Immanants and Total Positivity

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Immanants

Definition

For a function $f : S_n \to \mathbb{R}$, $\mathrm{Imm}_f = \sum_{w \in S_n} f(w) x_{1,w(1)} x_{2,w(2)} \cdots x_{n,w(n)}$.

Example

If $f(w) = \mathrm{sgn}(w)$, then Imm_f is just the determinant of the $n \times n$ matrix (x_{ij}). If $f(w) = 1$, then Imm_f is just the permanant.

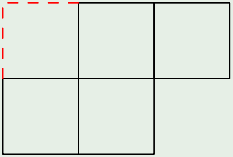
Skew Tableaux

Definition

For two tableaux $\mu \subset \lambda$, the skew tableau λ/μ consists of all boxes in λ but not in μ. (Here, we align λ, μ to share the same upper left corner.)

Example

The skew tableau $(3,2)/(1)$ is



%-Immanants

Definition

For a skew tableau λ/μ, $\mathsf{Imm}^{\%}_{\lambda/\mu} = \sum_{\sigma \in A} \mathsf{sgn}(\sigma) x_{1,\sigma(1)} x_{2,\sigma(2)} \cdots x_{n,\sigma(n)}$, where $\sigma \in A$ iff. $\forall i$, $(i, \sigma(i)) \in \lambda/\mu$.

Example

$$\mathrm{Imm}^{\%}_{(4,4,3,3)/(2,1,1)} = \begin{vmatrix} 0 & 0 & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & x_{32} & x_{33} & 0 \\ x_{41} & x_{42} & x_{43} & 0 \end{vmatrix}$$

%-immanant generated by a permutation

Any permutation w generates a %-immanant, by tracing out the skew tableau marked out by all $(i, w(i))$.

Example

$\mathrm{Imm}^{\%}_{2413} = \begin{vmatrix} 0 & * & * & * \\ 0 & * & * & * \\ * & * & * & 0 \\ * & * & * & 0 \end{vmatrix}$ is generated by 2413: $\begin{vmatrix} 0 & x & * & * \\ 0 & * & * & x \\ x & * & * & 0 \\ * & * & x & 0 \end{vmatrix}$

Complementary Minors

We will define Temperley-Lieb immanants as a basis for $\mathcal{V}$, the vector space spanned by all products of complementary minors.

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Definition

For an $n \times n$ matrix (x_{ij}) and subsets $I, J \subset \{1, 2, \cdots, n\}$ with $|I| = |J|$, define $\Delta_{I,J}$ as the determinant of the minor with rows indexed by I and columns indexed by J.

A product of complementary minors is a product $\Delta_{I,J}\Delta_{\bar{I},\bar{J}}$ for some I, J.

Example

If $I = \{1\}, J = \{3\}$, then $\Delta_{I,J}\Delta_{\overline{I},\overline{J}} = x_{13} \begin{vmatrix} x_{21} & x_{22} \\ x_{31} & x_{32} \end{vmatrix} = \begin{vmatrix} 0 & 0 & x_{13} \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & 0 \end{vmatrix}$.

321-avoiding permutations and Non-crossing matchings

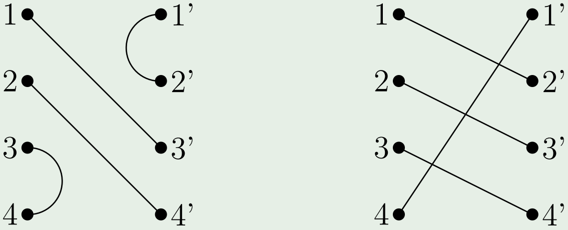
Our TL immanants will be indexed by 321-avoiding permutations in S_n.

Proposition

321-avoiding permutations of $\{1, 2, \cdots, n\}$ are in bijection with non-crossing matchings of $2n$ vertices (and there are C_n of them).

Example

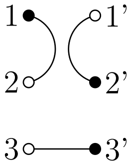
The non-crossing matching corresponding to the 321-avoiding permutation 2341.



Compatible Matching

Definition

A black or white coloring of vertices $1, 2, ..., n, 1', 2', ..., n'$ is *compatible* with a non-crossing matching if every black vertex is matched with a white vertex.



TL-immanants

Definition

A black or white coloring of vertices $1, 2, ..., n, 1', 2', ..., n'$ is *compatible* with a non-crossing matching if every black vertex is matched with a white vertex.

We now define a basis for $\mathcal{V}$ called Imm_w, where w ranges over non-crossing matchings (equivalently over 321-avoiding permutations).

Theorem-Definition (Rhoades-Skandera)

Let $I, J \subseteq [n]$. Color I black, $\overline{I}$ white on the left and J white, $\overline{J}$ black on the right, then

$$\Delta_{I,J}\Delta_{\bar{I},\bar{J}} = \sum_{w \text{ compatible with coloring}} \mathrm{Imm}_w$$

TL-immanants

Example

For $I = \{1\}$ and $J = \{1\}$, we have:

$$x_{11} \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} = \Delta_{1,1}\Delta_{23,23}$$

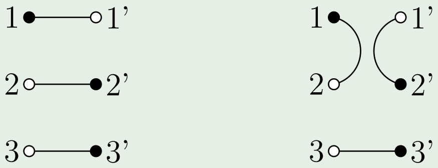
$$\begin{array}{ccc} 1\bullet & & \circ 1' \\ 2\circ & & \bullet 2' \\ 3\circ & & \bullet 3' \end{array}$$

TL-immanants

Example

For $I = \{1\}$ and $J = \{1\}$, we have:

$$x_{11} \begin{vmatrix} x_{22} & x_{23} \\ x_{32} & x_{33} \end{vmatrix} = \Delta_{1,1}\Delta_{23,23} = \mathrm{Imm}_{123} + \mathrm{Imm}_{213}$$



A curious observation

Example

We can compute

$$\mathrm{Imm}_{123} = \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}, \qquad \mathrm{Imm}_{213} = -\begin{vmatrix} 0 & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix}.$$

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- When is a TL-immanant equal to $\pm$ %-immanant?

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- When is a TL-immanant equal to $\pm$ %-immanant?
- When is Imm_w equal to the sum of two $\pm$ %-immanants?

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Question

- When is a TL-immanant equal to $\pm$ %-immanant?
- When is Imm_w equal to the sum of two $\pm$ %-immanants?
- Can we compute Imm_w?

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Theorem (Chepuri-Sherman-Bennett $\Leftarrow$, LRSW $\Rightarrow$)

Let w be a 321-avoiding permutation. Then Imm_w is a %-immanant up to sign if and only if w avoids both 1324 and 2143. In that case, $\mathrm{Imm}_w = \mathrm{sgn}(w)\,\mathrm{Imm}_w^{\%}$.

Example

$$\mathrm{Imm}_{41523} = \begin{vmatrix} 0 & 0 & 0 & x_{14} & x_{15} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{43} & 0 & 0 \\ x_{51} & x_{52} & x_{53} & 0 & 0 \end{vmatrix}$$

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Now what about two %-immanants?

Theorem (LRSW)

If a permutation w avoids the patterns 321, 1324, 24153, 31524, 231564, and 312645, then w can be written as the sum of at most two % immanants.

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Proof Sketch.

- Characterize the structure of w.

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Proof Sketch.

- Characterize the structure of w.
- Compute the immanant coefficients $f_w(u)$, for $u \in S_n$.
- Guess a candidate sum of two %-immanants, and show it works by comparing $f_w(u)$ to the coefficient of $x_{1u(1)}x_{2u(2)} \cdots x_{nu(n)}$ in $\%_1 + \%_2$. □

Structure of w

Proposition

Suppose that w is a permutation that avoids 1324 and 321, but not 2143. Then one of the following holds:

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Suppose that w *is a permutation that avoids* 1324 *and* 321, *but not* 2143. *Then one of the following holds:*

- w*'s one line notation has at most five ascending strings of consecutive integers in* $[2][1][3][5][4]$ *block form, e.g.* $[345][12][67][9][8]$

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Proposition

Suppose that w is a permutation that avoids 1324 and 321, but not 2143. Then one of the following holds:

- *w's one line notation has at most five ascending strings of consecutive integers in $[2][1][3][5][4]$ block form, e.g. $[345][12][67][9][8]$*
- *w consists of at most six ascending strings of consecutive integers in $[3][5][1][6][2][4]$ block form, e.g. $[34][78][1][9][2][56]$.*

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Suppose that w is a permutation that avoids 1324 and 321, but not 2143. Then one of the following holds:

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- *w consists of at most six ascending strings of consecutive integers in $[3][5][1][6][2][4]$ block form, e.g. $[34][78][1][9][2][56]$.*

Furthermore, if w avoids 24153 and 31524, then the second case can't hold, and we must be in the first case.

Computing the Immanant Coefficients

For a given w, we explicitly construct a set A of pairs $I, J \subset \{1, 2, \cdots, n\}$ such that for some function f,

$$\mathrm{Imm}_w = \sum_{(I,J)\in A} \Delta_{I,J}\Delta_{\overline{I},\overline{J}}(-1)^{f(I,J)}.$$

Extracting the coefficient of $x_{1u(1)}x_{2u(2)}\cdots x_{nu(n)}$ from both sides, we get:

Theorem (LRSW)

Define the intervals $I_1 = [1, w^{-1}(1) - 1], I_2 = [w^{-1}(n) + 1, n], J_1 = [1, w(1) - 1], J_2 = [w(n) + 1, n]$.
Let
$A = |u(I_1) \cap J_2|, B = |u(I_2) \cap J_1|, C = |u(I_1) \cap J_1)|, D = |u(I_2) \cap J_2)|$.
Then

$$f_w(u) = \begin{cases} \mathrm{sgn}(w)\,\mathrm{sgn}(u)\binom{A+B}{A}, & C = D = 0; \\ 0, & \textit{otherwise}. \end{cases}$$

Example

$w = 2136745$. Suppose we want to find $f_w(u)$. First, create the %-immanant cut out by w. Then, label the upper-right corner a's and the lower-left corner b's.

$$\begin{vmatrix} 0 & * & * & * & * & a & a \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ b & * & * & * & * & 0 & 0 \\ b & * & * & * & * & 0 & 0 \end{vmatrix}$$

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Rule: mark all $(i, u(i))$ red. If a zero is red, then $f_w(u) = 0$. Otherwise, let A be the number of a's marked, and B be the number of b's marked. Then $|f_w(u)| = \binom{A+B}{A}$.

Example

$w = 2136745, u = 3524716$

$$\begin{vmatrix} 0 & * & * & * & * & a & a \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ b & * & * & * & * & 0 & 0 \\ b & * & * & * & * & 0 & 0 \end{vmatrix}$$

$f_w(u) = 0$

Example

$w = 2136745, u = 6243751$

$$\begin{vmatrix} 0 & * & * & * & * & a & a \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ b & * & * & * & * & 0 & 0 \\ b & * & * & * & * & 0 & 0 \end{vmatrix}$$

$A = 1, B = 1, f_w(u) = \binom{2}{1} = 2$

Sum of two %-immanants

We can now show for $w = 2136745$,

$$-\operatorname{Imm}_w = \begin{vmatrix} 0 & * & * & * & * & a & a \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ b & * & * & * & * & 0 & 0 \\ b & * & * & * & * & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & a & a \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ b & * & * & * & * & 0 & 0 \\ b & * & * & * & * & 0 & 0 \end{vmatrix}$$

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Example

$u = 6243751$
LHS: $\binom{2}{1} = 2$ using the rule
RHS: u fits in both %-immanants, so $1 + 1 = 2$

The converse

Theorem (LRSW)

Given a $321-$avoiding permutation w, Imm_w is a linear combination of %-immanants if and only if w avoids the patterns $1324, 24153, 31524, 231564, 312645$.

Proof Sketch for $w = 51324$.

Let $w = 51324$ and $w' = 51234$. Then $f_w(w') = 0$ and $f_w(w) = 1$. However, the coefficient of w and w' in any %-immanant must be negatives of each other. This must also be true in any linear combination of %-immanants, contradiction. □

$$\begin{vmatrix} 0 & 0 & * & * & x \\ x & * & * & * & * \\ * & * & x & * & * \\ * & x & * & * & * \\ * & * & * & x & * \end{vmatrix} \qquad \begin{vmatrix} 0 & 0 & * & * & x \\ x & * & * & * & * \\ * & x & * & * & * \\ * & * & x & * & * \\ * & * & * & x & * \end{vmatrix}$$

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Totally non-negative

Definition

A matrix is totally non-negative if all of its minors are non-negative.

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Example

$A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$ are not TNN, but $C = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 4 & 16 \\ 1 & 8 & 64 \end{pmatrix}$ is TNN.

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An immanant is TNN if it is always non-negative when evaluated on TNN matrices.

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Definition

An immanant is TNN if it is always non-negative when evaluated on TNN matrices.

Question

For a permutation w, when will one of $\mathrm{Imm}_w^{\%}$ or $-\mathrm{Imm}_w^{\%}$ be TNN?

Our conjecture

Proposition

If w contains one of $1324, 24153, 31524, 426153$, there exist TNN matrices A, B such that $\mathrm{Imm}_w^{\%}(A) > 0$ and $\mathrm{Imm}_w^{\%}(B) < 0$.

Our conjecture

Proposition

If w contains one of $1324, 24153, 31524, 426153$, there exist TNN matrices A, B such that $\mathrm{Imm}_w^{\%}(A) > 0$ and $\mathrm{Imm}_w^{\%}(B) < 0$.

Conjecture

If w avoids $1324, 24153, 31524, 426153$, then either $\mathrm{Imm}_w^{\%}$ or $-\mathrm{Imm}_w^{\%}$ is TNN.

Our conjecture

Proposition

If w *contains one of* $1324, 24153, 31524, 426153$, *there exist TNN matrices* A, B *such that* $\mathrm{Imm}_w^{\%}(A) > 0$ *and* $\mathrm{Imm}_w^{\%}(B) < 0$.

Conjecture

If w *avoids* $1324, 24153, 31524, 426153$, *then either* $\mathrm{Imm}_w^{\%}$ *or* $-\mathrm{Imm}_w^{\%}$ *is TNN.*

Verified for $n \leq 7$ using computer.

A partial result

Proposition

If w *avoids* $321, 1324, 24153, 31524, 34127856$*, then either* $\mathrm{Imm}_w^{\%}$ *or* $-\mathrm{Imm}_w^{\%}$ *is TNN.*

A partial result

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If w avoids $321, 1324, 24153, 31524, 34127856$, then either $\mathrm{Imm}_w^{\%}$ or $-\mathrm{Imm}_w^{\%}$ is TNN.

Proof Sketch.

We can express $\mathrm{Imm}_w^{\%}$ as the sum of the TL immanant Imm_w and another Kazhdan-Lusztig immanant KL_u for some u, and it's known that KL immanants and TL immanants are TNN. □

A partial result

Proposition

If w *avoids* $321, 1324, 24153, 31524, 34127856$, *then either* $\mathrm{Imm}^{\%}_w$ *or* $-\mathrm{Imm}^{\%}_w$ *is TNN.*

Proof Sketch.

We can express $\mathrm{Imm}^{\%}_w$ as the sum of the TL immanant Imm_w and another Kazhdan-Lusztig immanant KL_u for some u, and it's known that KL immanants and TL immanants are TNN. □

Remark

$\mathrm{Imm}^{\%}_{34127856}$ is actually a sum of a TL immanant and two KL immanants.

Summary

- We found necessary and sufficient conditions for a TL-immanant to be a sum of one or two %-immanants.
- We found an explicit combinatorial formula for the coefficients $f_w(u)$ of a TL-immanant if w avoids 321 and 1324.
- We showed that if w avoids a certain family of patterns, then $\mathrm{Imm}_w^{\%}$ is TNN.

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For those patiently waiting: The end of this presentation is immanant!