Shelling AugBerg and the Weak Lefschetz Property

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- *Independent sets*: sets of linearly independent vectors.
  *Flats*: closed under linear span
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- Okay... what are matroids?
- Intuitively: a matroid is an object that stores information about a set of vectors and their dependencies.
- Independent sets: sets of linearly independent vectors.
  Flats: closed under linear span
- A matroid can be equiv. defined by its independent sets or by its flats
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1. $\text{Berg}(\mathcal{M})$ is a simplicial complex in which faces correspond to chains of flats (excluding $\emptyset$ and $E$)

2. $\mathcal{I}(\mathcal{M})$ is a simplicial complex in which faces correspond to independent sets of $\mathcal{M}$
What is AugBerg?

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- augmented Bergman complex $\text{AugBerg} (\mathcal{M})$ is a simplicial complex on vertices $\{y_1, \ldots, y_n\} \cup \{x_F\}_{F \in \mathcal{F}(\mathcal{M}) - \{E\}}$
- Simplices are given by $\{y_i\}_{i \in I} \cup \{x_{F_1}, \ldots x_{F_k}\}$ where $I \in \mathcal{I}(\mathcal{M})$ and $I \subseteq F_1 \subset F_2 \subset \ldots \subset F_k$
AugBerg Example

\( \mathcal{F}(M) \)

\{1, 2, 3\}

\{1, 2\} \quad \{1, 3\} \quad \{2, 3\}

\{1\} \quad \{2\} \quad \{3\}

\emptyset
AugBerg Example

\[ \mathcal{F}(\mathcal{M}) \]

\{1, 2, 3\}

\{1, 2\} \quad \{1, 3\} \quad \{2, 3\}

\{1\} \quad \{2\} \quad \{3\}

\emptyset

\[ \mathcal{I}(\mathcal{M}) \]

\[ F(M) \]

\( y_1 \)

\( y_2 \) \quad \( y_3 \)

\[ \operatorname{Berg}(\mathcal{M}) \]

\( x_1 \) \quad \( x_{12} \) \quad \( x_2 \)

\( x_{13} \) \quad \( x_3 \) \quad \( x_{23} \)
AugBerg Example

$\mathcal{F}(\mathcal{M})$

\{1, 2, 3\}

\{1\} \quad \{2\} \quad \{3\}

$\emptyset$

$\mathcal{I}(\mathcal{M})$

$y_1$

$y_2$ \quad $y_3$

Berg($\mathcal{M}$)

AugBerg($\mathcal{M}$) \ \setminus \ B \ \setminus \ \{x_0\}
Our question

• Already well known that the independent set and Bergman complexes of a matroid are **shellable**
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Our question

• Already well known that the independent set and Bergman complexes of a matroid are **shellable**
  • we can order facets in such a way that these complexes are *very connected*

• Also known that AugBerg is gallery connected, a weaker property than shellable [1]

A Natural Question

Is AugBerg shellable?
Theorem

AugBerg(M) is shellable. Furthermore, we have

- a shelling that shells Cone(Berg(M)) first and $I(M)$ last.
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AugBerg(M) is shellable. Furthermore, we have

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Idea

We leverage the following two well-known facts.

- For the “base case,” apply the lexicographic shelling of \( I(M) \)
- For the “inductive step,” apply the lexicographic shelling of Berg(\( M' \)) for some “quotient” of \( M \)
The Shelling Order

Shell in increasing order based on rank of independent set.
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Shell in increasing order based on rank of independent set. Consider facets of AugBerg(M) given by

\[ T_i = I \subseteq F_1^i \subseteq \cdots \subseteq F_m^i \]
\[ T_j = J \subseteq F_1^j \subseteq \cdots \subseteq F_n^j \]
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1. If \( \#I < \#J \), order \( T_i \) before \( T_j \).
2. If \( \#I = \#J \) but \( I \neq J \),
   Apply the lexicographic order on \( I \) and \( J \).

Apply the shelling order on \( \text{Berg}(M/F) \).
The Shelling Order

Shell in increasing order based on rank of independent set. Consider facets of AugBerg(M) given by

\[ T_i = I \subseteq F^i_1 \subsetneq \cdots \subsetneq F^i_m \]
\[ T_j = J \subseteq F^j_1 \subsetneq \cdots \subsetneq F^j_n \]

1. If \( \#I < \#J \), order \( T_i \) before \( T_j \).
2. If \( \#I = \#J \) but \( I \neq J \),
   Apply the lexicographic order on \( I \) and \( J \).
3. If \( I = J \), then \( F^i_1 = F^j_1 = \text{span}\{I\} =: F \)
   Define the contraction matroid
   \[ M/F = (E \setminus F, \{I : I \cup F \in I(M)\}) \]
   Then \( \{\text{Flats in } M \text{ containing } F \} \leftrightarrow \{\text{Flats in } M/F\} \).
   Apply the shelling order on Berg(M/F).
The Shelling Order
Shell in *decreasing* order based on rank of independent set!
Let $M$ be a matroid of rank $r(M)$. Recall the Tutte Polynomial:

$$T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)}(y - 1)^{|A| - r(A)}$$
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- $Cone(Berg(M))$ is homotopy equiv. to a wedge of $T_M(1, 0)$ spheres of dimension $r(M) - 2$ (Garsia [2])
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T_M(x, y) = \sum_{A \subseteq E} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}
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- $I(M)$ is homotopy equiv. to a wedge of $T_M(0, 1)$ spheres of dimension $r(M) - 1$ (Provan and Billera [3]).
- $\text{Cone}(\text{Berg}(M))$ is homotopy equiv. to a wedge of $T_M(1, 0)$ spheres of dimension $r(M) - 2$ (Garsia [2])

**Our Result**

$\text{AugBerg}(M)$ is homotopy equiv. to a wedge of $T_M(1, 1)$ spheres of dimension $r(M) - 1$. 
At this point in the research we switched gears:
Moving on...

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Now introducing:
the Weak Lefschetz Property
Some Background (Stanley-Reisner Ring)

- \( \Delta \) is simplicial complex with vertices \( \{1, \ldots, n\} \)
- \( I_\Delta \) is the ideal generated by monomials supported on non-faces of \( \Delta \)

Example

Taking \( \Delta \) to be the boundary of a tetrahedron, we have \( K[\Delta] = K[x_1, x_2, x_3, x_4]/(x_1 x_2 x_3 x_4) \).
Some Background (Stanley-Reisner Ring)

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- $I_\Delta$ is the ideal generated by monomials supported on non-faces of $\Delta$
- the **Stanley-Reisner ring** is $K[\Delta] := K[x_1, \ldots, x_n] / I_\Delta$
- the Stanley-Reisner ring is isomorphic to the $K$-span of monomials whose support is a face of $\Delta$
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**Example**

![Diagram of a tetrahedron with vertices labeled 1, 2, 3, 4]
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Example

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\begin{center}
\begin{tabular}{c}
\begin{tikzpicture}
\draw[thick, fill=gray!50] (0,0) -- (1,1) -- (2,0) -- cycle;
\draw[thick, fill=gray!50] (0,0) -- (1,1) -- (0.5,0.5) -- cycle;
\draw[thick, fill=gray!50] (0,0) -- (1,1) -- (1,0) -- cycle;
\end{tikzpicture}
\end{tabular}
\end{center}
Definition

A linear system of parameters (LSOP) $\theta$ is a set of $\theta_i \in K[\Delta]$ that are linear in the $x_j$'s such that $K[\Delta]/(\theta)$ is finite dimensional over $K$.

$$M(\theta) = \begin{bmatrix} \theta_1 & \cdots & \theta_r \end{bmatrix}$$
Fact

If $\Delta$ is the boundary of a simplicial polytope, then we can get an LSOP as follows:

$$M(\theta) = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

Example

Now $K[\Delta]/(\theta) = K[t]/t^4$. 

$$M(\theta) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \theta_1 = x_1 - x_4$$

$$\theta_2 = x_2 - x_4$$

$$\theta_3 = x_3 - x_4$$

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$\theta_1 = x_1 - x_4$
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Now $K[\Delta]/(\theta) = K[t]/t^4$. 
• Let $A = K[\Delta]/(\theta)$ be the Stanley-Reisner ring of a the simplicial complex $\Delta$ quotiented out by an LSOP $\theta$. 

**Definition**

Given an $\ell \in A_1$, we say that $\ell$ is Weak-Lefschetz (WL) if and only if the multiplication by $\ell$ map $(\cdot \ell)$ from $A_i$ to $A_{i+1}$ is full rank for all $i \in \{0, \ldots, d-1\}$. In particular, if $\Delta$ is the boundary of a convex simplicial polytope, then $\ell$ is WL iff $(\cdot \ell)$ from $A_i$ to $A_{i+1}$ is injective for $i < \frac{r}{2}$ and surjective otherwise, since the dimensions of the $A_i$'s are symmetric and unimodal.
Weak Lefschetz

- Let $A = K[\Delta]/(\theta)$ be the Stanley-Reisner ring of a simplicial complex $\Delta$ quotiented out by an LSOP $\theta$.
- $A$ is $\mathbb{N}$ graded, say with graded components $A_i$ for $i \in \{0, 1, \ldots, d\}$.
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What do we want to know?

Big Question

Is the WL property matroidal?
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Matroidal
Define $\hat{M}(\theta, \ell) = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_k \\ \ell \end{bmatrix}$. 
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**Matroidal**

Define $\hat{M}(\theta, \ell) = \begin{bmatrix} \theta_1 \\ \cdots \\ \theta_k \\ \ell \end{bmatrix}$.

Does WL property depend on minors of $\hat{M}(\theta, \ell)$?
Reduction to Middle Map

Proposition

- If $d$ odd, $\ell$ is WL $\iff \frac{A_{d-1}}{2} \cdot \ell \rightarrow A_{\frac{d+1}{2}}$ is injective.

- If $d$ even, $\ell$ is WL $\iff \frac{A_d}{2} - 1 \rightarrow \frac{A_d}{2}$ is injective $\iff \frac{A_d}{2} \cdot \ell \rightarrow A_{\frac{d+1}{2}}$ is surjective.
Reduction to Even Dimensions

### Bipyramid Construction

For a polytope $P$, let $P'$, its bipyramid, be the polytope with vertex set $\{x_1 \cdots x_n\} \cup \{x_{n+1}x_{n+2}\}$, where

- $x_{n+1}, x_{n+2} \not\in \text{span}\{x_1, \cdots, x_n\}$
- The line $x_{n+1}x_{n+2}$ goes through the origin
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Reduction to Even Dimensions

**Proposition**

- \( A' \cong A[x_{n+1}]/(x_{n+1}^2) \)
- \( A'_k \cong A_k \oplus x_{n+1}A_{k-1} \)
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- $A'_k \cong A_k \oplus x_{n+1}A_{k-1}$

Proposition

Let $d$ be odd.

$\sum_{i=1}^{n} \alpha_i x_i \in A_1$ is WL in $A \iff \sum_{i=1}^{n} \alpha_i x_i \in A'_1$ is WL in $A'$. 
### Stacking Construction

Let $P$ be a polytope and $F \in \mathcal{F}(P)$. To obtain $P'$ from $P$, add in a new vertex $x_{n+1}$ “close enough” to $F$ on the outside.
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Definition

$P$ is a stacked polytope if $P$ is obtained from a simplex through a sequence of stacking operations.
Stacked Polytopes

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Proposition

$\sum_{i=1}^{n+1} \alpha_i x_i \in A'_1$ is WL in $A'$ $\iff \begin{cases} \sum_{i=1}^{n} \alpha_i x_i \in A_1 \text{ is WL in } A \\ \alpha_{n+1} \neq 0 \end{cases}$
**Definition**

$C(n, d)$, the $d$-dimensional polytope on $n$ vertices is the convex hull of any $n$ points on the moment curve

$$t \mapsto \begin{bmatrix} t \\ t^2 \\ \vdots \\ t^d \end{bmatrix}$$
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Proposition

- Let $d$ even. $\ell$ is WL $\iff \ell \neq 0$
- Let $d$ odd. $\ell$ is WL $\iff$ all minors of $M((\theta), \ell)$ with columns indexed by $\{x_1, x_{i_1}, x_{i_2}, \ldots x_{i_{d-1}}, x_n\}$ are L.I., where $\{x_1, x_{i_1}, x_{i_2}, \ldots x_{i_{d-1}}\}$ runs through all facets not containing $x_n$. 
Cross Polytopes

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The $n$-dimensional cross polytope is the convex hull of \[ \{ e_i, -e_i, 1 \leq i \leq n \} \] (ie. square, octahedron)
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**Proposition**

Let \( \Delta \) be the boundary of the \( n \)-dimensional cross polytope. Then \( K[\Delta]/(\theta) \) is isomorphic to the \( K \)-span of all square-free monomials in \( x_1, \ldots, x_n \).
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**Proposition**

Let $\ell = \sum_{i=1}^{n} c_i x_i \in K[\Delta]/(\theta)$.

- If $n$ is odd, $\ell$ is WL if and only if $c_i \neq 0$ for all $i$.
- If $n$ is even, $\ell$ is WL if and only if $c_i = 0$ for at most one $i$. 
What We Found

Is the WL property matroidal in general?

Counterexample

Consider the following $\Delta$:

$x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ x_8$

with vertex LSOP:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\
-1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & -1 \\
\end{bmatrix}$$

Claim: The rank of $\ell: A_1 \rightarrow A_2$ is not determined by minors of $\hat{M}(\theta, \ell)$.
What We Found
Is the WL property matroidal in general? **No!**
Counterexample

What We Found

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Boundary of a Tetrahedron Counterexample

Consider the following $\Delta$:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 \\
-1 & 0 & 1 & 0 & 1 & 0 & 1 \\
-1 & 0 & 0 & 1 & 1 & 1 & 0 \\
\end{bmatrix}
\]

Claim: The rank of $\ell: A_1 \to A_2$ is not determined by minors of $\hat{M}(\theta, \ell)$. 
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Is the WL property matroidal in general? **No!**

Boundary of a Tetrahedron Counterexample

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```
 x_1
  |   |
  |   |
  |   |
```

with vertex LSOP:

```
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & -1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & -1 \\
\end{bmatrix}
```
Counterexample

What We Found
Is the WL property matroidal in general? **No!**

Boundary of a Tetrahedron Counterexample

Consider the following Δ:

![Triangle Diagram]

with vertex LSOP:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & -1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & -1 \\
\end{bmatrix}
\]

**Claim:** The rank of \( \ell : A_1 \rightarrow A_2 \) is not det. by minors of \( \hat{M}(\theta, \ell) \).
Thank you for watching and thank you to all the REU staff who were super thoughtful and encouraging throughout the research process, and especially to Vic for providing team 7 with a great problem to work on, and to Sasha and Trevor for their guidance!
