Puzzles, Ice, and Grothendieck Polynomials

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Outline

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Puzzles, Ice, and Grothendieck Polynomials

Chin and Davis

Introduction
Lattice Models
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Future Work

Partitions

**Definition**

A *partition* $\lambda$ is a string of weakly decreasing nonnegative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

**Definition**

A *skew partition* $\lambda/\mu$ is a set of two partitions $\lambda, \mu$ such that $\forall i, \lambda_i \geq \mu_i$.

**Example**

A skew partition diagram of shape $(4, 2)/(1, 0)$
Tableaux

**Definition**

A **semistandard tableau** of shape $\lambda/\mu$ is a filling of the Young/Ferrers diagram from $[n] = 1, \ldots, n$, with weakly increasing rows and strictly increasing columns.

**Definition**

Two **valued set tableaux** of shape $(4, 2)/(1, 0)$:

```
1 1 2
1 2
```

```
1 2
1 2
```
\[ j_{\lambda/\mu}(z, \alpha) = \sum_{T \in VST_{\lambda/\mu}} \alpha^{\lambda/\mu - |T|} z^{\text{wt}(T)} \]

**Example**

\[ \alpha^{\lambda/\mu - |T|} z^{\text{wt}(T)} \] for two valued set tableaux $T$

\[
\begin{array}{ccc}
1 & 2 & z_1^2 z_2^2 \\
1 & 2
\end{array}
\]

\[
\begin{array}{ccc}
1 & 2 & \alpha z_1^2 z_2^2 \\
1 & 2
\end{array}
\]
What is a Lattice Model?

- Model particle interactions within thin sheets of matter.
- Model classes of polynomials.
- We want to model Grothendieck polynomials to demonstrate certain polynomial identities.
  - Cauchy identities with families of dual polynomials
  - Littlewood-Richardson rule
  - Pieri/branching rules
What is a Lattice Model?

- Boundary conditions are fixed by skew partition $\lambda/\mu$.
- Vertices have a choice of weights.
- Edges are labeled with arrows/orientations, ICE.
- The **state** is a choice of orientation for each edge.
- The **system** is the set of all states.
Definition

The partition function of a lattice model over partition $\lambda/\mu$ is

$$Z(\mathcal{G}_{\lambda/\mu}(z)) = \sum_{S \in \mathcal{G}_{\lambda/\mu}} wt(S)$$

where $wt(S)$ is the product of weights of each vertex in the lattice model.
Partition Functions

Example

\[ \lambda = (3), \mu = (1) \]
A Model for $j_{\lambda/\mu}$

**Definition (Our Boltzmann Weights)**

1  \[\alpha + z_i\] 1  \[z_i\] 1  0

**Theorem (Bullock-Caplinger-C.-D.-Shemy)**

*Under our choice of Boltzmann weights, for skew partition $\lambda/\mu$,*

\[j_{\lambda/\mu}(z, \alpha) = Z(S_{\lambda/\mu}(z)) .\]
Finding a Compatible Model for $G_\lambda$

**Definition**

The stable symmetric Grothendieck polynomial for $\lambda$

$$G_\lambda(z) = \sum_{T \in SVT_\lambda} z^{wt(T)}$$

**Proposition (BCCDS)**

There are no top-bottom lattice models for $G_\lambda$ satisfying the following conditions:

- **Horizontal lattice lines are in direct correspondence with variables** $z_1, \ldots, z_n$.
- **ICE holds, with a 5-vertex model.**
- **There is a bijection between SSYTs and states in the lattice model.**
Schur Polynomials

Definition

The Schur polynomials can be defined as,

\[ s_\lambda(z) = \sum_{T \in \text{SSYT}(\lambda)} z^{\text{wt}(T)}. \]

These give a vector space basis for the symmetric polynomials in \( z_1, z_2, ..., z_n \).
Littlewood-Richardson Rule

There exists some unique expansion,

\[ s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu. \]

Theorem (Knutson, Tao, Woodward)

The Littlewood-Richardson coefficients, \( c_{\lambda\mu}^\nu \), count puzzle tilings with boundaries determined by \( \lambda \), \( \mu \), and \( \nu \).
Definition

A puzzle of size $n$ is a filling of an equilateral triangle of side length $n$ with KTW tiles such that adjacent edge labels match.

Example

For $\lambda = (2, 1, 0)$, $\mu = (3, 2, 0)$, and $\nu = (4, 3, 1)$ binary string of $\nu$ and puzzle tiling with boundary $\Delta_{\lambda\mu}^\nu$. 
The Connection

Theorem (KTW ’04)

For \( \lambda, \mu, \nu \) that fit in a \( k \times (n - k) \) ambient rectangle and \( |\nu| = |\lambda| + |\mu| \), the number of possible tilings of a puzzle with fixed boundary \( \Delta^\nu_{\lambda\mu} \) is \( c^\nu_{\lambda\mu} \).

Example

For \( \lambda = (2, 1, 0), \mu = (3, 2, 0), \nu = (4, 3, 1) \), we get \( c^\nu_{\lambda\mu} = 2 \).
Green Hexagons and $j_{\lambda\mu}^\nu$.

**Theorem (Pylyvaskyy, Yang ’18)**

For $\lambda, \mu, \nu$ that fit in a $k \times (n - k - 1)$ ambient rectangle and $|\nu| \leq |\lambda| + |\mu|$, the number of green hexagon tilings with boundary $\Delta_{\lambda\mu}^\nu$ is $d_{\lambda\mu}^\nu$.

**Littlewood-Richard Rule, $j_{\lambda\mu}^\nu$ specific**

There exists some unique expansion

$$j_{\lambda} \cdot j_{\mu} = \sum_{\nu} (-1)^{|\nu| - |\lambda| - |\mu|} d_{\lambda\mu}^\nu j_{\nu}.$$
Path Model

Path Tiles inspired by Zinn-Justin

We set $+$ = 1 and $-$ = 0. These correspond to KTW tiles.

This additional Z-J tile is needed to draw paths through green hexagons.
Puzzle to Lattice Model

Example

For $\lambda = (2, 0)$, $\mu = (2, 1)$, and $\nu = (2, 2)$.
Future Work

- Given a $\lambda$, $\mu$, $\nu$ we want a choice of Boltzmann weights which gives us the corresponding $d_{\lambda\mu}^{\nu}$.
- Attach puzzle lattice model to our $j_{\lambda\mu}^{\nu}$ lattice model so that it satisfies the Yang-Baxter equation.