma_k', \gamma_l \rangle} \right]$$

Since $j = 1, .., N$ and $N = 4$ for our case, we compute v_1, v_2 and v_3. We first find the set of (k,k') s.t. $i\gamma_k - i'\gamma_{k'} = -\alpha_1$ where $i = 1, i' = 1$. This gives us the pairs $(2,1), (4,3) \in S_1$. Similarly, we obtain the pairs $(3,1), (4,2) \in S_3$ and $\emptyset \in S_2$. So we have:

$$
\begin{aligned}
v_1 &= (-1)^2\eta_{1,1;\gamma_2,-\gamma_1}\cdot\frac{b_2c_1}{d_2d_1}\cdot\frac{1}{d_2^{\langle\alpha_1,\gamma_2\rangle}}\frac{1}{d_3^{\langle\alpha_1,\gamma_3\rangle}}\frac{1}{d_4^{\langle\alpha_1,\gamma_4\rangle}}+(-1)^2\eta_{1,1;\gamma_4,-\gamma_3}\cdot\frac{b_4c_3}{d_4d_3}\cdot\frac{1}{d_4^{\langle\alpha_1,\gamma_4\rangle}}\\
&= \frac{b_4c_3}{d_3}+\frac{b_2c_1d_4}{d_3d_1}\\
v_2 &= \frac{c_4}{d_4}\\
v_3 &= (-1)^2\eta_{1,1;\gamma_4,-\gamma_2}\cdot\frac{b_4c_2}{d_4d_2}\cdot\frac{1}{d_4^{\langle\alpha_3,\gamma_4\rangle}}\frac{1}{d_3^{\langle\gamma_2,\gamma_3\rangle}}+(-1)^2\eta_{1,1;\gamma_3,-\gamma_1}\cdot\frac{b_3c_1}{d_3d_1}\cdot\frac{1}{d_4^{\langle\alpha_3,\gamma_4\rangle}}\frac{1}{d_3^{\langle\alpha_3,\gamma_3\rangle}}\frac{1}{d_2^{\langle\gamma_1,\gamma_2\rangle}}\\
&= -(\frac{b_4c_2}{d_2}+\frac{b_3c_1d_4}{d_1d_2}).
\end{aligned}
$$

To see where we got $\eta_{1,1;\gamma_2,-\gamma_1}$, for instance, we compute $e_{\gamma_2}(s)e_{-\gamma_1}(t)e_{\gamma_2}(s)^{-1}$, which yields us with the matrix product

$$
\begin{pmatrix}1&0&0&0\\0&1&0&s\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\t&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&s\\0&0&1&0\\0&0&0&1\end{pmatrix}^{-1}.
$$

Evaluating the product of matrices yields us with the matrix

$$
\begin{pmatrix}1&0&0&0\\st&1&0&0\\0&0&1&0\\t&0&0&1\end{pmatrix}.
$$

At the same time, noticing that $\gamma_2-\gamma_1=-\alpha_1$, we evaluate

$$
e_{-\gamma_1}(t)e_{-\alpha_1}(\eta_{1,1;\gamma_2,-\gamma_1}st)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\t&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\\eta_{1,1;\gamma_2,-\gamma_1}st&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\\eta_{1,1;\gamma_2,-\gamma_1}st&1&0&0\\0&0&1&0\\t&0&0&1\end{pmatrix},
$$

where we see that $\eta_{1,1;\gamma_2,-\gamma_1}=1$. Again, we will perform this computation in more generality later on, which will reduce the amount of matrix multiplication we need to explicitly perform.

3. Computing an exponential sum from A_n

In this section, we describe our method of computing the exponential sums which arise from removing the rth node in a Dynkin diagram of A_n. In order to do this, we use facts about the geometry of the roots appearing in the unipotent radical of the chosen parabolic to simplify computing v_j. Then, we draw an associated directed graph which allows us to better understand the sum. Finally, we use the graph as a guide to reindex the sum into a nicer form.

Let Φ be the root system, Φ_M the root system of our Levi subgroup, and Φ_P the set of (positive) roots in our unipotent radical. Note that $\Phi^+=\Phi_M^+\sqcup\Phi_P$.

In this section, we work in the case that each d_i is a power of the same prime and we write $d_i = p^{l_i}$. As discussed in Theomerem 6.5 of [BF15], $H(\mathbf{d},\mathbf{t})$ satisfies a "twisted multiplicativity" condition, so we only need to evaluate $H(\mathbf{d},\mathbf{t})$ on prime powers.

Further, in the following section there will be times when we want to index d_is linearly, and other times when we want to do so according to the position of γ_i in the matrix. To accomplish this, we use the following notation: Define the function I which takes a position in our matrix to the index of the corresponding γ_i, meaning for $\gamma_i = e_a - e_b$ that

$$I(a,b) = i$$

We will simplify this notation by writing

$$\gamma_{i,j} := \gamma_{I(i,j)},\ d_{i,j} := d_{I(i,j)}$$

.

Throughout this section we describe how our methods apply to the following example:

Example 3.1. Let $G = \mathrm{GL}_6(\mathbb{C})$, with associated root system Φ of type A_5. We "remove the second simple root" of Φ, meaning that we fix our Levi subalgebra $M \cong \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_4(\mathbb{C})$. In this section, we simply state facts about this example which will be justified in section 4

3.1. **Computing D_j's geometrically.** Here, we discuss how to view the computation of D_j's geometrically. Since D_j depends on the relationship between the γ_ks, we first discuss their geometry.

First, fix the standard coordinates for A_n in $\mathbb{R}^{n+1}$ as described in Section 2.2. We can associate any positive simple root to the i,jth position of an $(n+1)\times(n+1)$ matrix. Our $\gamma_1,\dots,\gamma_N$ are all in a $r \times (n+1-r)$ rectangle in the upper right corner of such a matrix, where r is the index of the node we remove.

Example 3.2. In Example 3.1, the positioning of the γ_js looks like

$$\begin{bmatrix} * & * & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\ * & * & \gamma_8 & \gamma_7 & \gamma_6 & \gamma_5 \\ & & * & * & * & * \\ & & * & * & * & * \\ & & * & * & * & * \\ & & * & * & * & * \end{bmatrix},$$

where the asterisks represent the A_1 and A_3 parts that are left over.

We formalize this observation with the following

Proposition 3.3. *$\gamma = e_a - e_b \in \Phi_P$ if and only if $a < b$, $a \le r$, and $b \ge r+1$.*

Proof. First, say γ is a root in the unipotent radical. Since γ is positive, $a < b$. If we had $a > r$, then $r < a < b$, so γ is a sum of positive simple roots $\alpha_a + \alpha_{a+1} + \cdots + \alpha_{b-1}$ which avoids α_r. Then, $\gamma \in \Phi_M$, a contradiction since $\Phi_M \cap \Phi_P = \emptyset$. Similarly, if we had $b < r+1$, then $a < b < r+1$ and again $\gamma \in \Phi_M$.

Now, let γ satisfy the conditions of the theorem. Since $a < b$, $\gamma \in \Phi^+$. Further, $a \le r$ and $b \ge r+1$ so writing

$$\gamma = \alpha_a + \cdots + \alpha_{b-1},$$

this expansion must include α_r. Thus, $\gamma \notin \Phi_M$, so $\gamma \in \Phi_P$. ☐

Lemma 3.4.

$$\langle \gamma_i, \gamma_j \rangle = \begin{cases} 2 & \gamma_i = \gamma_j \\ 1 & \gamma_i, \gamma_j \textit{ are in the same row or same column} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Assume $\gamma_i \neq \gamma_j$. Let $\gamma_i = e_a - e_b$, $\gamma_j = e_c - e_d$. Since γ_i, γ_j reside in a $r \times (n+1-r)$ matrix in the upper right hand corner, we must have $a, c \geq r+1$ and $b, d \leq r$. Then, $a \neq d, b \neq c$. If γ_i, γ_j are in the same row, this means $a = c$. However, as $\gamma_i \neq \gamma_j$, we have $b \neq d$ which implies

$$\langle \gamma_i, \gamma_j \rangle = \langle e_a, e_c \rangle - \langle e_a, e_d \rangle - \langle e_b, e_c \rangle - \langle e_b, e_d \rangle = 1 - 0 - 0 + 0 = 1$$

Similarly, if γ_i, γ_j are in the same row, then their inner product is 1. However, if they are not in the same row or column, then $a \neq c$ and $b \neq d$. Since $a \neq d$ and $b \neq c$, their inner product is just 0. ☐

We also need an additional lemma about the ordering of the roots. Suppose that we pick a maximal parabolic $P \subset GL_n(\mathbb{C})$, corresponding to removing the root α_i.

Lemma 3.5. *If $\gamma_j > \gamma_k$ in the usual ordering on the roots then $j < k$.*

Proof. Recall from subsection 2.3 that our ordering of the roots arises from decomposing the longest word of our Weyl group, w_0 into $w_M w^P$, where P is our maximal parabolic and M is the Levi subgroup. In this case, our Weyl group is S_n.

To find the ordering of the roots, we thus must find a nice decomposition for w^P. In order to do this, recall that our M in this case is going to be $GL_i(\mathbb{C}) \times GL_{n-i}(\mathbb{C})$. In this case, our Weyl group associated to this Levi subgroup of P is $S_i \times S_{n-i}$. meaning that in particular our longest word w_M has the one-line notation

$$i(i-1)(i-2)\dots 21n(n-1)(n-2)\dots(i+1),$$

in essence combining the longest word of S_i with that of S_{n-i}.

From here, we claim that the following decomposition for w^P yields us with $w_0 = w_M w^P$:

$$s_i s_{i+1} s_{i+2} \dots s_{n-1} s_{i-1} s_i s_{i+1} \dots s_{n-2} \dots s_{n-i}.$$

To see this, we first consider what the permutation $w_{j,k} = s_j s_{j+1} s_{j+2} \dots s_k$ does, for $j < k \leq n-1$. It's not hard to see that this permutation sends $k+1$ to j, increases each of $j, j+1, \dots, k$ by one, and fixes the rest of the elements; this is just the cycle $(j \quad j+1 \quad \dots \quad k+1)$. We now consider the combined effect of these permutations, which can be expressed as $w_{i,n-1} w_{i-1,n-2} \dots w_{1,n-i}$. Notice that the first permutation sends $1, 2, \dots, n-i$ to $2, 3, \dots, n-i+1$, respectively, per our description above. But by construction, we see that we may repeat this; indeed, if after a of these $w_{j,k}$ permutations (going from right to left) we end up at $a+1, a+2, \dots, n-i+a$, then notice that by construction the $a+1$st one to apply is $w_{a+1,n-i+a}$, which increases each of these elements by one, meaning that the first $a+1$ send $1, 2, \dots, n-i$ to $a+2, a+3, \dots n-i+a+1$.

In total, we see that this permutation sends x to $i+x$ for $x = 1, 2, \dots, n-i$. Therefore, the product of these has at least $(n-i)i$ inversions; the pairs (x, y) where $x < i < y$ are inversions (the last $n-i$ elements are the first $n-i$). But our word only has length $(n-i)i$, so there are exactly that many inversions. This in particular means that our permutation above is just the one with one-line notation $(n-i+1)(n-i+2)\dots n12\dots(n-i)$. But then this is w^P, since $w_M w^P$ is precisely w_0.

We now explicitly state the ordering of the roots with this particular decomposition. In this case, our ordering will be

$$\gamma_1 = w_M(\alpha_i), \gamma_2 = w_M s_i(\alpha_{i+1}), \dots, \gamma_{(n-i)i} = w_M s_i s_{i+1} \dots s_{n-i-1}(\alpha_{n-i}).$$

We will turn to coordinates for this part. Notice that the action of s_i, corresponding to $\alpha_i = e_i - e_{i+1}$, sends the vector $\sum_{j=1}^{n} v_j e_j$ to $\sum_{j=1}^{n} v_j e_j - (v_i - v_{i+1})(e_i - e_{i+1})$, which swaps the ith and $i+1$st coordinates. In other words, we can see that the action of the Weyl group is to permute the coordinates. For instance, as $\alpha_i = e_i - e_{i+1}$, we have that $w_M(e_i - e_{i+1}) = e_1 - e_n$.

We will use this to show, in fact, that the following is our ordering of roots:

$$\gamma_{k(n-i)+l} = e_{k+1} - e_{n-l+1}, \tag{1}$$

where $1 \le l \le n-i$ and $0 \le k < i$. To prove this, observe that by definition, we have that

$$\gamma_{k(n-i)+l} = w_M w_{i,n-1} w_{i-1,n-2} \dots w_{i-k+1,n-k} w_{i-k,i-k+l-2} \alpha_{i-k-1+l}$$

(if $k = 0$ we remove all the middle terms, leaving $w_M w_{i,i+l-1} a_{i+l-1}$, and similarly if $l = 1$ we remove the $w_{i-k+1,n-k} w_{i-k,i-k+l-2}$ term).

We see that, from our observation about the $w_{a,b}$ being cycles, how they act on the coordinates, and how $\alpha_{i-k-1+l} = e_{i-k-1+l} - e_{i-k+l}$, we have that $w_{i-k,i-k+l-2}\alpha_{i-k-1+l} = e_{i-k} - e_{i-k+l}$. Notice if $l = 0$ that this does nothing, which is also what we expect.

For the next terms, we see that

$$w_{i,n-1} w_{i-1,n-2} \dots w_{i-k+1,n-k}(e_{i-k} - e_{i-k+l}) = e_{i-k} + e_{i+l}.$$

For instance, we can see that $w_{i-k+1,n-k}(e_{i-k} - e_{i-k+l}) = e_{i-k} - e_{i-k+l+1}$, since $i-k+l \le n-k$. It's also not hard to see that this process repeats for the other cycles. Note that if $k = 0$ this again does nothing, which is also consistent with what we want (in the case where $k = 0$, none of these terms exist, so we expect to just get back the same root, which we do).

Finally, we have that $w_M(e_{i-k} - e_{i-k+l}) = e_{k+1} - e_{n-l+1}$, which is exactly what we claimed. Thus, the ordering that we specified is indeed the ordering arising from this decomposition.

From here, the lemma isn't hard to prove: if $\gamma_i > \gamma_j$, where $\gamma_i = e_a - e_b, \gamma_j = e_c - e_d$, we have that this inequality holds if and only if $e_a + e_d - e_b - e_c$ is a positive root, which holds precisely when either $a = c$ and $d < b$, or $b = d$ and $a < c$.

In the first case, $\gamma_j > \gamma_k$ implies that $\gamma_j = e_a - e_b, \gamma_k = e_a - e_d$, which implies that $j = (a-1)(n-i) + (n+1-b) < (a-1)(n-i) + (n+1-d) = k$ and in the second case we have $\gamma_j = e_a - e_d, \gamma_k = e_c - e_d$, meaning that from $j = (a-1)(n-i) + (n+1-d) < (c-1)(n-i) + (n+1-d) = k$, using the ordering given in equation (1). This proves the lemma. $\square$

We can use these lemmas to compute $D_j = D_{a,b}$ in a geometric way:

Proposition 3.6. *We have that*

$$D_{a,b} = d_{a,b} \prod_{c<a} d_{c,b} \prod_{c>b} d_{a,c}$$

Remark. If we place the d_j's in a matrix according to the positions of their corresponding γ_js, then we can understand the preceding proposition as saying

$$D_j = d_j \cdot d_k\text{s below } d_j \text{ in the same column } \cdot d_k\text{s to the left of } d_j \text{ in the same row}$$

Proof of Proposition 3.6. The definition of D_j is

$$D_j = d_j \prod_{k>j} d_k^{\langle \gamma_j, \gamma_k \rangle}$$

Using Lemma 3.4, we know that $\langle \gamma_j, \gamma_k \rangle$ is always 1 or 0. Then, letting

$$S := \{k : k > j, \langle \gamma_j, \gamma_k \rangle\}$$

we see that

$$D_j = d_j \prod_{k \in S} d_j$$

Now, making use of our index function I, define

$$S' = \{I(c,b) : \ c < a\} \cup \{I(a,c) : \ c > b\}$$

Showing $S = S'$ will prove the claim.

First, say $k = I(s,t) \in S$. Then, $\langle \gamma_i, \gamma_j \rangle = 1$, so by Lemma 3.4, γ_k is in the same row or same column as γ_j, meaning $a = s$ or $b = t$. For simplicity, assume they are in the same row, meaning $a = s$.

Assume for contradiction that $t \leq b$. Since $k \neq j$, we know $b \neq t$, so $t < b$. Then, we have

$$\gamma_k - \gamma_j = (e_a - e_b) - (e_s - e_t) = e_t - e_b,$$

which is a positive root since $t < b$. Then, $\gamma_k > \gamma_j$. But, Lemma 3.5 then implies $k < j$, a contradiction. Thus, $t > b$ so $k \in S'$. The case where instead $b = t$ identical.

Now, say $k = I(s,t) \in S'$, and again assume we are in the case $a = s$. Then, γ_j, γ_k are in the same row, so Lemma 3.4 implies $\langle \gamma_j, \gamma_k \rangle = 1$.

We then have $t > b$, so

$$\gamma_j - \gamma_k = (e_s - e_t) - (e_a - e_b) = e_b - e_t$$

is a positive root. Then, $\gamma_k > \gamma_j$ so by Lemma 3.5, we get $k > j$. Thus, $k \in S$, and the case when $b = t$ is again identical. $\square$

3.2. **Computing the structure coefficients η.**

Proposition 3.7. *Let Φ be a root system of type A_n using the usual coordinates. Let $\alpha \in \Phi^+$, $\beta \in \Phi^-$, $\alpha + \beta \neq 0$, such that $x\alpha + y\beta \in \Phi^-$ for some $x, y \in \mathbb{Z}^+$. Then, $x = y = 1$, and letting $\alpha = e_i - e_j, \beta = e_k - e_l$, we must have either $j = k$ or $i = l$ and*

$$\eta_{\alpha,\beta;1,1} = \begin{cases} +1 & j = k \\ -1 & i = l \end{cases}$$

Proof. First, remember that under our coordinates, every root in Φ is given by $e_a - e_b$ for some a, b. Then, say we have

$$x\alpha + y\beta = x(e_i - e_j) + y(e_k - e_l) \in \Phi^-,$$

Say for contradiction that $x \geq 2$. Recall that every root in Φ is of the form $e_s - e_t$. Then, to ensure $x\alpha + y\beta$ is of this form we need $i = l$ and $j = k$. This says $\alpha + \beta = 0$, which we assumed was not the case. Then, $x < 2$, meaning $x = 1$, and similarly $y = 1$. Further, if we had $i \neq k$ and $j \leq l$ then $\alpha + \beta$ would be a sum of 4 basis vectors and would also not be in Φ. Then, we have exactly one of $i = l$ or $j = k$.

Now, we can rewrite (21) from [BF15] as

$$e_\alpha(s)e_\beta(t)e_\alpha(s)^{-1} = e_\beta(t)e_{\alpha+\beta}(\eta_{\alpha,\beta;1,1}st) \tag{2}$$

We will drop the $1, 1$ and simply write $\eta_{\alpha,\beta}$ to mean $\eta_{\alpha,\beta;1,1}$. First assume we are in the case where $i = l$ and $j \neq k$. We see that

$$e_\alpha(s) = I_{n+1} + sE_{ij},\ e_\beta(t) = I_{n+1} + tE_{kl}.$$

We compute the LHS of (2)

$$\begin{aligned} e_\alpha(s)e_\beta(t)e_\alpha(s)^{-1} &= (I_{n+1} + sE_{ij})(I_{n+1} + tE_{kl})(I_{n+1} - sE_{ij}) \\ &= (I_{n+1} + sE_{ij})(I_{n+1} + tE_{kl} - sE_{ij} - stE_{kj}) \\ &= I_{n+1} + tE_{kl} - sE_{ij} - stE_{kj} + sE_{ij} \\ &= I_{n+1} + tE_{kl} - stE_{kj}. \end{aligned}$$

Since $i = l$, we have that

$$\alpha + \beta = e_i - e_j - (e_k - e_l) = e_k - e_j.$$

The RHS of (2) is then

$$\begin{aligned} e_\beta(t)e_{\alpha+\beta}(\eta_{\alpha,\beta}st) &= (I_{n+1} + tE_{kl})(I_{n+1} + \eta_{\alpha,\beta}stE_{kj}) \\ &= I_{n+1} + tE_{kl} + \eta_{\alpha,\beta}stE_{kj} \end{aligned}$$

So we see that we must have $\eta_{\alpha,\beta} = -1$. The case where $j = l$ is similar and we end up with $\eta_{\alpha,\beta} = 1$. $\square$

Corollary 3.8. *For γ, γ' in the same row or same column and $\gamma - \gamma' \in \Phi^-$*

$$\eta_{1,1;\gamma,-\gamma'} = \begin{cases} -1 & \gamma, \gamma' \text{ are in the same row} \\ +1 & \gamma, \gamma' \text{ are in the same column} \end{cases}$$

Proof. Set $\gamma = e_i - e_j$, $\gamma' = e_l - e_k$ so that $-\gamma' = e_k - e_l$. Then, if γ, γ' are in the same row, $i = l$, so we are in the first case of Proposition 3.7. Similarly, if γ, γ' are in the same column, we are in the second case of Proposition 3.7. □

3.3. **Computing v_js.**

Definition 4 (Brubaker-Friedberg, [BF15], (26))**.** We define

$$(3) \qquad v_j = \sum_{(k,k')\in S_j} \left[(-1)^{i+i'}\eta_{i,i',k,-k'}(b_k d_k^{-1})^i (c_{k'} d_{k'}^{-1})^{i'} \prod_{l\geq k}(d_l^{-1})^{\langle \alpha_j, \gamma_l^\vee\rangle} \prod_{k'<l<k}(d_l^{-1})^{i'\langle \gamma_k', \gamma_l^\vee\rangle}\right]$$

where S_j is the set of pairs (k, k'), $k > k'$, such that $i\gamma_k - i'\gamma_{k'} = -\alpha_j$ for some $i, i' \in \mathbb{Z}^{>0}$.

Proposition 3.9 (similar to Brubaker-Friedberg, [BF15], Lemma 6.3)**.** *We can simplify the definition for v_j by defining it in terms of the D_js as follows:*

$$v_j = \sum_{(k,k')\in S_j} (-1)^{i+i'}\eta_{i,i',k,-k'} b_k^i c_{k'}^{i'} \frac{D_k^i}{D_{k'}^{i'}}$$

Proof. We use v_j as defined in (3)

Since $(k, k') \in S_j$ as defined above, we have that $i\gamma_k - i'\gamma_{k'} = -\alpha_j$. Then, we make the following simplification

$$\begin{aligned}\prod_{l\geq k}(d_l^{-1})^{\langle \alpha_j, \gamma_l^\vee\rangle} \prod_{k'<l<k}(d_l^{-1})^{i'\langle \gamma_k', \gamma_l^\vee\rangle} &= \prod_{l\geq k}(d_l^{-1})^{\langle i'\gamma_{k'} - i\gamma_k, \gamma_l^\vee\rangle} \prod_{k'<l<k}(d_l^{-1})^{i'\langle \gamma_k', \gamma_l^\vee\rangle} \\ &= \prod_{l\geq k}(d_l)^{i\langle \gamma_k, \gamma_l^\vee\rangle} \prod_{l>k'}(d_l^{-1})^{i'\langle \gamma_k', \gamma_l^\vee\rangle}\end{aligned}$$

Now, we look at each of these two products and get

$$\prod_{l\geq k}(d_l)^{i\langle \gamma_k, \gamma_l^\vee\rangle} = (d_k)^{i\langle \gamma_k, \gamma_k^\vee\rangle} \prod_{l>k}(d_l)^{i\langle \gamma_k, \gamma_l^\vee\rangle} = d_k^{2i} \prod_{l>k}(d_l)^{i\langle \gamma_k, \gamma_l^\vee\rangle} = d_k^i D_k^i$$

where the middle equality follows from the fact that that

$$\langle \gamma_k, \gamma_k^\vee\rangle = \left\langle \gamma_k, \frac{2\gamma_k}{\langle \gamma_k, \gamma_k\rangle}\right\rangle$$

for any root system. Then, by definition

$$\prod_{l>k'}(d_l^{-1})^{i'\langle \gamma_k', \gamma_l^\vee\rangle} = d_{k'}^{i'} D_{k'}^{-i'}$$

Putting this together, we have

$$
\begin{aligned}
v_j &= \sum_{(k,k')\in S_j} \left[(-1)^{i+i'} \eta_{i,i',k,-k'} (b_k d_k^{-1})^i (c_{k'} d_{k'}^{-1})^{i'} d_k^i D_k^i d_{k'}^{i'} D_{k'}^{-i'} \right] \\
&= \sum_{(k,k')\in S_j} (-1)^{i+i'} \eta_{i,i',k,-k'} b_k^i c_{k'}^{i'} \frac{D_k^i}{D_{k'}^{i'}}
\end{aligned}
$$

□

Further, we can say a lot about which terms $b_k c_{k'}$ appear in some v_j. We have that

Proposition 3.10. *For type A_n, $(k,k') \in S_j$ for some j if and only if $\gamma_{k'}$ covers γ_k in the poset on Φ^+.*

Proof. First, we show the "only if" direction. Say $(k,k') \in S_j$, meaning $i'\gamma_{k'} - i\gamma_k = \alpha_j$ for $i, i' \in \mathbb{Z}^{>0}$. By Proposition 3.7, we must have $i = i' = 1$, so $\gamma_{k'} - \gamma_k = \alpha_j$. This says that $\gamma_{k'} > \gamma_k$. Further, if we have some $\delta \in \Phi$ such that $\gamma_{k'} > \delta > \gamma_k$, then

$$(\gamma_{k'} - \delta) + (\delta - \gamma_k) = \alpha_j$$

presents α_j as a sum of two positive roots, a contradiction.

Now, we handle the "if" direction. say we have $\gamma_{k'}$ covers γ_k. Then, we have $\gamma_{k'} - \gamma_k = \beta \in \Phi^+$. For contradiction say that β is not simple, so we can write $\beta = \beta_1 + \beta_2$. However, then we have

$$\gamma_{k'} > \gamma_k + \beta_1 > \gamma_k,$$

a contradiction. Thus, $\beta = \alpha_j$ for some j and $(k,k') \in S_j$. □

However, recall that for the v corresponding to the removed root we have $v_r = t_r \frac{c_N}{d_N}$, so we need to ensure that we never have $(k,k') \in S_r$.

Lemma 3.11. *For $\gamma, \gamma' \in \Phi_P$,*

$$\gamma - \gamma' \neq \alpha_r$$

Proof. Say $\gamma = \alpha_a + \cdots + \alpha_{b-1}$ and $\gamma' = \alpha_c + \cdots + \alpha_{d-1}$. Using the fact that α_is are linearly independent, if

$$\gamma - \gamma' = \alpha_r$$

were true, we would need $\alpha_c, \ldots, \alpha_{d-1}$ to omit α_r. However, this says that $\gamma' \in \Phi_M$, a contradiction. □

3.4. **Divisibility Conditions.** We know that $H(\mathbf{d}, \mathbf{t})$ is zero unless certain divisibility conditions on the d_is hold. These conditions are instrumental in solving $H(\mathbf{d}, \mathbf{t})$ and in understanding its support.

Lemma 3.12 (Brubaker-Friedberg, [BF15], Lemma 6.1)**.** *$H(\mathbf{d},\mathbf{t})$ vanishes unless, for each simple root α_j,*

$$t_j \prod_{i=1}^{N} d_i^{-\langle \alpha_j, \gamma_i^\vee \rangle} \in \mathbb{Z}$$

We now re-interpret these conditions in terms of rows and columns in the matrix of γ_js.

Proposition 3.13. *Let*

$$R(a) = \{I(a,b) : r+1 \leq b \leq n+1\},\ C(b) = \{I(a,b) : 1 \leq a \leq r\}$$

which are the indices of γ_js in a given row or column. Then, the divisibility conditions hold if and only if for each a, $1 \leq a \leq r-1$ and b, $r+1 \leq b \leq n$, we have

(1) $\prod_{i \in R(a)} d_i \mid t_a \prod_{i \in R(a+1)} d_i$
(2) $\prod_{i \in C(b+1)} d_i \mid t_b \prod_{i \in C(b)} d_i$

Remark. $R(a)$ corresponds to the roots in the ath row, and $C(b)$ corresponds to roots in the bth column. If we fill the matrix of γ_js with their corresponding d_js, we can think about these conditions as relating the products of rows and columns.

Proof of Proposition 3.13. First, we need the following

Lemma 3.14.

$$\{\alpha_j | \text{ there exist } \gamma, \gamma' \in \Phi_P \text{ such that } \gamma - \gamma' = \alpha_j\} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \setminus \{\alpha_r\}$$

Proof. We know $\subset$ from Lemma 3.11. For the other direction, let α_i, $i \neq r$ arbitrary. When $i < r$, let

$$\gamma = e_i - e_{n+1},\ \gamma' = e_{i+1} - e_{n+1}.$$

When $i > r$, let

$$\gamma = e_1 - e_{i+1},\ \gamma' = e_1 - e_i.$$

In both cases, Proposition 3.3 shows that $\gamma, \gamma' \in \Phi_P$. □

We now transform each of the conditions in Lemma 3.12 to one of our conditions. First, consider $j \leq r-1$ and we'll show (1) holds. Then, using Lemma 3.14, let $\gamma = e_j - e_{n+1}, \gamma' = e_{j+1} - e_{n+1}$, and we have $\gamma - \gamma' = \alpha_j$. Then, we have

$$\begin{aligned} \mathbb{Z} \ni t_j \prod_{i=1}^{N} d_i^{-\langle \alpha_j, \gamma_i \rangle} &= t_j \prod_{i=1}^{N} d_i^{\langle \gamma' - \gamma, \gamma_i \rangle} \\ &= t_j \frac{\prod_{i=1}^{N} d_i^{\langle \gamma', \gamma_i \rangle}}{\prod_{i=1}^{N} d_i^{\langle \gamma, \gamma_i \rangle}} \qquad (4) \end{aligned}$$

(5)

In light of Lemma 3.4, this is

$$\begin{aligned}(4) &= t_j \frac{\prod_{i\in R(j+1)} d_i \prod_{i\in C(n+1)} d_i}{\prod_{i\in R(j)} d_i \prod_{i\in C(n+1)} d_i} \\ &= t_j \frac{\prod_{i\in R(j+1)} d_i}{\prod_{i\in R(j)} d_i} \in \mathbb{Z},\end{aligned}$$

which is condition (1). For $j \geq r+1$, set $\gamma = e_1 - e_{i+1}, \gamma' = e_1 - e_i$, and using the same method we get the condition

$$t_j \frac{\prod_{i\in C(j)} d_i}{\prod_{i\in C(j+1)} d_i} \in \mathbb{Z},$$

We have described how each one of the conditions in Lemma 3.12 is equivalent to each of our conditions. Thus, this is an if and only if.

□

Example 3.15. In Example 3.1, these conditions are

$$\begin{aligned} d_1d_2d_3d_4 &\mid d_5d_6d_7d_8t_1 \\ d_3d_7 &\mid d_4d_8t_3 \\ d_2d_6 &\mid d_3d_7t_4 \\ d_1d_5 &\mid d_2d_6t_5 \end{aligned}$$

3.5. **The Dependency Graph.** In order to better model the exponential sum, we associate it to a graph which we call the "dependency graph". Our exponential sum takes the form

$$H(\mathbf{d};\mathbf{t}) = \sum_{c_i \bmod D_i} \underbrace{\varphi\left(\sum_i t_i v_i\right)}_{\text{exponential part}} \overbrace{\prod_{k=1}^{N} \left(\frac{c_k}{d_k}\right)_2^{q_k}}^{\text{quadratic residues}}, \tag{6}$$

In the A_n case, we can capture the exponential part of this sum in a directed graph. The argument of the exponential part (the "exponent") will look like

$$\sum_i t_i v_i = t_r \frac{c_N}{d_N} + \text{ terms of the form } \frac{b_i c_j D}{D'} \text{ with } i > j \text{ and } D, D' \text{ integers,}$$

where r is the index of the root we remove. Then, for N the number of roots in the unipotent radical of the chosen parabolic, create a graph on vertex set $[N]$ with an edge $i \to j$ if the term $\frac{b_i c_j D}{D'}$ appears in the sum. We call this graph the "dependency graph". The A_5 in the case where we remove the second root ($r = 2$), we get the dependency graph

$$\begin{tikzcd}
4 \arrow[r] & 3 \arrow[r] & 2 \arrow[r] & 1 \\
8 \arrow[u] \arrow[r] & 7 \arrow[u] \arrow[r] & 6 \arrow[u] \arrow[r] & 5 \arrow[u]
\end{tikzcd}$$

The graph captures every time in the exponent, except for the term $t_2\frac{c_8}{d_8}$. We have circled the vertex 8 in order to indicate this, although this circle is not formally a part of our graph. This graph lets us visually understand the terms appearing in the sum.

We want to use this graph to re-index the sum by assigning variables to some edges. However, as written, it is insufficient. Define the weight of an edge $i \to j$ as $\mathrm{wt}(i \to j)$ and define the weight of a path through the complete graph on $[N]$ as the product of it's edges. Our hope would be that for a path from a to b that $\mathrm{wt}(P) = \mathrm{wt}(a \to b)$. For example, given the path $7 \to 3 \to 6$, it's weight is $b_7c_3b_3c_6$ which we might expect to equal b_7c_6 since we think about b_3, c_3 as being inverses (we know that $b_3c_3 \equiv -1 \bmod d_3$). However, nuance in the sum prevents this from being exactly true. First, we notice that b_7, c_3, c_6 all run over different moduli. Further, if $d_3 = 0$, there are no conditions on needing $(c_3, p) = 1$. In this case, we no longer have $b_3c_3 = 1$. We later explore a more complicated graph that will capture this nuance and allow us to do such a reindexing.

3.6. **Reparametrizing the Exponential Sum.** In our exponential sum, if $d_i = 1$, there there is no condition on c_i being relatively prime to p. As we saw in the previous section, this can cause issues when we try to use our dependency graph for re-indexing. Then, we want to be able to rewrite the sum in some way that removes this dependence on whether individual $d_i = 1$. Here, we reparametrize the sum in a way that will support later re-indexing through an augmented dependency graph.

Given input data $\mathbf{d} = (d_1, \ldots, d_N), \mathbf{t} = (t_1, \ldots, t_n)$, the sum we desire to compute is

$$H(\mathbf{d};\mathbf{t}) = \sum_{c_i \bmod D_i} \varphi\left(\sum_i t_iv_i\right) \prod_{k=1}^{N} \left(\frac{c_k}{d_k}\right)_2^{q_k}, \tag{7}$$

with $\varphi(z) = e^{2\pi iz}$ and where the notation $c_i \bmod D_i$ means summing over vectors $(c_1, \ldots, c_N)$ with each $c_i \in \mathbb{Z}/D_i\mathbb{Z}$. To define $H(\mathbf{d}, \mathbf{t})$ we perform the following procedure:

- Compute the D_i's in terms of the d_i's.
- For each i, choose mappings $a_i, b_i, c_i \colon \mathbb{Z}/D_i\mathbb{Z} \to \mathbb{Z}$ such that for a residue s in $\mathbb{Z}/D_i\mathbb{Z}$, $c_i(s) \equiv s \mod D_i$, and the matrix $\left(\begin{smallmatrix} a_i(s) & b_i(s) \\ c_i(s) & d_i \end{smallmatrix}\right)$ is in $\mathrm{SL}_2(\mathbb{Z})$, i.e. $a_id_i - b_ic_i = 1$. We denote this matrix by $g_i(s)$.
- Then

$$H(d;t) = \sum_{(s_1,\ldots,s_N)\in\mathbb{Z}/D_1\mathbb{Z}\times\cdots\times\mathbb{Z}/D_N\mathbb{Z}} \prod \left(\frac{c_j(s_j)}{d_j}\right)_2 e^{2\pi i\sum_{i=1}^n v_it_i},$$

where v_i is a function in the c_j's and b_j's.

In particular, any valid mappings will give the same exponential sum [BF15, Prop. 5.9]. As discussed at the beginning of the section, we work in the case where each d_i is a power of a prime p, and write $d_i = p^{l_i}$.

In the above sum, if $l_i > 0$, then the summand is 0 if $p \mid c_i$, so we can assume that always $(c_i, p) = 1$. However, if $l_i = 0$, then there is no such condition. We'd like to be able to say that always $(c_i, p) = 1$, but in order to do so we'd need to do cases on each l_i. Instead, we will perform a re-indexing of the sum that takes care of this for us. Choose functions $x_i, y_i : \mathbb{Z}/D_i\mathbb{Z} \to \mathbb{Z}$ such that $c_i = x_i + d_i y_i$. The advantage is that

$$\left(\frac{c_i}{d_i}\right)_2 = \left(\frac{x_i + d_i y_i}{d_i}\right)_2 = \left(\frac{x_i}{d_i}\right)_2,$$

and this holds even in the case $l_i = 0$ since we define $\left(\frac{0}{1}\right)_2 = 1$. Then, we can in every case assume $(x_i(s), p) = 1$. We would like to be able to take the modulus of each s_i to the same power. To do this, take M to be a sufficiently large power of p ($M = D_1 \cdots D_8$ works), and fix some function $W : \mathbb{Z}/M\mathbb{Z} \to \mathbb{Z}$ which sends elements of $\mathbb{Z}/M\mathbb{Z}$ to a chosen integer representative (we require $W(a) \equiv a \bmod M$).

Proposition 3.16. *Let C be the number of l_i which are 0. Then,*

$$H(\mathbf{d}, \mathbf{t}) = \frac{d_1 \cdots d_N}{M^N} \left(\frac{p}{p-1}\right)^C \sum_{x_j \bmod M:\ (x_j,p)=1,\ y_j \bmod D_j/d_j} \varphi\left(\sum_j t_j v_j\right) \prod_{k=1}^{N} \left(\frac{x_k}{d_k}\right)_2^{q_k},$$

where v_j depends on c_j, b_j and we set $c_j = W(x_j + d_j y_j)$ and $b_j = -W(x_j^{-1})$

Proof. For each $d = p^l$ such that $d \mid M$, fix a map $W_d : \mathbb{Z}/d\mathbb{Z} \to \mathbb{Z}/M\mathbb{Z}$ where $W_d(s) \equiv s \bmod d$. Define $W_1 \equiv 0$. Consider the index set

$$S = \{(m_1, \ldots, m_N \mid 0 \le m_j < M/d_i, (m_j, p) = 1 \text{ if } d_i = 1\}$$

For each tuple $(m_1, \ldots, m_N) \in S$, construct the tuple of mappings $a_{j,m_j}, b_{j,m_j}, c_{j,m_j} : \mathbb{Z}/D_i\mathbb{Z} \to \mathbb{Z}$, $x_{j,m_j} : \mathbb{Z}/d_i\mathbb{Z} \to \mathbb{Z}/M\mathbb{Z}$, $y_j : \mathbb{Z}/(D_i/d_i)\mathbb{Z} \to \mathbb{Z}/M\mathbb{Z}$, $1 \le j \le N$ via

$$\begin{aligned}
x_{j,m_j}(s) &= W_{d_j}(s) + d_j m_j \\
y_j(s) &= \frac{1}{d_j}\left(W_{D_j}(s) - x_{j,m_j}(s)\right) \\
c_{j,m_j}(s) &= W(x_{j,m_j}(s) + d_j y_j(s)) \\
b_{j,m_j}(s) &= -W(x_{j,m_j}^{-1}(s)) \\
a_{j,m_j}(s) &= \frac{1 + b_{j,m_j}(s) c_{j,m_j}(s)}{d_j},
\end{aligned}$$

where we can assume that $(s_i, p) > 1$ where $l_i > 0$. We argue that a, b, c are valid mappings. By inspection, we see that $c_{j,m_j}(s) \equiv s \bmod D_j$ since $D_j <= M$. Further, $x_{j,m_j}(s)^{-1}$ is well-defined since if $l_i > 0$ then $(x_{j,m_j}(s), p) = 1$ since $(s, p) = 1$ and if $l_i = 0$ then $(x_{j,m_j}(s), p) = 1$ since in this case $(m_j, p) = 1$. Then, b_{j,m_j} is well-defined and $b_{j,m_j}(s) c_{j,m_j}(s) \equiv -1 \bmod d_j$, so $a_{j,m_j}(s)$ is an integer. By construction $a_{j,m_j}(s) d_j - b_{j,m_j}(s) c_{j,m_j}(s) = 1$.

Then, for any $(m_1, \ldots, m_N)$, we get the same sum. Consider the sum

$$H'(\mathbf{d}, \mathbf{t}) = \sum_{(m_1,\ldots,m_N)\in S} \sum_{s_i \bmod D_i} \varphi\left(\sum_i t_i v_i\right) \prod_{k=1}^{N} \left(\frac{c_{k,m_k}(s_k)}{d_k}\right)_2^{q_k},$$

which we will compute two ways. By independence of the sum for different choices of $(m_1, \ldots, m_N)$,

$$H'(\mathbf{d}, \mathbf{t}) = |S| H(\mathbf{d}, \mathbf{t})$$

Further, observe that the cardinality of S is

$$|S| = \prod_{d_j : d_j = 1} \frac{M(p-1)}{p} \prod_{d_j : d_j > 1} \frac{M}{d_j} = \left(\frac{p-1}{p}\right)^C \frac{M^N}{d_1 \cdots d_N}$$

Now, write $x_j = x_{j,m_j}(s), y_j = y_j(s)$. Considering this as a sum over all variables simultaneously, we see that x_j ranges over all values of $(\mathbb{Z}/M\mathbb{Z})^\times$ exactly once and that y_j ranges over all values of $\mathbb{Z}/(D_j/d_j)\mathbb{Z}$ exactly once. Justifying the second part requires using Proposition 3.9 to notice that every term in our sum involving a y_j looks like

$$\pm t_k \frac{x_i^{-1} y_j D_j D_i}{D_j}.$$

Then, if we chose a different representative for y_j in $\mathbb{Z}/(D_j/d_j)\mathbb{Z}$, this term would change by an integer. Since the term is in $e^{2\pi i(\cdots)}$, this does not cause the sum to change.

Under the re-indexing, we have

$$H'(\mathbf{d}, \mathbf{t}) = \sum_{x_j \bmod M:\ (x_j, p)=1,\ y_j \bmod D_j/d_j} \varphi\left(\sum_k t_k v_k\right) \prod_{k=1}^{N} \left(\frac{x_k}{d_k}\right)_2^{q_k},$$

where in each v_k we have $c_j = W(x_j + d_j y_j), b_j = W(x_j^{-1})$. Setting the two ways of computing $H'(\mathbf{d}, \mathbf{t})$ equal proves the claim. □

Remark. We can choose M to be large enough such that it is bigger than any denominator we see in any term in the exponential sum. Then, if we take $x_i + M$ instead of x_i, this will only add an integer to the exponent, which will not change the value of $e^{2\pi i(\cdots)}$.

3.7. **The augmented dependency graph.** We now describe how to associate an exponential sum to a directed graph in the A_n case. Refer to the input to $\varphi(\cdot)$ as the "exponent". Using Proposition 3.9, and in light of Proposition 3.7, we know that always $i = i' = 1$ in this case. Then, the exponent is of the form:

$$\sum_k t_k v_k = t_R \frac{c_N}{d_N} + \text{ terms of the form } \frac{b_i c_j D}{D'} \text{ with } i > j \text{ and } D, D' \in \mathbb{Z}$$

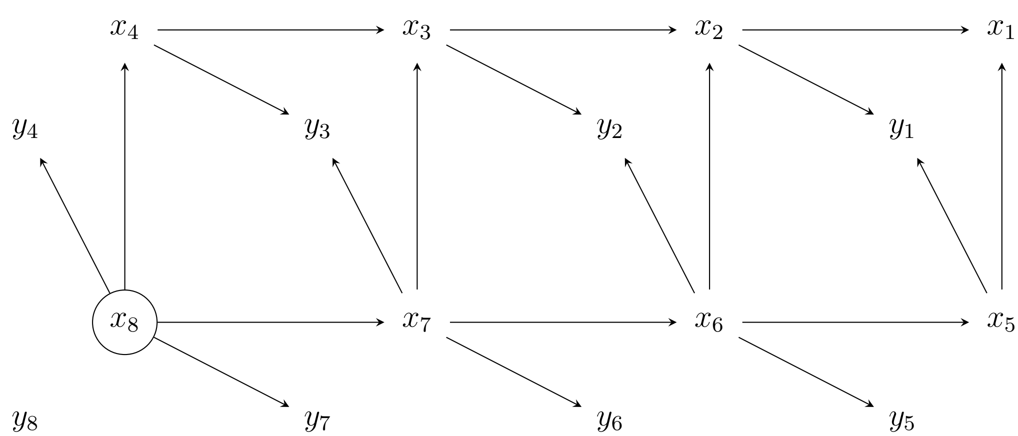
Under our reparameterization into x_i, y_i and in light of Proposition 3.9, terms in the sum other than $t_r \frac{x_N + d_8 y_N}{d_8}$ take one of the following two forms:

$$\pm 1 \cdot t_k \frac{x_i^{-1} x_j D_i}{D_j}, \quad \pm 1 \cdot t_k \frac{x_i^{-1} y_j D_i d_j}{D_i} \quad \text{where } i > j \tag{8}$$

Then, let G be the augmented dependency graph on vertices $V(G) = \{x_1, \ldots, x_N, y_1, \ldots, y_N\}$ with a directed edge $u \to v \in E(G)$ if there is a $u^{-1}v$ term in the sum. Further, we define the maps wt, t, ν on $E(G)$. Define $\mathrm{wt}(u \to v) = u^{-1}v$.

The map t sends an edge to its corresponding t_k and ν to it's corresponding structure constant such that for an edge $e = x_i \to x_j$, $\nu(e)t(e)\,\mathrm{wt}(e)\frac{D_i}{D_j}$ is the term in (8), and for $e = x_i \to y_j$, $\nu(e)t(e)\,\mathrm{wt}(e)\frac{D_i d_j}{D_j}$ is the term.

Example 3.17. For Example 3.1 the augmented dependency graph is

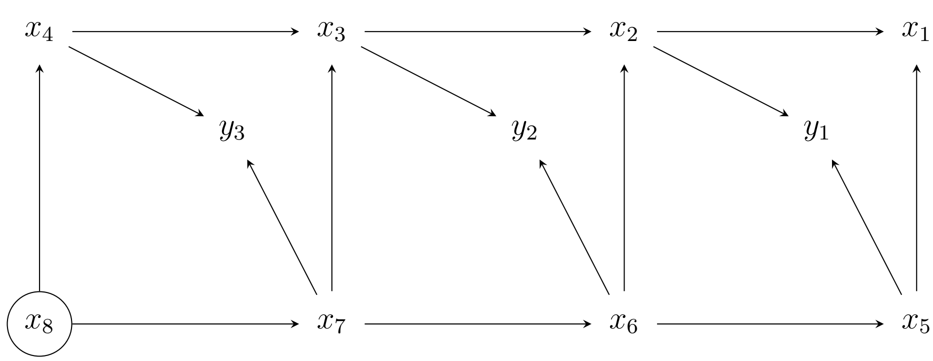


We have circled the x_N node to indicate the $t_r \frac{x_N}{d_N}$ term which is not otherwise represented in the exponent. In fact, the exponent is just this first term plus the terms represented by each edge in the graph.

Further, note that we have laid the graph out to correspond to the geometry in Proposition 3.6, and using this we can visually compute the D_j's given the d_j's.

Remark. We can verify that the terms corresponding to the variables on the left and bottom edges of the graph $(y_4, y_8, y_7, y_6, y_5)$ are in fact all integers. Thus, the sum is independent of the values of these variables. Thus, after removing them, we can write

$$\frac{D_4 D_5 D_6 D_7 D_8}{d_4 d_5 d_6 d_7 d_8} \times$$



where the factor added to the front of the graph is the number of copies we have of the simplified graph after removing y_4, y_5, y_6, y_7, y_8.

3.8. **Reindexing the sum.** Loosely, the sum as written is hard to work with for two reasons: 1) we must deal with x_i^{-1} terms and 2) the structure of the dependency graph is relatively complicated, i.e. the degree of each x_i node is at least 2. Also, we desire to have a graph with multiple connected components, which would allow us to factor the sum.

We will use the graph to reindex the sum. First, we need some definitions.

Definition 5. A directed (rooted) tree T is a tree (as an undirected graph) with the additional requirement that for each vertex $v \in V(T)$, there exists at most one edge $u \to v \in E(T)$. The root of the tree is the unique node $x \in T$ such that there exist no $u \to x \in E(T)$ for any $u \in V(T)$.

Proof. We must justify that a unique root exists. We know that $|E(T)| = |V(T)| - 1$. Since each vertex has at most 1 edge going into it, we then have $|V(T)| - 1$ vertices with an edge going into them, leaving exactly one vertex with no edge going into it. This is our root. $\square$

To facilitate such a reindexing, draw a directed tree T on vertex set $V(T) = x_1, \ldots, x_N$. Say the T has root node x_r. Let $e_i = x_{j_i} \to x_{k_i}$, $1 \le i \le N-1$, be the edges of T in some order. Then, consider the change of variables

$$a_0 = x_r$$
$$a_j = \mathrm{wt}(e_j) = x_{j_i}^{-1} x_{k_i}$$

We take $a_j \bmod M$ and $(a_j, p) = 1$, so that the cardinality of the set of a_js and of x_js are the same.

Proposition 3.18. *Such a reindexing is always a bijection between the set of a_js and the set of x_js. Further, we can write any edge in our dependency graph as a product of a_js, $1 \le j \le N-1$ and their inverses.*

Proof. We prove this second part of the claim first. For this we need

Lemma 3.19. *Let $P = x_{i_1} \to \cdots \to x_{i_m}$, $i_1 = a, i_m = b$ be a (directed) path in the complete graph on $x_1, \ldots, x_N$ from x_a to x_b. Define the weight of a path, $\mathrm{wt}(P)$, as the product of the weights of its edges. Then, $wt(P) \equiv wt(x_a \to x_b) \bmod M$.*

Proof. We prove this by induction on $|P|$ (number of edges in the path). If $|P| = 1$, $P = x_a \to x_b$ and we are done. If $|P| > 1$ and $P = x_a \to \cdots \to x_t \to x_b$, we know by induction

$$\mathrm{wt}(P) \equiv \mathrm{wt}(x_a \to x_t)\, \mathrm{wt}(x_t \to x_b) \bmod M$$

However, we then have

$$\begin{aligned}\mathrm{wt}(x_a \to x_t)\, \mathrm{wt}(x_t \to x_b) &\equiv x_a^{-1} x_t x_t^{-1} x_b \bmod M \\ &\equiv x_a^{-1} x_b \bmod M \equiv \mathrm{wt}(x_a \to x_b) \bmod M\end{aligned}$$

$\square$

Then, let $x_a \to x_b$ an arbitrary edge. We can find some path P from $x_a \to x_b$ with all edges in T. Then, the weight of P will be a product of a_js and their inverses, depending on the orientation of the edges in T, $1 \le j \le N-1$. From here, we can construct the inverse map

$$x_i = \begin{cases} a_0 & i = r \\ \mathrm{wt}(x_r \to x_i)a_0 & i \neq r \end{cases}$$

This proves the change of variables is injective, and since both sets have the same cardinality, it is a bijection. $\square$

We have described how to write the terms of the exponential sum in terms of our new reindexing, and now describe what the quadratic residues look like with the new a_js.

Proposition 3.20. *For edges $(i \to j) \in T$ define*

$$S(i \to j) = \{k \in [N] : x_k \text{ is a successor node of } x_j \text{ in } T\} \cup \{j\}$$

Then,

$$\prod_{k=1}^{N} \left(\frac{c_k}{d_k}\right)_2^{q_k} = \left(\prod_{k=1}^{N} \left(\frac{a_0}{d_k}\right)_2^{q_k}\right) \left(\prod_{e_j \in T} \prod_{k \in S(e_j)} \left(\frac{a_j}{d_k}\right)_2^{q_k}\right)$$

Proof. We use the inverse map defined in Proposition 3.18. Then, we see that

$$\prod_{k=1}^{N} \left(\frac{c_k}{d_k}\right)_2^{q_k} = \left(\frac{a_0}{d_N}\right)_2^{q_N} \prod_{k \neq r} \left(\frac{\mathrm{wt}(x_r \to x_k)a_0}{d_k}\right)_2^{q_k} \tag{9}$$

Using the multiplicativity of of the quadratic residue, we see that

$$(9) = \left(\prod_{k=1}^{N} \left(\frac{a_0}{d_k}\right)_2^{q_k}\right) \left(\prod_{(j,k) \in Q} \left(\frac{a_j}{d_k}\right)_2^{q_k}\right)$$

where

$$Q = \{(j,k) \mid e_j \text{ is in the path from } x_r \to x_k\}$$

If we can show the following, it will prove our claim:

Lemma 3.21.

$$Q = \{(j,k) | k \in S(e_j)\}$$

Proof. First, let $(j,k) \in Q$. Then e_j is in the path from $x_r \to x_k$. Say $e_j = (x_a \to x_b)$. If $k = b$, by definition $k \in S(e_j)$. If $k \neq b$, then x_k comes after x_b in a path and thus is a successor of x_b, so $k \in S(e_j)$.

Now, say we have $k \in S(e_j)$, with $e_j = (x_a \to x_b)$. If $k = b$, take a path from x_r to x_a and append $x_a \to x_b$ to get a path from x_r to x_k containing e_j. If $k \neq b$, then x_k is a successor of x_b, meaning there is some path from x_k to x_b. Prepend the path from x_r to x_b onto this path and we get a path from x_r to x_k, which necessarily contains the edge e_j since it goes through x_b. $\square$

$\square$

Now, we show the conditions under which the reindexing from our tree T is favorable.

Proposition 3.22. *We can write every edge of of our dependency graph as a product of a_js, $1 \leq j \leq N$, without inverses, if and only if*

(1) *T is a path*
(2) *The path respects the ordering on the roots, meaning if γ_j covers γ_i then i must come before j in the path.*

Proof. We will make use of the following

Lemma 3.23. *There is an edge $x_i \to x_j$ if and only if γ_j covers γ_i in the usual ordering on Φ^+.*

Proof. Corollary of Proposition 3.10 □

First, let T satisfy (1) and (2). Let $x_i \to x_j$ be an arbitrary edge. By Lemma 3.23, this means γ_j covers γ_i, so since the path respects the ordering on the roots, i must come before j in the path. Then, our path looks something like

$$c_N \to \cdots \to x_i \to \cdots \to x_j \to \cdots$$

so we have that

$$\mathrm{wt}(x_i \to x_j) = \mathrm{wt}(x_i \to \cdots \to x_j)$$

is a product of a_ks.

Now, assume for contradiction that T does not satisfy (1). This means we have at least two x_is that are leaves. Since A_n has a unique maximal root, we can choose i such that γ_i is not maximal, and we have some edge $e_k = (x_s \to x_i)$. Then, there is some γ_j covering γ_i, so by Lemma 3.23 we have an edge $x_i \to x_j$. Then, let P be the path in T from $x_i \to x_j$. However, since x_i is a leaf, this path must go $x_i \to x_s \to \cdots \to x_j$, meaning that

$$\mathrm{wt}(x_i \to x_j) = \mathrm{wt}(x_i \to x_s)\,\mathrm{wt}(x_s \to \cdots \to x_j) = a_k^{-1}\,\mathrm{wt}(x_s \to \cdots \to x_j),$$

which is a contradiction.

Now, assume T does not satisfy (2). Let γ_j cover γ_i. By Lemma 3.23 we have an edge $x_i \to x_j$. Since T is a path, it either looks like one of the following cases

$$\text{Case 1: } T = x_s \to \cdots \to x_i \to \cdots \to x_j \to \cdots$$
$$\text{Case 2: } T = x_s \to \cdots \to x_j \to \cdots \to x_i \to \cdots$$

In case 2, $\mathrm{wt}(x_i \to x_j)$ is a product of inverses of a_ks, which is a contradiction. Then, we are in case 1, so we see that i comes before j in the path. □

Example 3.24. In Example 3.1, we can re-index as follows:

$$
\begin{tikzcd}
x_4 \arrow[r, "a_2a_3"] \arrow[dr, dashed, "a_2"] & x_3 \arrow[r, "a_4a_5"] \arrow[dr, dashed, "a_4"] & x_2 \arrow[r, "a_6a_7"] \arrow[dr, dashed, "a_6"] & x_1 \\
a_0 \; \boxed{x_8} \arrow[u, "a_1"] \arrow[r, "a_1a_2"'] & x_7 \arrow[u, "a_3"] \arrow[r, "a_3a_4"'] & x_6 \arrow[u, "a_5"] \arrow[r, "a_5a_6"'] & x_5 \arrow[u, "a_7"]
\end{tikzcd}
$$

This means, perform the following re-indexing:

$$a_0 = x_8, a_1 = x_8^{-1}x_7, a_2 = x_7^{-1}x_4, \ldots, a_7 = x_5^{-1}x_1$$

and we see that the remaining edges become

$$x_8^{-1}x_7 = a_1a_2, x_4^{-1}x_3 = a_2a_3, \ldots, x_2^{-1}x_1 = a_6a_7$$

We also want to consider re-indexing the y_is. Fixing $x_1, \ldots, x_8$, we consider the re-indexing

$$y_1' := x_7^{-1}y_1,\ y_2' := x_6^{-1}y_2,\ y_3' := x_5^{-1}y_3.$$

This re-indexing is injective, thus bijective, since x_5, x_6, x_7 are invertible $\bmod p$. Then, we for the full augmented depenendency graph, the re-indexing looks like

$$
\begin{tikzcd}
x_4 \arrow[rr, "a_2a_3"] \arrow[dr, "a_2y_3'"] & & x_3 \arrow[rr, "a_4a_5"] \arrow[dr, "a_4y_2'"] & & x_2 \arrow[rr, "a_6a_7"] \arrow[dr, "a_6y_1'"] & & x_1 \\
& y_3 & & y_2 & & y_1 & \\
a_0 \; x_8 \arrow[uu, dashed, "a_1"] \arrow[rr, "a_1a_2"'] & & x_7 \arrow[uu, dashed, "a_3"] \arrow[ul, "y_3'"'] \arrow[rr, "a_3a_4"'] & & x_6 \arrow[uu, dashed, "a_5"] \arrow[ul, "y_2'"'] \arrow[rr, "a_5a_6"'] & & x_5 \arrow[uu, dashed, "a_7"] \arrow[ul, "y_1'"']
\end{tikzcd}
$$

4. Removing the Second Root in A_5

We can perform a similar computation to what we did above, but now we remove the second root in A_5. In this case, this corresponds to the following M :

$$\begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{pmatrix}.$$

Our goal is to compute the exponential sum

$$H(d;t)=\sum_{c_1 \bmod D_1}\cdots\sum_{c_8 \bmod D_8}\prod\left(\frac{c_j}{d_j}\right)_2 e^{\sum_{i=1}^5 v_i t_i}$$

(in the case where t is all 1).

We now find the v_is and D_is. We see that $w_M = s_1s_3s_4s_3s_5s_4s_3$, the longest word in the Weyl group associated to this subgroup M. Note that the one line notation of this permutation is 216543. This means that the longest word for S_6, the Weyl group associated to A_5, can be written as the following reduced word decomposition:

$$w_0 = s_1s_3s_5s_4s_5s_3s_4s_2s_3s_4s_5s_1s_2s_3s_4.$$

In particular, we have that $w^P = s_2s_3s_4s_5s_1s_2s_3s_4$. From here, we compute our ordering on the roots by computing $\gamma_1 = w_M(\alpha_2), \gamma_2 = w_Ms_2(\alpha_3), ..., \gamma_7 = w_Ms_2s_3s_4s_5s_1s_2(\alpha_3), \gamma_8 = w_Ms_2s_3s_4s_5s_1s_2s_3(\alpha_4)$. This yields us with the following computation:

(1) $\gamma_1 = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5$,
(2) $\gamma_2 = \alpha_1+\alpha_2+\alpha_3+\alpha_4$,
(3) $\gamma_3 = \alpha_1+\alpha_2+\alpha_3$,
(4) $\gamma_4 = \alpha_1+\alpha_2$,
(5) $\gamma_5 = \alpha_2+\alpha_3+\alpha_4+\alpha_5$,
(6) $\gamma_6 = \alpha_2+\alpha_3+\alpha_4$,
(7) $\gamma_7 = \alpha_2+\alpha_3$,
(8) $\gamma_8 = \alpha_2$.

The positions of the γ_is in a matrix is then

$$\begin{bmatrix} * & * & \gamma_4 & \gamma_3 & \gamma_2 & \gamma_1 \\ * & * & \gamma_8 & \gamma_7 & \gamma_6 & \gamma_5 \\ & & * & * & * & * \\ & & * & * & * & * \\ & & * & * & * & * \\ & & * & * & * & * \end{bmatrix}$$

From these positions, we use Proposition 3.6 to compute the D_js. The relevant positions of the d_js are

d_4	d_3	d_2	d_1
d_8	d_7	d_6	d_5

so we get

(1) $D_1 = d_1d_2d_3d_4d_5$,
(2) $D_2 = d_2d_3d_4d_6$,
(3) $D_3 = d_3d_4d_7$,
(4) $D_4 = d_4d_8$,
(5) $D_5 = d_5d_6d_7d_8$,
(6) $D_6 = d_6d_7d_8$,
(7) $D_7 = d_7d_8$,
(8) $D_8 = d_8$.

Now, we compute the v_js. We compute S_j as defined in (3) for $j = 1, 3, 4, 5$. In light of Proposition 3.7, we always have $i = i' = 1$. We see that

(1) $S_1 = \{((5,1), (6,2), (7,3), (8,4)\}$,
(2) $S_3 = \{(4,3), (8,7)\}$,
(3) $S_4 = \{(3,2), (7,6)\}$,
(4) $S_5 = \{(2,1), (6,5)\}$.

Then, using Proposition 3.9 and Corollary 3.8, we get

$$\begin{aligned} v_1 &= -\frac{b_5c_1d_6d_7d_8}{d_1d_2d_3d_4} - \frac{b_6c_2d_7d_8}{d_2d_3d_4} - \frac{b_7c_3d_8}{d_3d_4} - \frac{b_8c_4}{d_4}, \\ v_2 &= \frac{c_8}{d_8}, \\ v_3 &= \frac{b_4c_3d_8}{d_3d_7} + \frac{b_8c_7}{d_7}, \\ v_4 &= \frac{b_3c_2d_7}{d_2d_6} + \frac{b_7c_6}{d_6}, \\ v_5 &= \frac{b_2c_1d_6}{d_1d_5} + \frac{b_6c_5}{d_5}. \end{aligned}$$

Note that we initially take $b_ic_i \equiv -1 \bmod d_i$. Here, for simplicity, we instead take $b_ic_i \equiv 1 \bmod d_i$. For this reason, the signs of v_1, v_3, v_4, v_5 are the opposite of what Corollary 3.8 gives us directly.

We impose certain domain restrictions on d_i. We have the following lemma:

Lemma 4.1 (Brubaker-Friedberg, [BF15], Lemma 6.1)**.** *$H(\mathbf{d}, \mathbf{t})$ vanishes unless, for each simple root α_j,*

$$t_j \prod_{i=1}^{N} d_i^{-\langle \alpha_j, \gamma_i^\vee \rangle} \in \mathbb{Z}.$$

We call these the *divisibility conditions*, which in our case are

$$\ell_4 + \ell_8 \geq \ell_3 + \ell_7 \geq \ell_2 + \ell_6 \geq \ell_1 + \ell_5,$$
$$\ell_8 + \ell_7 + \ell_6 + \ell_5 \geq \ell_4 + \ell_3 + \ell_2 + \ell_1.$$

4.1. **The Comb Reparametrization.** Recall our exponential sum

$$H(d_1, \ldots, d_8) = \sum_{c_i \bmod D_i} e^{2\pi i \sum_{j=1}^{5} v_j} \prod_{i=1}^{8} \left(\frac{c_i}{d_i}\right)_2.$$

For convenience we call $\sum_{j=1}^5 v_j$ the *exponent*. Note that we can ignore any integer part of $\sum_{j=1}^5 v_j$, since $e^{2\pi i n} = 1$ for $n \in \mathbb{Z}$. We claim that if $4 \leq i \leq 8$, then the summand in H does not depend on c_i. Indeed, if $\ell_i = 0$ for a general i, then d_i is 1. In this case, we can set $b_i = 0$ so that all terms in the exponent with a b_i disappear. Furthermore, every term in the exponent with a factor c_i is a fraction over d_i for $4 \leq i \leq 8$, so these terms become integers and can be ignored.

Since c_i does not affect the summand if $\ell_i = 0$ for $4 \le i \le 8$, in the case that such an ℓ_i is 0, we can calculate H by only summing over c_i not divisible by p, and then multiply by $\frac{p}{p-1}$. For $1 \le i \le 3$, if $\ell_i \ne 0$, we sum over c_i relatively prime to p, since otherwise the $\left(\frac{c_i}{d_i}\right)_2$ term gives 0. However if $\ell_i = 0$, then we must sum over all c_i modulo D_i.

We can therefore rewrite H as

$$H(d_1, \dots, d_8) = \sum_{\substack{c_i \bmod D_i \\ p\nmid c_i \\ 4\le i\le 8}} \sum_{\substack{c_i \bmod D_i \\ p\nmid c_i \text{ if } \ell_i\ne 0 \\ 1\le i\le 3}} \left(\frac{p}{p-1}\right)^C e^{2\pi i \sum_{j=1}^5 v_j} \prod_{i=1}^8 \left(\frac{c_i}{d_i}\right)_2,$$

where C is the number of $i \in \{4,5,6,7,8\}$ with $\ell_i = 0$.

For a large power of p that we denote M, we perform the "raise to M" trick on all the c_i: we sum over c_i modulo M rather than modulo D_i, and average by using a $\frac{D_1\cdots D_8}{M^8}$ scaling factor, as explained in Section 3.6. For any $\ell_i = 0$ with $1 \le i \le 3$, we set b_i to 0; for other i in $\{1,2,3\}$ and for all $i \in \{4,5,6,7,8\}$, we set b_i to the inverse of c_i modulo M. We can then write

$$H(d_1, \dots, d_8) = \frac{D_1\cdots D_8}{M^8} \sum_{c_i \bmod M} e^{2\pi i \sum_{j=1}^5 v_j} \prod_{i=1}^8 \left(\frac{c_i}{d_i}\right)_2.$$

Consider the following reparametrization:

$$(c_1, c_2, \dots, c_8) \mapsto (b_5c_1, b_6c_2, b_7c_3, b_8c_4, b_6c_5, b_7c_6, b_8c_7, c_8) =: (a_1, \dots, a_8).$$

We call this reparametrization the *comb reparametrization* because besides a_8, the a_i form 7 edges of the dependency graph described in Section 3.5, forming a comb shape. This reparametrization is actually a bijection from the set of possible $(c_1, c_2, \dots, c_8)$ (a subset of $(\mathbb{Z}/M\mathbb{Z})^8$) to itself. We can write the summand in H in terms of the a_i, noting that it changes based on the cases of ℓ_1, ℓ_2, ℓ_3 being 0 due to the presence of certain b_is or lack thereof with $1 \le i \le 3$.

We first consider the simplest case, which is $\ell_1 = \ell_2 = \ell_3 = 0$. Then all the b_1, b_2, b_3 terms disappear, and using the comb reparametrization, we get

$$H = \frac{D_1D_2\cdots D_8}{M^8} \sum_{\substack{a_i \bmod M \\ p\nmid a_i}} \left(\frac{p}{p-1}\right)^C \left(\frac{a_8}{d_8}\right)_2 \left(\frac{a_8a_7}{d_7}\right)_2 \left(\frac{a_8a_4}{d_4}\right)_2 \left(\frac{a_8a_7a_6}{d_6}\right)_2 \left(\frac{a_8a_7a_6a_5}{d_5}\right)_2$$

$$\left(\frac{a_8a_7a_3}{d_3}\right)_2 \left(\frac{a_8a_7a_6a_2}{d_2}\right)_2 \left(\frac{a_8a_7a_6a_5a_1}{d_1}\right)_2 e^{2\pi i\left(\frac{a_8}{d_8} - \frac{a_4}{d_4} + \frac{a_7}{d_7} - \frac{a_3d_8}{d_3d_4} + \frac{a_6}{d_6} - \frac{a_2d_7d_8}{d_2d_3d_4} + \frac{a_5}{d_5} - \frac{a_1d_6d_7d_8}{d_1d_2d_3d_4} + a_4^{-1}a_7a_3\frac{d_8}{d_3d_7}\right)},$$

where C is the number of $\ell_4, \dots, \ell_8$ that are 0. Note that since $b_5, \dots, b_8$ are all relatively prime to p, the variable c_i being divisible by p is equivalent to a_i being divisible by p, so the domain which we sum a_i over is the same as which we sum c_i over. Furthermore, a_i^{-1} is the residue that is the inverse of a_i modulo M. Then $a_4^{-1}a_7a_3 \equiv (b_8c_4)^{-1}(b_8c_7)(b_7c_3) \equiv (b_4c_8)(b_8c_7)(b_7c_3) \equiv b_4c_3 \bmod M$, so $a_4^{-1}a_7a_3\frac{d_8}{d_3d_7}$ corresponds to $\frac{b_4c_3d_8}{d_3d_7}$. We have modifications if $\ell_i = 0$ for $i \in \{1,2,3\}$, which are as follows.

- If $\ell_3 \neq 0$, then we must add the b_3c_2 term to the summand. Note that in this case, we sum over c_3 not divisible by p, so b_3 is also relatively prime to p and modulo M, we have $b_3c_2 = a_3^{-1}a_6a_2$. In the sense of the comb graph, we are adding the "notch" going along vertices 3,7,6,2. Then the exponent has an extra term $a_3^{-1}a_6a_2\frac{d_7}{d_2d_6}$. We sum over all a_3 modulo M (instead of just those relatively prime to p).
- If $\ell_2 \neq 0$, then we add the "notch" going along vertices 2,6,5,1, giving an extra term in the exponent $a_2^{-1}a_5a_1\frac{d_6}{d_1d_5}$. We sum over all a_2 modulo M (instead of just those relatively prime to p).
- If $\ell_1 \neq 0$, then since there is no b_1 term, the summand does not change. We sum over all a_i modulo M (instead of just those relatively prime to p).

These modifications are all independent, i.e., they stack.

Using this reparametrization, we first demonstrate bounds on the support of H. Note that we continually make use of a method where we "sum over all roots of unity," as shown in Example 2.1 in Section 2. This method allows us to conclude that H is 0 in a wide variety of cases, when we have a reduced fraction in the exponent with denominator divisible by p^2. For convenience we refer to this method as the *root of unity method*. We will not be able to rule out the case $(\ell_3, \ell_4, \ell_7, \ell_8) = (2, 1, 0, 1)$, for which we write $(d_1, \ldots, d_8) \in \alpha$.

We first demonstrate a more in-depth example of this method. For convenience, we denote $v_p(a)$ as the largest power of p that divides an integer a; for example, $v_3(18) = 2$. We also write $v_p(a/b) = v_p(a) - v_p(b)$.

Example 4.2. The exponent in our exponential sum H has an a_3-dependent part

$$a_3\left(\frac{a_4^{-1}a_7d_8}{d_3d_7} - \frac{d_8}{d_3d_4}\right).$$

Suppose that we can write this expression as a fraction $a_3\frac{k}{p^\ell}$ for ℓ an integer at least 2, and k relatively prime to p; in other words, $v_p\left(\frac{a_4^{-1}a_7d_8}{d_3d_7} - \frac{d_8}{d_3d_4}\right) \leq -2$. Then if we write $a_3 = x + py$, any quadratic residue symbols dependent on a_3 only depend on x, not y. So, we have

$$H = \sum_{\cdots}\sum_{x}(\cdots)\sum_{0\leq y<M/p} e^{2\pi i(x+py)\frac{k}{p^\ell}},$$

where the $\cdots$ indicate expressions not dependent on y. But

$$\sum_{0\leq y<M/p} e^{2\pi i(x+py)\frac{k}{p^\ell}} = e^{2\pi i x}\sum_{0\leq y<M/p} e^{2\pi i\frac{yk}{p^{\ell-1}}} = 0,$$

since we sum over all $p^{\ell-1}$th roots of unity multiple times, and $p^{\ell-1} > 1$. Then H is 0.

In general, if we can show that the a_i-dependent part of the exponent, when reduced, is a fraction over p^ℓ for $\ell \geq 2$, and there is no a_i^{-1} term in the exponent, then H is 0. In other words, if there is no a_i^{-1} in the exponent and a_i is multiplied by a fraction with v_p at most -2, then H is 0; this is the essence of the root of unity method.

Example 4.3. If we have two fractions $\frac{a}{p^k}$ and $\frac{c}{p^\ell}$, then their sum can be written as a fraction over $p^{\max(k,\ell)}$, meaning $v_p\left(\frac{a}{p^k} + \frac{c}{p^\ell}\right) \geq \min\left(v_p\left(\frac{a}{p^k}\right), v_p\left(\frac{c}{p^\ell}\right)\right)$. In fact, this inequality must be an equality for $k \neq \ell$. So in Example 4.2, $v_p\left(\frac{d_8}{d_3d_4}\right) \neq v_p\left(\frac{d_8}{d_3d_7}\right)$, or $\ell_8 - \ell_3 - \ell_4 \neq \ell_8 - \ell_3 - \ell_7$, implies that as long as either $\ell_3 + \ell_4 - \ell_8$ or $\ell_3 + \ell_4 - \ell_7$ is at least 2, then summing over a_3 yields 0 by the root of unity method. This observation essentially will force many ℓ_i to be equal to save H from being 0, e.g. in 4.6 below.

In some cases, we can use the root of unity method even when there is an a_i^{-1} in the exponent.

Example 4.4. The a_4-dependent part of the exponent is

$$a_4^{-1} a_7 a_3 \frac{d_8}{d_3 d_7} - \frac{a_4}{d_4}.$$

Suppose that $\ell_4 \geq 2$, but $\ell_3 + \ell_7 - \ell_8 \leq 1$. If we write $a_4 = x + py$, the expression $a_4^{-1} a_7 a_3 \frac{d_8}{d_3 d_7}$ is not dependent on y, since it only depends on a_4^{-1} modulo p, which is determined by a_4 modulo p. Then like in Example 4.2, we have

$$H = \sum_{\cdots} \sum_{x} (\cdots) \sum_{0 \leq y < M/p} e^{2\pi i (x+py) \frac{1}{p^{\ell_4}}},$$

with

$$\sum_{0 \leq y < M/p} e^{2\pi i (x+py) \frac{1}{p^{\ell_4}}} = e^{2\pi i x} \sum_{0 \leq y < M/p} e^{2\pi i \frac{y}{p^{\ell_4 - 1}}} = 0.$$

In fact, this method works as long as ℓ_4 is strictly greater than $\ell_3 + \ell_7 - \ell_8 - v_p(a_3)$, since we can write $a_4 = x + p^{\ell_4 - 1} y$ and perform the same calculation.

We now use the above examples to bound the support of H. Inspired by the geometric nature of the dependency graph in 3.5, we informally call vertices $3, 4, 7, 8$ and associated parameters the *left box*, and vertices $1, 2, 5, 6$ and associated parameters the *right box*. We first focus on the left box, and show that ℓ_8, ℓ_3 are generally small, though we might have ℓ_4 or ℓ_7 large if $\ell_4 = \ell_7$. We then prove a similar result for the right box, showing that ℓ_1, ℓ_6 are generally small, though we might have ℓ_2 or ℓ_5 large if $\ell_2 = \ell_5$.

Proposition 4.5. *For $(d_1, \ldots, d_8) \notin \alpha$ in the support of H, we have $\ell_8 \leq 1$.*

Proof. Note that in the comb reparametrization, the a_8-dependent term is $\frac{a_8}{d_8}$. If $\ell_8 \geq 2$, then summing over a_8 yields 0 by the root of unity method. $\square$

Proposition 4.6. *For $(d_1, \ldots, d_8) \notin \alpha$ in the support of H, if $\ell_7 \geq 2$ or $\ell_7 + \ell_3 - \ell_8 \geq 2$, we have $\ell_3 = \ell_8$.*

Proof. In the comb reparametrization, the a_7 exponent term is $a_7(\frac{1}{d_7} + a_4^{-1} a_3 \frac{d_8}{d_3 d_7})$. Since a_4^{-1} is not divisible by p, we have $v_p\left(\frac{1}{d_7}\right) = -\ell_7$ and $v_p\left(a_4^{-1} a_3 \frac{d_8}{d_3 d_7}\right) = -\ell_7 - \ell_3 + \ell_8 + v_p(a_3)$. As in 4.3, if these two are not equal, then the sum of the fractions has v_p equal to $\min(-\ell_7, -\ell_7 - \ell_3 + \ell_8 + v_p(a_3))$. We need this value to be at least -1 for H to be nonzero.

If $k \neq 0$, then $\ell_3 = 0$ so $\ell_7 - \ell_8 - k$ is strictly less than ℓ_7 anyway, implying that ℓ_7 and $\ell_7 + \ell_3 - \ell_8 - k$ cannot be equal, and neither ℓ_7 nor $\ell_7 + \ell_3 - \ell_8$ can be at least 2 for $(d_1, \ldots, d_8)$ in the support of H. So if H is nonzero and $\ell_7 \geq 2$ or $\ell_7 + \ell_3 - \ell_8 \geq 2$, we must have $k = 0$, implying that $-\ell_7 = -\ell_7 - \ell_3 + \ell_8 + v_p(a_3) = -\ell_7 - \ell_3 + \ell_8$, so $\ell_3 = \ell_8$. $\square$

Proposition 4.7. *For $(d_1, \ldots, d_8) \notin \alpha$ in the support of H, if $\ell_4 \geq 2$, we need $\ell_4 = \ell_7$.*

Proof. In the comb reparametrization, the a_4-dependent exponent term is $a_4\frac{1}{d_4}+a_4^{-1}a_7a_3\frac{d_8}{d_3d_7}$. If the denominators of the two fractions are not equal, and the larger has exponent ℓ at least 2, then H is zero by the root of unity method as the sum of the fractions can be written as a reduced fraction over p^ℓ (see Example 4.3). Then $\ell_3+\ell_7-\ell_8=\ell_4\geq 2$. But then by Proposition 4.6, we have $\ell_3=\ell_8$, so $\ell_4=\ell_7$. □

Corollary 4.8. *For $(d_1,\ldots,d_8)\notin\alpha$ in the support of H, if $\ell_7\geq 2$ then $\ell_7=\ell_4$. If $\ell_4\geq 2$ then $\ell_4=\ell_7$. In both cases, we have $\ell_3=\ell_8$.*

Proof. If $\ell_7\geq 2$, then by 4.6 we get $\ell_3=\ell_8$, so by the divisibility conditions we get $\ell_4\geq\ell_7$, which implies $\ell_4=\ell_7$ by 4.7. The second part of the corollary follows from 4.7 and 4.6. □

Corollary 4.9. *For $(d_1,\ldots,d_8)\notin\alpha$ in the support of H, if $\ell_3+\ell_7\geq 2$, then $\ell_4+\ell_8=\ell_3+\ell_7$ and $\ell_3+\ell_4=\ell_7+\ell_8$. In particular, this implies $\ell_4=\ell_7$ and $\ell_3=\ell_8$.*

Proof. We first show the first equality. By divisibility conditions, we get $\ell_4+\ell_8\geq\ell_3+\ell_7\geq 2$. But if $\ell_4+\ell_8=2$ then we are done, and if $\ell_4+\ell_8\geq 3$ then $\ell_4\geq 2$, and we are done by Corollary 4.8.

Now we show the second equality. If $\ell_3,\ell_4,\ell_7,\ell_8$ are all at most 1, then they all equal 1 and we are done. If $\ell_4\geq 2$ or $\ell_7\geq 2$, then we are done by Corollary 4.8. We cannot have $\ell_8\geq 2$ by Proposition 4.5. The final case is if $\ell_3\geq 2$. Then since $\ell_8\leq 1$ by Proposition 4.5, we must have $\ell_4,\ell_7\leq 1$ (since otherwise Corollary 4.8 implies $\ell_3=\ell_8\leq 1$. The divibility condition $\ell_4+\ell_8\geq\ell_3+\ell_7$ implies that the only possibility is $\ell_3=2,\ell_4=\ell_8=1$, $\ell_7=0$. But this case is in α so we are done. □

We now focus on the right box.

Proposition 4.10. *For $(d_1,\ldots,d_8)\notin\alpha$ in the support of H, we have $\ell_6\leq 1$.*

Proof. Suppose $\ell_6+\ell_2-\ell_7\leq 1$. Then if $\ell_6\geq 2$, the $\frac{a_6}{d_6}$ term gives 0 by the root of unity method, so $\ell_6\leq 1$.

Otherwise, suppose $\ell_6+\ell_2-\ell_7\geq 2$. Since $\ell_7+\ell_3\geq\ell_2+\ell_6$, we need $\ell_3\geq 2$, so by 4.9 we get $\ell_4+\ell_8\geq 2$, so $\ell_4\geq 1$ since $\ell_8\leq 1$ by 4.5.

If $\ell_7\geq 2$, we would get $\ell_3=\ell_8\leq 1$ by Proposition 4.6, so we need $\ell_7\leq 1$ and therefore $\ell_4\leq 1$ by 4.7, giving $\ell_4=\ell_8=1,\ell_3=2,\ell_7=0$. Then $\ell_6+\ell_2=2$. If ℓ_6 is not at most 1, we get $\ell_2=0,\ell_6=2$, so $(d_1,\ldots,d_8)\in\alpha$ and we are done. □

Corollary 4.11. *For $(d_1,\ldots,d_8)\notin\alpha$ in the support of H, if $\ell_5\geq 2$ then $\ell_5=\ell_2$. If $\ell_2\geq 2$ then $\ell_2=\ell_5$. In both cases, we have $\ell_1=\ell_6$.*

Proof. Suppose $\ell_5\geq 2$. Consider the a_5 term in the exponent, which is $a_5(\frac{1}{d_5}-\frac{a_2^{-1}a_5a_1d_6}{d_1d_5})$, or just $\frac{a_5}{d_5}$ if $\ell_2=0$. Then summing over a_5 yields 0 by the root of unity method unless $(\frac{1}{d_5}-\frac{a_2^{-1}a_5d_6}{d_1d_5})$ can be written as a fraction over p (in particular, this implies $\ell_2\neq 0$). So, we need $v_p\left(\frac{1}{d_5}\right)=v_p\left(\frac{a_2^{-1}a_5d_6}{d_1d_5}\right)$, or $d_1=d_6$. But then the a_1 term in the exponent is $a_1(a_2^{-1}a_5\frac{d_6}{d_1d_5}-\frac{d_6d_7d_8}{d_1d_2d_3d_4})=a_1(a_2^{-1}a_5\frac{1}{d_5}-\frac{1}{d_2})$ by Corollary 4.9, so for the summation over a_1 to not be 0, we need $d_5=d_2$.

Now suppose $\ell_2\geq 2$. By the divisibility conditions, Corollary 4.9 applies, so the $\frac{a_2d_7d_8}{d_2d_3d_4}$ term becomes $\frac{a_2}{d_2}$.

If $\ell_6 > \ell_1$, then $\ell_6 \geq 1$ so $\ell_4 + \ell_8 \geq \ell_2 + \ell_6 \geq 3$, implying that $\ell_4 = \ell_7 \geq 2$ and $\ell_3 = \ell_8 \leq 1$. We would then get that the $a_3^{-1} a_6 a_2 \frac{d_7}{d_2 d_6}$ term can be written as a fraction over p. Then the remaining a_2 terms, namely $\frac{a_2}{d_2}$ and $a_2^{-1} a_5 a_1 \frac{d_6}{d_1 d_5}$, need to have the same power of p in the denominator when simplified, as otherwise summing over a_2 yields 0 by the root of unity method. So $\frac{1}{d_2} = \frac{1}{d_2} \frac{d_2 d_6}{d_1 d_5}$, which implies $\ell_5 > \ell_2$ since $\ell_6 > \ell_1$. But then $\ell_5 \geq 3$, which by the above implies $\ell_5 = \ell_2$, which is a contradiction.

Therefore $\ell_6 \leq \ell_1$. The a_1 terms are $a_1 \frac{d_6}{d_1 d_2}$ and $a_2^{-1} a_5 a_1 \frac{d_6}{d_1 d_5}$, so for summing over a_1 to not yield 0, we need $d_5 = d_2$, in which case we are done by the $\ell_5 \geq 2$ case above. ☐

Proposition 4.12. *For $(d_1, \ldots, d_8) \notin \alpha$ in the support of H, we have $\ell_1 \leq 1$.*

Proof. Note that the a_1-dependent term in the exponent is

$$a_1 \left(\frac{d_6 d_7 d_8}{d_1 d_2 d_3 d_4} + \frac{a_2^{-1} a_5 d_6}{d_1 d_5} \right),$$

or just $\frac{a_1 d_6 d_7 d_8}{d_1 d_2 d_3 d_4}$ if $\ell_2 = 0$. Suppose $\ell_1 \geq 2$. Then by the divisibility conditions, we get $\ell_3 + \ell_7 \geq 2$, so 4.9 implies that $\ell_3 + \ell_4 = \ell_7 + \ell_8$, and we can write the a_1-dependent term as

$$a_1 \left(\frac{d_6}{d_1 d_2} + \frac{a_2^{-1} a_5 d_6}{d_1 d_5} \right).$$

Since Proposition 4.10 implies $\ell_6 \leq 1$, we cannot have $\ell_2 = \ell_5 = 0$ by divisibility conditions (since $\ell_1 = 2$). Then $\min \left(v_p \left(\frac{d_6}{d_1 d_2} \right), v_p \left(\frac{d_6}{d_1 d_5} \right) \right) \leq -2$. For H not to be 0, we must then have that $\frac{d_6}{d_1 d_2} = \frac{d_6}{d_1 d_5}$, or $d_2 = d_5$. But $d_2 = d_5 \leq 1$ is not possible by divisibility conditions, and $d_2 = d_5 \geq 2$ implies $d_1 = d_6$ by Corollary 4.11. So we must have $\ell_1 \leq 1$. ☐

We summarize our results in the following theorem.

Theorem 4.13. *For $(d_1, \ldots, d_8) \notin \alpha$ in the support of H, if H is not 0, we either have all $\ell_i \leq 1$; or $\ell_4 = \ell_7, \ell_3 = \ell_8 \leq 1$, all other $\ell_i \leq 1$; or $\ell_1 = \ell_6 \leq 1, \ell_7 = \ell_8 \leq 1, \ell_2 = \ell_5 \leq \ell_4 + 1 = \ell_7 + 1$.*

Proof. If some ℓ_i is greater than 1, then that ℓ_i must be $\ell_2, \ell_3, \ell_4, \ell_5$, or ℓ_7 by Propositions 4.10, 4.5, and 4.12. By Corollary 4.9, we cannot have $\ell_3 \geq 2$, since it would imply $\ell_8 \geq 2$ which is not possible. The Corollaries 4.11 and 4.8 in conjunction with the divisibility conditions force $(\ell_1, \ldots, \ell_8)$ to then satisfy $\ell_4 = \ell_7$ and $\ell_3 = \ell_8$, with $\ell_2, \ell_5 \leq \ell_4 + 1$, and if ℓ_2 or $\ell_5 \geq 2$, then $\ell_2 = \ell_5$ and $\ell_1 = \ell_6$.

Note that the case $\ell_2 = \ell_5 = \ell_4 + 1 = \ell_7 + 1$ is only possible if $\ell_1 = \ell_6 = 0, \ell_3 = \ell_8 = 1$ by the divisibility conditions; this case is Case 2 of 4.2.1. ☐

4.2. **Solving H.** We focus on solving the cases with $\ell_7 \geq 2$, since otherwise there are a small number of finite cases due to divisibility cases and Proposition 4.5; we call these the infinite support cases. If $\ell_3 = 0$, then calculating H is much easier, as there is no interaction between a_i and a_j for $i \in \{1, 2, 5, 6\}$, $j \in \{3, 4, 7, 8\}$ (since the only possible exponent term that contains a_i from both sets, namely $a_3^{-1} a_6 a_2 \frac{d_7}{d_2 d_6}$, is not actually in the exponent). If $\ell_3 \geq 0$, but $\ell_2 + \ell_6 \leq \ell_7$, then there is also no such interaction since $a_3^{-1} a_6 a_2 \frac{d_7}{d_2 d_6}$ is an integer. We first tackle the non-interaction case, and we then solve the other infinite support cases (the cases with interaction). Note that we use many of the results on Gauss sums from Section 2.

We formalize the non-interaction in the following theorem. We first recall the definition of the exponential sum when removing the second root in the A_3 case.

Definition 6. For e_1, e_2, e_3, e_4 powers of a prime p with $\frac{e_3 e_4}{e_2 e_1}$ and $\frac{e_2 e_4}{e_3 e_1}$ integers (the A_3 divisibility conditions), we define

$$H_{A_3}(e_1, e_2, e_3, e_4) = \sum_{\substack{c_1 \bmod e_1 e_2 e_3 \\ c_2 \bmod e_2 e_4 \\ c_3 \bmod e_3 e_4 \\ c_4 \bmod e_4}} \prod_{i=1}^{4} \left(\frac{c_i}{e_i}\right)_2 e^{2\pi i\left(\frac{c_4}{e_4} + \frac{b_4 c_2}{e_2} - \frac{b_4 c_3}{e_3} + \frac{b_3 c_1 e_4}{e_1 e_2} - \frac{b_2 c_1 e_4}{e_1 e_3}\right)}.$$

We have a mini comb reparametrization $(b_2 c_1, b_4 c_2, b_4 c_3, c_4) =: (a_1, a_2, a_3, a_4)$. Then, for M a large power of p, and $E_1 = e_1 e_2 e_3, E_2 = e_2 e_4, E_3 = e_3 e_4, E_4 = e_3$, we can write

$$H_{A_3} = \frac{E_1 E_2 E_3 E_4}{M^4} \left(\frac{p}{p-1}\right)^C \sum_{a_i \bmod M} \left(\frac{a_4}{e_4}\right)_2 \left(\frac{a_4 a_3}{e_3}\right)_2 \left(\frac{a_4 a_2}{e_2}\right)_2 \left(\frac{a_4 a_2 a_1}{e_1}\right)_2 e^{2\pi i\left(\frac{a_4}{e_4} + \frac{a_2}{e_2} - \frac{a_3}{e_3} - \frac{a_1 e_4}{e_2 e_3} + \frac{a_3^{-1} a_2 a_1 e_4}{e_1 e_2}\right)},$$

where C is the number of $i \in \{1, 3, 4\}$ with $e_i = 1$.

Theorem 4.14. *Suppose $\ell_7 \geq \ell_2 + \ell_6$ (in particular, this holds if $\ell_3 = 0$ by the divisibility conditions), and that $\ell_7 + \ell_8 = \ell_3 + \ell_4$ hold. Then we have*

$$H_{A_5}(d_1, \dots, d_8) = H_{A_3}(d_1, d_5, d_2, d_6) H_{A_3}(d_3, d_7, d_4, d_8)(d_3 d_4 d_7 d_8)^2.$$

Proof. Note that the assumptions of the theorem statement imply that there is no relevant $a_3^{-1} a_6 a_2 \frac{d_7}{d_2 d_6}$ term in H_{A_5}. Also crucially, if $\ell_2 = 0$, then the sum is not dependent on c_2, so when calculating H_{A_5} in the left hand side of the theorem statement, we can add the $a_2^{-1} a_5 a_1 \frac{d_6}{d_1 d_5}$ term to the exponent and only sum over a_2 relatively prime to p, scaling by a $\frac{p}{p-1}$ factor. We multiply the two H_{A_3} functions, using the mini comb reparametrizations

$$(b_5 c_1, b_6 c_5, b_6 c_2, c_6) =: (a_1, a_5, a_2, a_6)$$

and

$$(b_7 c_3, b_8 c_7, b_8 c_4, c_8) =: (a_3, a_7, a_4, a_8),$$

giving exactly $H_{A_5}(d_1, \dots, d_8)$ (with the above modification) except for a missing factor $(d_3 d_4 d_7 d_8)^2$. $\square$

To calculate a common H_{A_3} case, we will make use of the following proposition.

Proposition 4.15. *For $m \in \{0, 1\}$, $\ell \geq 2$, we have*

$$\sum_{x,y,z \in (\mathbb{Z}/p^\ell \mathbb{Z})} e^{2\pi i \frac{-x+y-z+x^{-1}yz}{p^\ell}} \left(\frac{x}{p^\ell}\right)_2 \left(\frac{y}{p^{\ell+m}}\right)_2 \left(\frac{z}{p^m}\right)_2 = (p-1)p^{2\ell-1} \left(\frac{-1}{p}\right)_2^m.$$

Remark. The value x^{-1} is shorthand for the inverse of x mod p^ℓ. The sum is well-defined because if p divides x, then the summand is 0 due to the $\left(\frac{x}{p^\ell}\right)_2$ term.

Proof. First suppose $m = 0$. Then we are summing $e^{2\pi i z \frac{x^{-1}y-1}{p^\ell}}$ for all z mod p^ℓ, which is 0 unless $\frac{x^{-1}y-1}{p^\ell}$ is an integer. So we can set $y = x$, which cancels out the whole exponent and quadratic residues and makes the summand simply equal 1 (assuming $p \nmid x$). There are $(p-1)p^{\ell-1}$ choices for x and p^ℓ choices for z, giving a sum $(p-1)p^{2\ell-1}$.

Now suppose $m = 1$. Then summing over z yields 0 by the root of unity method unless $x^{-1}y - 1$ is divisible by $p^{\ell-1}$. Then we can set $y = x + kxp^{\ell-1}$, and sum over $0 \le k \le p-1$. The sum becomes

$$\begin{aligned}\sum_{x,z\in(\mathbb{Z}/p^\ell\mathbb{Z}),k\in\mathbb{Z}/p\mathbb{Z}} e^{2\pi i\frac{kx+kz}{p}}\left(\frac{x}{p}\right)_2\left(\frac{z}{p}\right)_2 =&p^{2\ell-2}\sum_{k\in\mathbb{Z}/p\mathbb{Z}} j_1(k,p)^2\\ =&p^{2\ell-2}(p-1)\left(\frac{-1}{p}\right)_2 p\\ =&(p-1)p^{2\ell-1}\left(\frac{-1}{p}\right)_2.\end{aligned}$$

□

Corollary 4.16. *We have*

$$H_{A_3}(p^m,p^\ell,p^\ell,p^m) = p^{3\ell+3m-1}(-1)^m\left(\frac{-1}{p}\right)_2^m(p-1)$$

for $\ell \ge 2$ and $m \in \{0,1\}$.

Proof. Consider the mini comb reparametrization. We can factor out the a_4 part; summing over a_4 yields M for this part if $m = 0$, and $-M/p$ if $m = 1$. Then the rest of the sum, by Proposition 4.15, is $(\frac{M}{p^\ell})^3$ times $(p-1)p^{2\ell-1}\left(\frac{-1}{p}\right)_2^m$. We then get

$$\begin{aligned}H_{A_3}(p^m,p^\ell,p^\ell,p^m) =&\frac{M^4}{p^{3\ell+m}}(-1)^m\left(\frac{-1}{p}\right)_2^m(p-1)p^{2\ell-1}\frac{p^{4m+4\ell}}{M^4}\\ =&p^{3\ell+3m-1}(-1)^m\left(\frac{-1}{p}\right)_2^m(p-1).\end{aligned}$$

□

Remark. This calculation demonstrates an error in Proposition 10.3 of [BF15] (which uses Lemma 2.4 of the supplementary calculations) which implies that

$$H_{A_3}(p^0,p^\ell,p^\ell,p^0) + H_{A_3}(p^1,p^{\ell-1},p^{\ell-1},1) = 0;$$

this equality only holds for $p \equiv 1 \bmod 4$. Thus for $p \equiv 3 \bmod 4$, the infinite cases do not actually cancel, likely leading to a Dirichlet series with infinite support in both the A_3 and A_5 cases.

4.2.1. *Solving the infinite support infinite support.* **Case 1**: Suppose $2 \le \ell = \ell_7 = \ell_4, 2 \le \ell' = \ell_2 = \ell_5, \ell_1 = \ell_6 = m' \in \{0,1\}, \ell_3 = \ell_8 = m \in \{0,1\}$ with $\ell \ge \ell' + m'$.

In this case, we can factor the sum using Theorem 4.14.! Since $(d_3d_4d_7d_8)^2 = p^{4\ell+4m}$, we get

$$H_{A_5}(d_1,\ldots,d_8) = p^{7\ell+7m+3\ell'+3m'-2}(p-1)^2(-1)^{m+m'}\left(\frac{-1}{p}\right)_2^{m+m'}.$$

Case 1.5: Suppose $2 \le \ell = \ell_7 = \ell_4, \ell_3 = \ell_8 = m \in \{0,1\}$, with $\ell_1, \ell_2, \ell_5, \ell_6 \le 1$. Then we can factor out the left box as above to get a factor

$$p^{7\ell+7m-1}(p-1)(-1)^m\left(\frac{-1}{p}\right)_2^m,$$

and our sum reduces to the A_3 case in d_1, d_2, d_5, d_6.

Case 2: Suppose $3 \le \ell = \ell_2 = \ell_5 = \ell_7 + 1 = \ell_4 + 1$, with $\ell_3 = \ell_8 = 1$ and $\ell_1 = \ell_6 = 0$. We get

$$H = \frac{D_1\cdots D_8}{M^8}\sum\left(\frac{p}{p-1}\right)^C\left(\frac{a_7}{p^\ell}\right)_2\left(\frac{a_4}{p^{\ell-1}}\right)_2\left(\frac{a_5}{p^\ell}\right)_2\left(\frac{a_3}{p}\right)_2\left(\frac{a_2}{p^\ell}\right)_2$$

$$e^{2\pi i(\frac{a_8}{p}-\frac{a_4}{p^{\ell-1}}+\frac{a_7}{p^{\ell-1}}-\frac{a_3}{p^{\ell-1}}+\frac{a_6}{p}-\frac{a_2}{p^\ell}+\frac{a_5}{p^\ell}-\frac{a_1}{p^\ell}+\frac{a_4^{-1}a_7a_3}{p^{\ell-1}}+\frac{a_3^{-1}a_6a_2}{p}+\frac{a_2^{-1}a_5a_1}{p^\ell})}.$$

Summing over a_8 gives $-M/p$. Summing over a_1 gives 0 unless $a_2^{-1}a_5 - 1$ is divisible by p^ℓ, in which case it gives M. Then, we can pretend a_5 is equal to a_2, and multiply by M/p^ℓ to compensate for the lack of generality in a_5. Simplifying gives

$$H = \frac{-D_1\cdots D_8}{p^{\ell+1}M^5}\sum_{a_2,a_3,a_4,a_6,a_7}\left(\frac{p}{p-1}\right)^C\left(\frac{a_7}{p^\ell}\right)_2\left(\frac{a_4}{p^{\ell-1}}\right)_2\left(\frac{a_3}{p}\right)_2$$

$$e^{2\pi i(-\frac{a_4}{p^{\ell-1}}+\frac{a_7}{p^{\ell-1}}-\frac{a_3}{p^{\ell-1}}+\frac{a_6}{p}+\frac{a_4^{-1}a_7a_3}{p^{\ell-1}}+\frac{a_3^{-1}a_6a_2}{p})}.$$

Note that the a_2/p^ℓ and a_5/p^ℓ terms cancel. Summing over a_2 gives $-M/p$. Then summing over a_6 gives $-M/p$ again. We get

$$H = \frac{-D_1\cdots D_8}{p^{\ell+3}M^3}\sum_{a_3,a_4,a_7}\left(\frac{p}{p-1}\right)^C\left(\frac{a_7}{p^\ell}\right)_2\left(\frac{a_4}{p^{\ell-1}}\right)_2\left(\frac{a_3}{p}\right)_2$$

$$e^{2\pi i(-\frac{a_4}{p^{\ell-1}}+\frac{a_7}{p^{\ell-1}}-\frac{a_3}{p^{\ell-1}}+\frac{a_4^{-1}a_7a_3}{p^{\ell-1}})}.$$

By Proposition 4.15, summing over a_3, a_4, a_7 gives a number $(M/p^{\ell-1})^3$ times $(p-1)p^{2\ell-3}\left(\frac{-1}{p^\ell}\right)_2$. Then using $C = 1$, we have

$$H = \frac{-D_1\ldots D_8(p-1)p^{2\ell-3}}{p^{4\ell}}\left(\frac{-1}{p}\right)_2\frac{p}{p-1} = \frac{-D_1\cdots D_8}{p^{2\ell+2}}\left(\frac{-1}{p}\right)_2 = -p^{10\ell-2}\left(\frac{-1}{p}\right)_2.$$

Case 3: Suppose $2 \le \ell = \ell_7 = \ell_4 = \ell_2 = \ell_5$, with $1 = \ell_3 = \ell_8 = \ell_1 = \ell_6$.

We get

$$H = \frac{D_1 \cdots D_8}{M^8} \sum \left(\frac{p}{p-1}\right)^C \left(\frac{a_7}{p^{\ell+1}}\right)_2 \left(\frac{a_4}{p^\ell}\right)_2 \left(\frac{a_3}{p}\right)_2 \left(\frac{a_5}{p^{\ell+1}}\right)_2 \left(\frac{a_2}{p^\ell}\right)_2 \left(\frac{a_1}{p}\right)_2$$

$$e^{2\pi i(\frac{a_8}{p} - \frac{a_4}{p^\ell} + \frac{a_7}{p^\ell} - \frac{a_3}{p^\ell} + \frac{a_6}{p} - \frac{a_2}{p^\ell} + \frac{a_5}{p^\ell} - \frac{a_1}{p^\ell} + \frac{a_4^{-1}a_7a_3}{p^\ell} + \frac{a_3^{-1}a_6a_2}{p} + \frac{a_2^{-1}a_5a_1}{p^\ell})}.$$

Note the presence of the $\frac{a_3^{-1}a_6a_2}{p}$ term linking the two boxes. We proceed in a manner similar to the proof of Proposition 4.15 in the left and right boxes. Summing over a_8 yields $(-M/p)$. By looking at the $\frac{a_2^{-1}a_5a_1}{p^\ell}$ and $\frac{a_1}{p^\ell}$ terms, we see that summing over all a_1 congruent to a fixed residue mod p yields 0 unless $a_2^{-1}a_5 - 1$ is divisible by $p^{\ell-1}$, so we can set $a_5 = a_2 + k_1 a_2 p^{\ell-1}$ for $0 \le k_1 \le p-1$, and multiply by a factor (M/p^ℓ) to compensate for the loss of generality in a_5. Similarly, by looking at the $\frac{a_4^{-1}a_7a_3}{p^\ell}$ and $\frac{a_3}{p^\ell}$ terms, summing over a_3 tells us that we can set $a_7 = a_4 + k_2 a_4 p^{\ell-1}$ for $0 \le k_2 \le p-1$, and multiply by a factor M/p^ℓ.

Our sum simplifies to

$$H = \frac{-D_1 \cdots D_8}{M^5 p^{2\ell+1}} \sum_{k_1,k_2,a_1,a_2,a_3,a_4,a_6 \in (\mathbb{Z}/p\mathbb{Z})^\times} \left(\frac{p}{p-1}\right)^C \left(\frac{a_4}{p}\right)_2 \left(\frac{a_3}{p}\right)_2$$

$$\left(\frac{a_2}{p}\right)_2 \left(\frac{a_1}{p}\right)_2 \left(\frac{M}{p}\right)^5 e^{2\pi i(\frac{a_6}{p} + \frac{k_2a_4}{p} + \frac{k_2a_3}{p} + \frac{k_1a_2}{p} + \frac{k_1a_1}{p} + \frac{a_3^{-1}a_6a_2}{p})}.$$

Note that the $\left(\frac{M}{p}\right)^5$ term comes from the fact that we are now summing a_1, a_2, a_3, a_4, a_6 mod p rather than mod M.

Summing over a_1 gives $j_1(k_1, p) = \left(\frac{k_1}{p}\right)_2 j_1(1,p)$, and summing over a_4 gives $j_1(k_2,p) = \left(\frac{k_2}{p}\right)_2 j_1(1,p)$; note that these are 0 for k_1 or k_2 being 0, so we can assume that they are not divisible by p. The product of these is $\left(\frac{-k_1k_2}{p}\right)_2 p$. Then (using $C = 0$) our sum becomes

$$H = \frac{-D_1 \cdots D_8}{p^{2\ell+5}} \sum_{k_1,k_2,a_2,a_3,a_6 \in (\mathbb{Z}/p\mathbb{Z})^\times} \left(\frac{-k_1k_2a_3a_2}{p}\right)_2 e^{2\pi i(\frac{a_6}{p} + \frac{k_2a_3}{p} + \frac{k_1a_2}{p} + \frac{a_3^{-1}a_6a_2}{p})}.$$

Summing over a_6 yields $j_0(a_3^{-1}a_2 + 1, p)$, so letting $a_2 = k_3 a_3$ for $k_3 \in [1, p-1]$, we get

$$\begin{aligned}
H =& \frac{-D_1 \cdots D_8}{p^{2\ell+5}} \sum_{k_1,k_2,k_3,a_3 \in (\mathbb{Z}/p\mathbb{Z})^\times} \left(\frac{-k_1k_2k_3}{p}\right)_2 e^{2\pi i(\frac{(k_2+k_1k_3)a_3}{p})} j_0(k_3+1,p) \\
=& \frac{-D_1 \cdots D_8}{p^{2\ell+5}} \sum_{k_3,a_3 \in [1,p-1]} j_1(a_3,p) j_1(k_3a_3,p) j_0(k_3+1,p) \left(\frac{-k_3}{p}\right)_2 \\
=& \frac{-D_1 \cdots D_8}{p^{2\ell+5}} \sum_{k_3,a_3 \in [1,p-1]} p j_0(k_3+1,p) \\
=& \frac{-D_1 \cdots D_8}{p^{2\ell+4}} (p-1) \\
=& -p^{10\ell+8}(p-1),
\end{aligned}$$

using

$$\sum_{k_3\in[1,p-1]} j_0(k_3+1,p)=1.$$

Case 4 Suppose our ℓ_is fall outside of the above cases.

Then for $(\ell_1,\ldots,\ell_8)$ to lie in the support of H, we need all $\ell_i\le 1$, except for the case $(0,2,1,1,2,0,1,1)$, and the cases in α.

5. The Complete Dirichlet Series

Recall that the formula for our Whittaker coefficient for the maximal parabolic Eisenstein series, from [BF15, Theorem 4.1] is the following:

$$\mathcal{W}_{f_1,f_2,s}(1)\sum_{\substack{d_j\in\mathfrak{o}_S/\mathfrak{o}_S^\times,d_j\neq 0\\ j=1,2,\ldots,N}} H(d_1,d_2,\ldots,d_N;\mathbf{t})\delta_P^{s+1/2}(\mathfrak{D})\Psi(\mathfrak{D})\zeta_{\mathfrak{D}}c^{\psi}_{f_1,f_2}(\mathfrak{D}).$$

So far our attention has been focused on the function $H(d_1,d_2,\ldots,d_N;\mathbf{t})$. Our goal is to now understand the rest of this Dirichlet series. This will be useful in understanding the connection our series has with that in [Chi05].

Our first goal is to compute the matrix $\mathfrak{D}$. From Proposition 5.9 from [BF15], we have the formula for $\tilde{\mathfrak{D}}$ the product of a section applied to elements of the diagonal subgroup, given by

$$\tilde{\mathfrak{D}}=\mathbf{s}(h_{\gamma_1}(d_1^{-1}))\cdots\mathbf{s}(h_{\gamma_N}(d_N^{-1})), \tag{10}$$

where $\mathbf{s}:G\to\tilde{G}$ is defined by $\mathbf{s}(g)=(g,1)$,

$$h_\beta(x)=e_\beta(x)e_{-\beta}(-x^{-1})e_\beta(x)(e_\beta(1)e_{-\beta}(-1)e_\beta(1))^{-1},$$

and $\tilde{\mathfrak{D}}=(\mathfrak{D},\zeta_{\mathfrak{D}})$. Recall also that $N=8$ here.

The multiplication rule in $\tilde{G}$ is given underneath [BF15, Theorem 2.1] as

$$(g_1,\zeta_1)(g_2,\zeta_2)=(g_1g_2,\sigma_v(g_1,g_2)\zeta_1\zeta_2).$$

Using this multiplication rule, we can transform equation 10 into the equation

$$(\mathfrak{D},\zeta_{\mathfrak{D}})=\left(h_{\gamma_1}(d_1^{-1})h_{\gamma_2}(d_2^{-1})\cdots h_{\gamma_N}(d_N^{-1}),\prod_{i=2}^{N}\sigma_v\left(\prod_{j=1}^{i-1}h_{\gamma_i}(d_1^{-1}),h_{\gamma_j}(d_1^{-1})\right)\right). \tag{11}$$

Comparing the first coordinate yields us with the equation

$$\mathfrak{D}=h_{\gamma_1}(d_1^{-1})\cdots h_{\gamma_N}(d_N^{-1}).$$

To compute $\mathfrak{D}$, we will compute $h_\alpha(x)$ in general. To do this, recall from subsection 2.4 that if $\beta=e_i-e_j$ is a root, then $e_\beta(x)=I+xE_{i,j}$.

Performing the matrix multiplication, notice that the second matrix is a column operation on the first (adding $-x^{-1}$ times the jth column to the ith), meaning that our resulting matrix is the identity, but the (i,i)th entry is 0, entry (j,i) is $-x^{-1}$, and entry (i,j) is x. Similarly, the third matrix tells us to add x times the ith column to the jth column, yielding us with a matrix that is the identity, except entry $(i,i),(j,j)$ are zero, entry (j,i) is $-x^{-1}$, and entry (i,j) is x.

Now, notice that the last three matrices are similar, but with $x = 1$. We also want the inverse of this matrix; but notice that the resulting matrix is the matrix that swaps columns i and j when acting on the right, and that negates column j. Thus, its inverse negates column i before swapping columns i, j. But our final matrix is thus the diagonal matrix with 1s along the diagonal, except with an x at (i, i) and x^{-1} at (j, j).

For instance, for $h_{\gamma_1}(d_1^{-1})$, our process yields us with the matrix product

$$h_{\gamma_1}(d_1^{-1}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & d_1^{-1} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -d_1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & d_1^{-1} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right)^{-1},$$

which evaluates to

$$\begin{pmatrix} d_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_1 \end{pmatrix}.$$

Repeating this procedure yields us with

$$\mathfrak{D} = \begin{pmatrix} (d_1d_2d_3d_4)^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & (d_5d_6d_7d_8)^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & d_4d_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_3d_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_2d_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_1d_5 \end{pmatrix}.$$

Now recall that our goal is to find the Dirichlet series. Theorem 4.1 of [BF15] tells us that the Whittaker coefficient of the maximal parabolic Eisenstein series, which is what we are trying to find, is given by

$$\mathcal{W}_{f_1,f_2,s}(1) \sum_{\substack{d_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times, d_j \neq 0 \\ j=1,\ldots,N}} H(d_1, \ldots, d_N) \delta_P^{s+1/2}(\mathfrak{D}) \Psi(\mathfrak{D}) \zeta_{\mathfrak{D}} c^{\psi}_{f_1,f_2}(\mathfrak{D}).$$

We now evaluate what $\delta_P^{s+1/2}(\mathfrak{D})$ is here. We first review what the function δ_P is. For a block upper-triangular matrix with blocks A, B in our parabolic (which in this case demands $A \in GL_2(\mathbb{C}), B \in GL_4(\mathbb{C})$), δ_P is equal to $|\det(A)|^2 |\det(B)|^{-1}$. Why is this true? Therefore, $\delta_P^{s+1/2}(\mathfrak{D}) = (|d_1d_2d_3d_4d_5d_6d_7d_8|^{-2} |d_1d_5d_2d_6d_3d_7d_4d_8|^{-1})^{s+1/2} = |d_1d_2d_3d_4d_5d_6d_7d_8|^{-3s-3/2}$.

Our Dirichlet series for A_5, with our ordering of removal of the roots, is equal to

$$\mathcal{W}_{f_1,f_2,s}(1) \sum_{\substack{d_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times, d_j \neq 0 \\ j=1,2,\ldots,8}} \frac{H_{\text{removing second root from } A_5}(d_1, d_2, \ldots, d_8; t_1, \ldots, t_5)}{(d_1d_2d_3d_4d_5d_6d_7d_8)^{3s_1+3/2}} \Psi(\mathfrak{D}) \zeta_{\mathfrak{D}} c^{\psi}_{f_1,f_2}(\mathfrak{D}).$$

Note that the summand depends on the entries of $\mathfrak{D}$; we hope that the dependence is not too strong, so that we can get some cancellation.

We now compute what $c^{\psi}_{f_1,f_2}(\mathfrak{D})$ is. As defined in Section 4 of [BF15], $c^{\psi}_{f_1,f_2}(\mathfrak{D})$ comes from the computation of removing roots in the other two blocks, meaning that we just need what this coefficient would be in the example of A_1 and A_3.

In particular, we can write

$$c^{\psi}_{f_1,f_2}(\mathfrak{D}) = \mathcal{W}_{f_3,f_4,s_2}(1) \sum_{\substack{d_j \in \mathfrak{o}_S/\mathfrak{o}_S^{\times}, d_j \neq 0 \\ j=1,\ldots,N}} H(d_1,\ldots,d'_N;\mathbf{t}')\delta_P^{s_2+1/2}(\mathfrak{D}')\Psi(\mathfrak{D}')\zeta_{\mathfrak{D}'}c^{\psi'}_{f_3,f_4}(\mathfrak{D}').$$

However, in this inductive piece, we are computing the coefficient at a different character. In our case, this corresponds to having a different $\mathbf{t}'$ in the exponential sum.

In particular, from Section 4 [BF15], the $c^{\psi}_{f_1,f_2}(\mathfrak{D})$ is equal to the Whittaker function evaluated at the character associated to the one sending u_M to $\psi(u_M^{(\mathfrak{D}^{w_M})^{-1}})$, where we have the exponent given as conjugation.

But

$$u_M^{(\mathfrak{D}^{w_M})^{-1}} = (\mathfrak{D}^{w_M})^{-1}u_M(\mathfrak{D}^{w_M}) = w_M^{-1}\mathfrak{D}^{-1}w_M u_M w_M^{-1}\mathfrak{D}w_M.$$

In this particular case, we have that u_M, the unipotent component of u corresponding to the Levi subgroup $M = GL_2(\mathbb{C}) \times GL_4(\mathbb{C})$. But

$$u_M = \begin{pmatrix} 1 & x_{12} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x_{34} & x_{35} & x_{36} \\ 0 & 0 & 0 & 1 & x_{45} & x_{46} \\ 0 & 0 & 0 & 0 & 1 & x_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this formula, when w_M is written, what is meant is the matrix corresponding to the permutation w_M, namely

$$w_M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We can use Sage, as well as the value of $\mathfrak{D}$ that we computed above, to yield us with the product

$$\begin{pmatrix} 1 & \frac{d_5d_6d_7d_8x_{12}}{d_1d_2d_3d_4} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{d_2d_6x_{34}}{d_1d_5} & \frac{d_3d_7x_{35}}{d_1d_5} & \frac{d_4d_8x_{36}}{d_1d_5} \\ 0 & 0 & 0 & 1 & \frac{d_3d_7x_{45}}{d_2d_6} & \frac{d_4d_8x_{46}}{d_2d_6} \\ 0 & 0 & 0 & 0 & 1 & \frac{d_4d_8x_{56}}{d_3d_7} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

To find the new $\mathbf{t}$ vector, we look at the off-diagonal elements. The reason is that we care about a parametrization of upper-triangular matrices mod their commutator. Thus, when we evaluate the H coefficient on M, we evaluate with the new vector $\mathbf{t}' = (\frac{d_5d_6d_7d_8t_1}{d_1d_2d_3d_4}, \frac{d_2d_6t_3}{d_1d_5}, \frac{d_3d_7t_4}{d_2d_6}, \frac{d_4d_8t_5}{d_3d_7})$.

Notice that $\mathbf{t}'$ also consists of integers given that $\mathbf{t}$ consists of integers due to the divisibility conditions given Lemma 6.1 in [BF15].

Now, at this stage, we compute what the Whittaker function looks like for $A_1 \times A_3$, since we've removed the second simple root from A_5 already To do this, first recall that from Section 7 of [BF15], which we did in subsection 2.5, we have that the Whittaker coefficient for A_3 is equal to

$$\mathcal{W}_{f_1,f_2,s}(1) \sum_{\substack{d_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times, d_j \neq 0 \\ j=1,\ldots,N}} H(d_1,\ldots,d_4)|d_1d_2d_3d_4|^{-(1+2s)}\Psi(\mathfrak{D})\zeta_{\mathfrak{D}} c^{\psi}_{f_1,f_2}(\mathfrak{D}),$$

where H is the series given by

$$\sum_{c_1,c_2,c_3,c_4} \prod_{k=1}^{4} \left(\frac{c_k}{d_k}\right) \psi\left(t_1(\frac{b_2c_1d_4}{d_1d_3} + \frac{b_4c_3}{d_3}) + t_2\frac{c_4}{d_4} - t_3(\frac{c_1b_3d_4}{d_1d_2} + \frac{c_2b_4}{d_2})\right).$$

Now, for the combined $M = GL_2 \times GL_4$, our next parabolic comes from removing the fourth root. Notice here that w_M can be taken to be $s_1s_3s_5$, and $w^P = s_4s_3s_5s_4$, yielding us with the ordering of the roots being $(\alpha_3+\alpha_4+\alpha_5, \alpha_4+\alpha_5, \alpha_3+\alpha_4, \alpha_4)$, and so we end up with the evaluation of the coefficients as being

$$c^{\psi}_{f_1,f_2}(\mathfrak{D}) = \mathcal{W}_{f_3,f_4,s}(1) \sum_{\substack{d_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times, d_j \neq 0 \\ j=1,\ldots,N}} H(d_9,\ldots,d_{12}; t_1', t_2', t_3', t_4')\delta_P^{s+1/2}(\mathfrak{D}')\Psi(\mathfrak{D}')\zeta'_{\mathfrak{D}} c^{\psi}_{f_3,f_4}(\mathfrak{D}').$$

Notice, however, that the formula for $\mathfrak{D}$ is the same as that in the GL_4 case, except where the positive roots that we are enumerating over are shifted by 2. Therefore, we see that $\delta_P^{s+1/2}(\mathfrak{D}') = |d_9d_{10}d_{11}d_{12}|^{-(2s+1)}$. As for H, we see that, similarly, the formulas all depend simply on the enumeration of the positive roots, again with the same shifting of indices. But our formula for H is thus

$$H_{\text{removing third root from } A_1\times A_3}(d_9,\ldots,d_{12}; t_1',t_2',t_3',t_4') = H_{\text{removing second root from } A_3}(d_9,\ldots,d_{12}; t_2',t_3',t_4')$$

$$= \sum_{\substack{c_9 \pmod{d_9d_{10}d_{11}} \\ c_{10} \pmod{d_{10}d_{12}} \\ c_{11} \pmod{d_{11}d_{12}} \\ c_{12} \pmod{d_{12}}}} \prod_{k=9}^{12} \left(\frac{c_k}{d_k}\right) \psi\left(t_2'(\frac{b_{10}c_9d_{12}}{d_9d_{11}} + \frac{b_{12}c_{11}}{d_{11}}) + t_3'\frac{c_{12}}{d_{12}} - t_4'(\frac{c_9b_{11}d_{12}}{d_9d_{10}} + \frac{c_{10}b_{12}}{d_{10}})\right).$$

Notice that the coefficient of t_1' is zero because there are no positive roots in our enumeration whose difference is α_1.

Substituting in the value of t_i' we previously computed yields us with

$$c^{\psi}_{f_1,f_2}(\mathfrak{D}) = \mathcal{W}_{f_3,f_4,s}(1) \sum_{\substack{d_j \in \mathfrak{o}_S/\mathfrak{o}_S^\times, d_j \neq 0 \\ j=1,\ldots,N}} H_{\text{removing second root from } A_3}(d_9,\ldots,d_{12}; \frac{d_2d_6t_3}{d_1d_5}, \frac{d_3d_7t_4}{d_2d_6}, \frac{d_4d_8t_5}{d_3d_7})$$

$$\cdot |d_9d_{10}d_{11}d_{12}|^{-(2s_4+1)}\Psi(\mathfrak{D}')\zeta'_{\mathfrak{D}} c^{\psi}_{f_3,f_4}(\mathfrak{D}').$$

Here, we have that $c^{\psi}_{f_3,f_4}(\mathfrak{D}')$ is the coefficient when we treat the subgroup $GL_2\times GL_2\times GL_2$. At this point, however, using a similar logic to the above, removing any of the other three roots yields us with three products of Gauss sums.

Repeating the procedure above, and noting now that the three remaining roots that we have to remove have corresponding reflections that are orthogonal, we see that $c^{\psi}_{f_3,f_4}(\mathfrak{D})$ is proportional to $\frac{g(t''_1,d_{13})g(t''_2,d_{14})g(t''_3,d_{15})}{d_{13}^{s_3+1/2}d_{14}^{s_4+1/2}d_{15}^{s_5+1/2}}$. To compute what t''_1,t''_2,t''_3, we repeat the procedure that we've done above. Notice that $\mathfrak{D}$ with GL_4 is equal to

$$\tilde{\mathfrak{D}}' = \mathbf{s}(h_{\gamma_1}(d_9^{-1}))\cdots\mathbf{s}(h_{\gamma_4}(d_{12}^{-1})),$$

which using a similar procedure to the above yields

$$\mathfrak{D}' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_9^{-1}d_{11}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{10}^{-1}d_{12}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{11}d_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_9d_{10} \end{pmatrix}.$$

Now, to find the new character, we again compute

$$w_M^{-1}\mathfrak{D}'^{-1}w_Mu_Mw_M^{-1}\mathfrak{D}'w_M,$$

this time with

$$w_M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, u_M = \begin{pmatrix} 1 & x_{12} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & x_{34} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Again, computations with Sage allow us to see that

$$w_M^{-1}\mathfrak{D}'^{-1}w_Mu_Mw_M^{-1}\mathfrak{D}'w_M = \begin{pmatrix} 1 & x_{12} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{d_{10}d_{12}x_{34}}{d_9d_{11}} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{d_{11}d_{12}x_{56}}{d_9d_{10}} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence, (t''_1,t''_2,t''_3) corresponds to the off-diagonal entries $(1,2),(3,4)$ and $(5,6)$ in the matrix, since we are removing the root corresponding to t'_3. In other words, our inputs for the $\mathbf{t}-$vector are now $(\frac{d_5d_6d_7d_8t_1}{d_1d_2d_3d_4}, \frac{d_2d_6d_{10}d_{12}t_3}{d_1d_5d_9d_{11}}, \frac{d_4d_8d_{11}d_{12}t_5}{d_3d_7d_9d_{10}})$, so our final $c-$value that we need to compute has

$$c^{\psi}_{f_3,f_4}(\mathfrak{D}') = \frac{g(\frac{d_5d_6d_7d_8t_1}{d_1d_2d_3d_4},d_{13})g(\frac{d_2d_6d_{10}d_{12}t_3}{d_1d_5d_9d_{11}},d_{14})g(\frac{d_4d_8d_{11}d_{12}t_5}{d_3d_7d_9d_{10}},d_{15})}{d_{13}^{s_1+1/2}d_{14}^{s_3+1/2}d_{15}^{s_5+1/2}}\zeta_{d_{13}}\zeta_{d_{14}}\zeta_{d_{15}}\Psi(d_{13})\Psi(d_{14})\Psi(d_{15}).$$

Now, we need to compute the value of the ζ that appear in the coefficients. We recall the formula for $\zeta_{\mathfrak{D}}$ that we found way back in equation 11, by comparing the second coordinates this time. This yields us with the equation

$$\zeta_{\mathfrak{D}} = \prod_{i=2}^{N} \sigma_v \left(\prod_{j=1}^{i-1} h_{\gamma_i}(d_1^{-1}), h_{\gamma_j}(d_1^{-1}) \right).$$

We now evaluate this for each $\mathfrak{D}$ coming out of each root removal step.

Notice that for $\zeta_{d_{13}}, \zeta_{d_{14}}, \zeta_{d_{15}}$ we see that these will all be 1. To see this, after removing the first two roots we only have the positive roots $\alpha_1, \alpha_3, \alpha_5$; this means that removing each of the last three roots yields that our corresponding $\mathfrak{D}$ is just a matrix of the form $h_\gamma(d_i)$; but as $\tilde{\mathfrak{D}} = (h_\gamma(d_i), 1)$, we have ζ_{d_i} is 1 for those three values of i. From [BF15], we have that $\zeta_{\mathfrak{D}'}$ is equal to $(d_{10}, d_9)_S (d_{10}d_{12}, d_{11})_S$. We just need to compute the first step, when we were removing $\mathfrak{D}$. To do this, we revisit the multiplication rule from [BF15], Section 1.

In order to compute $\prod_{i=2}^{N} \sigma_v \left(\prod_{j=1}^{i-1} h_{\gamma_i}(d_1^{-1}), h_{\gamma_j}(d_1^{-1}) \right)$, we work one at a time. First, we begin with the matrices that we listed before. We start with $h_{\gamma_1}(d_1^{-1})$ and $h_{\gamma_1}(d_2^{-1})$. Here, we are considering the product of

$$h_{\gamma_1}(d_1^{-1}) = \begin{pmatrix} d_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_1 \end{pmatrix}$$

and

$$h_{\gamma_2}(d_2^{-1}) = \begin{pmatrix} d_2^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

per the formula given in [BF15], we get

$$\zeta_{h_{\gamma_1}(d_1^{-1})h_{\gamma_2}(d_2^{-1})} = (d_1^{-1}, d_2^{-1})_S (d_1^{-1}, 1)_S^4 (d_1^{-1}, d_2)_S (1,1)_S^{10} (1, d_2)_S^4 (d_1, 1)_S,$$

where the symbol $(a,b)_S$ denotes the Hilbert symbol.

From here, we use properties of this Hilbert symbol, such as in [Ser93]. For instance, Proposition 2 from [Ser93], part (i) tells us that anything with a 1 yields us with a 1, as $1^2 = 1$, giving the simplification

$$\zeta_{h_{\gamma_1}(d_1^{-1})h_{\gamma_2}(d_2^{-1})} = (d_1^{-1}, d_2^{-1})_S (d_1^{-1}, d_2)_S.$$

Parts (v) and (i) then tell us that $\sigma_v(h_{\gamma_1}(d_1^{-1}), h_{\gamma_2}(d_2^{-1})) = 1$. We next evaluate the product

$$(h_{\gamma_1}(d_1^{-1})h_{\gamma_2}(d_2^{-1})h_{\gamma_3}(d_3^{-1}), \sigma_v(h_{\gamma_1}(d_1^{-1}), h_{\gamma_2}(d_2^{-1}))\sigma_v(h_{\gamma_1}(d_1^{-1})h_{\gamma_2}(d_2^{-1}), h_{\gamma_1}(d_3^{-1})))$$

For the other products, we see that continuing our process of multiplication, which picks up these cocycle coefficients σ_v between our partial product and the next term, yields us with

$$\zeta_{\mathfrak{D}} = (d_4d_3d_2d_1, d_5)_S(d_4d_3d_2, d_6)_S(d_4d_3, d_7)_S(d_4, d_8)_S.$$

6. Littelman inequalities and the crystal basis

It turns out that yet another way of determining the Hs' in the Whittaker coefficients as Dirichlet series in several complex variables, is to attach a product of Gauss sums to each vertex in a crystal graph. These Gauss sums depend on some quantities called"string data" as mentioned in Littelmann [Lit98].

Given a specific factorization of the long Weyl group element into simple reflections, these data are the lengths of segments in a path from the given vertex to the vertex of lowest weight. In this section, we will borrow Littelmann's formulation of the adapted strings to understand the vertices of the polytope. We suspect that there is a correspondence between the support of the exponential sums in the Whittaker functions and the polytope obtained by the inequalities that define a rational polytope C_w^λ in [Lit98]. To this end, we explicitly computed the polytope using the following definition from Littelmann.

Given a dominant weight λ the bounds on the 15-dimensional rational polytope C_w^λ is defined by $a_p \leq \langle \lambda, \alpha_{i_p} \rangle$, $a_{p-1} \leq \langle \lambda - a_p\alpha_{i_p}, \alpha_{i_{p-1}} \rangle$,..., $a_1 \leq \langle \lambda - a_p\alpha_{i_p} - ... - a_2\alpha_{i_2}, \alpha_{i_1} \rangle$. For $p = 15$, the inequalities are inductively computed as follows:

$$\begin{aligned}
a_{15} &\leq \lambda_4 - \lambda_5 \\
a_{14} &\leq \lambda_3 - \lambda_4 + a_{15} \\
a_{13} &\leq \lambda_2 - \lambda_3 + a_{14} \\
a_{12} &\leq \lambda_1 - \lambda_2 + a_{13} \\
a_{11} &\leq \lambda_5 - \lambda_6 + a_{15} \\
a_{10} &\leq \lambda_4 - \lambda_5 - 2a_{15} + a_{14} + a_{11} \\
a_9 &\leq \lambda_3 - \lambda_4 - 2a_{14} + a_{15} + a_{13} + a_{10} \\
a_8 &\leq \lambda_2 - \lambda_3 - 2a_{13} + a_{14} + a_{12} + a_9 \\
a_7 &\leq \lambda_4 - \lambda_5 - 2a_{15} - 2a_{10} + a_{14} + a_{11} + a_9 \\
a_6 &\leq \lambda_3 - \lambda_4 - 2a_9 - 2a_{14} + a_{15} + a_{13} + a_{10} + a_8 + a_7 \\
a_5 &\leq \lambda_5 - \lambda_6 - 2a_{11} + a_{15} + a_{10} + a_7 \\
a_4 &\leq \lambda_4 - \lambda_5 - 2a_{15} - 2a_{10} - 2a_7 + a_{14} + a_{11} + a_9 + a_6 + a_5 \\
a_3 &\leq \lambda_5 - \lambda_6 - 2a_{11} - 2a_5 + a_{15} + a_{10} + a_7 + a_4 \\
a_2 &\leq \lambda_3 - \lambda_4 - 2a_{14} - 2a_9 - 2a_6 + a_{15} + a_{13} + a_{10} + a_8 + a_7 + a_4 \\
a_1 &\leq \lambda_1 - \lambda_2 - 2a_{12} + a_{13} + a_8
\end{aligned}$$

Using Sage, we computed the polytope to have 12,624 exterior vertices. Connections between the support of the H-functions and the vertices of this polytope still need to be established. We expect that the H-function is supported on a subset of all vertex points (both interior and exterior) of the polytope.

7. Sage Computations

I did some computations for $p = 3$.

$(\ell_1, \ldots, \ell_8)$	value
$(0,0,0,1,0,0,1,0)$	$2 \cdot 3^6$
$(0,1,0,1,1,0,1,0)$	$2^2 \cdot 3^8$
$(0,0,0,1,0,0,1,0)$	$2 \cdot 3^6$
$(0,0,0,2,0,0,2,0)$	$2 \cdot 3^{13}$
$(0,0,0,0,0,0,0,1)$	$3^4 i\sqrt{3}$
$(0,0,0,0,0,0,0,2)$	0
$(0,0,1,1,1,1,0,0)$	0

8. Future Directions

For one future direction, we would like to figure out a method for a change of variables so that we can compare the Whittaker coefficient with the Chinta polynomial. To develop this method, it might help us if we could understand the 15 zeta functions which got pulled out from the Chinta series when the denominator is multiplied by $(1+x)(1+y)(1+z)(1+w)(1+v)$, and how they coincide with the normalizing zeta factor of the Eisenstein series. This is suggested in the paper of Chinta [Chi05].

In addition, there exists another description of the same polynomial through "string data" defined in Littelmann [Lit98]. However, we have yet to find a connection between the support of the exponential sums and the Littelmann's inequalities. As such, another direction we could take would be to figure out how Littelmann's inequalities relate to our exponential sum H.

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