Weak Lefschetz Property for Boundaries of Simplicial Polytopes

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Abstract

We investigate the weak Lefschetz property for Artinian reductions $A = K[\Delta]/(\theta)$ of the Stanley-Reisner ring $K[\Delta]$ of a simplicial complex $\Delta$ by a linear system of parameters $\theta = (\theta_1, \ldots, \theta_d)$. It is known that deciding whether or not the ring $A$ is Artinian is matroidal, in the sense that it is controlled only by the matroid $M(\theta)$ for the coefficient matrix specifying $\theta$. We consider the weak Lefschetz property for a degree one element $\ell$ in $K[\Delta]$, meaning that multiplication by $\ell$ from $A_i \to A_{i+1}$ has full rank for all $i$, and ask when it is matroidal in the sense of being controlled only by the lifted matroid $\hat{M}(\theta, \ell)$ for the coefficient matrix of $\theta$ and $\ell$.

It is known from work of Stanley [3], McMullen and others that when $K = \mathbb{R}$ and $\Delta$ is the boundary of a simplicial polytope, there exists a choice $\theta$ and degree one elements $\ell$ having the weak Lefschetz property. In this context, we find some examples where the weak Lefschetz property is not matroidal, and others where it is matroidal.

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1 Introduction

Let $\Delta$ be a simplicial complex on $n$ vertices. Its Stanley-Reisner ring is the quotient

$$K[\Delta] := K[x_1, \ldots, x_n]/I_\Delta$$

where the ideal $I_\Delta$ is generated by all squarefree monomials $x_A := \prod_{i \in A} x_i$ for which $A$ is not a face of $\Delta$.

The early history and applications of $K[\Delta]$ in the theory of face numbers for convex polytopes make use of quotients $A = K[\Delta]/(\theta)$ in which $\theta = (\theta_1, \ldots, \theta_d)$ is a linear system of parameters (lsop), meaning that

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• each \( \theta_i \) is an element of \( K[\Delta] \) of degree one,

• the quotient \( A \) is Artinian (i.e., of Krull dimension zero; equivalently, finite-dimensional over \( K \)), and

• \( d \) is the smallest cardinality possible (which is one more than the dimension of \( \Delta \)).


The condition under which a candidate set of degree one elements \( \theta = (\theta_1, \ldots, \theta_d) \) form an isop is due to Kind and Kleinschmidt [3] Lemma III.2.4(a): the coefficient matrix \( (c_{ij}) \) in \( K^{d \times n} \) where \( \theta_i = \sum_{j=1}^{n} c_{ij} x_j \) should have the property that every submatrix obtained by restricting to a set of columns \( F \) indexing a face \( F \) of \( \Delta \) is of maximal rank \( \# F \). This condition is \emph{matroidal} in the sense that it depends only on the matroid \( M(\theta) \) associated with the matrix \( (c_{ij}) \), meaning the specification of which subsets of its columns are linear independent or dependent.

The interest in the Artinian reduction \( A = K[\Delta]/(\bar{\Phi}) \) has often centered on deciding whether it contains an element of degree one \( \ell = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \) for which the maps \( A_i \to A_{i+1} \) given by multiplication by \( \ell \) have maximal rank for all \( i \), where \( A_i \) is the \( i \)th graded component. In this case, one says that that \( \ell \) is a \emph{weak Lefschetz element}; see Migliore and Nagel [2] for more on the weak Lefschetz property. Having fixed \( \Phi \), one might ask whether this property depends solely on the matroid \( M(\Phi) \) and on the \emph{lifted matroid} \( M(\Phi, \ell) \) corresponding to the columns of the matrix in \( K^{(d+1) \times n} \) having the vectors of \( (\theta_1, \ldots, \theta_d, \ell) \) as its rows; in this case we will say that the weak Lefschetz property for \( \ell \) is \emph{matroidal}. This is the question we examine here.

\textbf{Question.} For which simplicial complexes \( \Delta \) and isop \( \Phi \) is the weak Lefschetz property for \( \ell \) matroidal?

From now on we will assume that \( \Delta \) is the boundary of a \( d \)-dimensional, convex, simplicial polytope \( P \) in \( \mathbb{R}^d \), containing the origin in its interior. It is not hard to see that in this setting, one can form an isop \( \Phi \) by taking the columns of the matrix \( (c_{ij}) \) to be the vectors \( v_1, \ldots, v_n \) which are the vertices of \( P \); we will call this the geometric choice of isop. Unless specified, we will always work with the geometric isop. Work of Stanley and McMullen (see [4] §III.1) shows that there always exist weak Lefschetz elements \( \ell \) in this context\footnote{In fact, they showed that there are always strong Lefschetz elements in this context, that is, those for which the maps \( A_i \to A_{i+m} \) which multiply by \( \ell^m \) all have maximal rank.}

\section{Preliminaries}

\textbf{Definition.} Recall that \( M(\theta) \) is the \( d \times n \) coefficient matrix of \( \theta \). Given any facet \( F \) of \( \Delta \), we use \( M(\theta)_F \) to denote the \( d \times d \) minor of \( M(\theta) \) with columns indexed by vertices in \( F \). Furthermore, given any vertex \( x \notin F \), we use \( M(\theta, \ell)_{F,x} \) to denote the \( (d+1) \times (d+1) \) minor of \( M(\theta, \ell) \) with columns indexed by vertices in \( F \cup \{x\} \).

\textbf{Lemma 2.1.} Let \( F \) be a facet of \( \Delta \). We have a canonical graded algebra isomorphism

\[ \phi_F : K[x_1, x_2, \cdots, x_n]/(\bar{\Phi}) \to K[x_1 : i \notin F] \]

given by \( x_i \mapsto x_i \) for all \( i \notin F \). This gives rises to a graded algebra isomorphism

\[ \psi_F : A \to K[x_i : i \notin F]/\phi_F(I_\Delta) \]

\textbf{Proof.} Notice that for any facet \( F \) of \( \Delta \), \( M(\Phi)_F \) is invertible. Thus, we can perform row reduction to express all elements in \( \{x_i : i \in F\} \) as a linear combination of \( \{x_i : i \notin F\} \). This gives us the isomorphism \( \phi_F \).

Furthermore, notice that \( A = K[\Delta]/(\bar{\Phi}) = K[x_1, x_2, \cdots, x_n]/(I_\Delta, \bar{\Phi}) \cong K[x_i : i \notin F]/\phi_F(I_\Delta) \) via \( \phi \).

\textbf{Lemma 2.2.} Let \( \ell = \sum_{i \in [n]} \alpha_i x_i \) (and resp., \( \sum_{i \notin F} \beta_i x_i \)) be any representative for some generic element in \( A_1 \) (and resp., its image under \( \psi_F \)). Then for any \( i_0 \notin F \), \( \beta_{i_0} = 0 \) if and only if \( \hat{M}(\theta, \ell)_{F,x_{i_0}} \) is invertible.
Proof. \( \beta_{i_0} = 0 \) if and only if applying row reduction to \( \tilde{M}(\tilde{\theta}, \ell)_{F, x_{i_0}} \) to zero out the last row with columns except \( x_{i_0} \) automatically zeros out \( x_{i_0} \). And this happens if and only if \( \tilde{M}(\tilde{\theta}, \ell)_{F, x_{i_0}} \).

The following well-known theorem is crucial for future calculations; a proof can be found in [4, II.6].

**Lemma 2.3** (Dehn-Sommerville equations). The \( h \)-vector of the boundary of a simplicial \( d \)-polytope satisfies

\[
h_k = h_{d-k}, \quad \text{for } k = 0, 1, \ldots, d.
\]

**Remark.** Since the \( h \)-vector enumerates the dimensions of the graded components of \( A \) as a \( K \)-vector space, we also have that \( \dim_C A_i = \dim_C A_{d-i} \).

A final lemma further characterizes the \( h \)-vector. This is Theorem 1.1(b) in [4, III.1].

**Lemma 2.4.** The \( h \)-vector for a \( d \)-dimensional simplicial polytope \( P \) is unimodal, and in particular

\[
h_1 \leq h_2 \leq \cdots \leq h_{\left\lfloor \frac{d}{2} \right\rfloor} \quad \text{and} \quad h_{\left\lfloor \frac{d}{2} \right\rfloor} \geq h_{\left\lceil \frac{d}{2} \right\rceil + 1} \geq \cdots \geq h_d.
\]

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### 3 Reducing the Weak Lefschetz Property

**A priori,** checking that our candidate choice of a degree one element \( \ell \in A_1 \) is weak Lefschetz requires verifying that every possible multiplication by \( \ell \) map \( A_i \xrightarrow{\ell} A_{i+1} \) is full rank. It would be ideal if we could simplify the criterion. In this section, we first show that it suffices to show that the middle map \( A_{\left\lfloor \frac{d}{2} \right\rfloor} \xrightarrow{\ell} A_{\left\lceil \frac{d}{2} \right\rceil + 1} \) is full rank. We then apply the bipyramid construction in order to reduce the odd case to the even case.

#### 3.1 The Gorenstein Property

It is known that \( A = K[\Delta]/(\theta) \) is a Gorenstein ring. Since \( A \) is a graded algebra over the field \( K \), this is equivalent to the product map \( \langle \cdot, \cdot \rangle: A_i \times A_{d-i} \to A_d \) being a nondegenerate bilinear form. In particular, given some basis \( \{a_j\} \) of \( A_i \) and basis \( \{b_j\} \) for \( A_{d-i} \), we have that

\[
\langle a_j, b_k \rangle = \delta_{jk}.
\]

Details can be found in Stanley [4, §II.5].

Using this fact, we can prove the following proposition that exploits this “duality.” In particular, for the weak Lefschetz property it will suffice to check that only half of the maps are full rank.

**Proposition 3.1.** Let \( A = K[\Delta]/(\theta) \) For any \( i \leq \left\lfloor \frac{d}{2} \right\rfloor \), the map \( A_i \xrightarrow{\ell} A_{i+1} \) is injective if and only if the map \( A_{d-i-1} \xrightarrow{\ell} A_{d-i} \) is surjective.

**Proof.** Let \( \{x\} \) be a basis for \( A_d \). Assume first that \( A_i \xrightarrow{\ell} A_{i+1} \) is injective. Fix a basis \( \{a_u\}_{u=1,2,\ldots,h_i} \) for \( A_i \). By the injectivity assumption we know that \( \ell \) maps \( A_i \) to an \( h_i \)-dimensional subspace of \( A_{i+1} \), so \( \{a_u \cdot \ell\}_{u=1,2,\ldots,h_i} \) is a linearly independent set in \( A_{i+1} \). Extend it to

\[
\left( \{a_u \cdot \ell\}_{u=1,2,\ldots,h_i} \right) \cup \left( \{b_u\}_{u=h_i+1,h_i+2,\ldots,h_{i+1}} \right)
\]

a basis of \( A_{i+1} \). There exists a basis \( \{c_v\}_{v=1,2,\ldots,h_{d-i-1}} \) for \( A_{d-i-1} \) that is dual to \( \left( \{a_u \cdot \ell\}_{u=1,2,\ldots,h_i} \right) \cup \left( \{b_u\}_{u=h_i+1,h_i+2,\ldots,h_{i+1}} \right) \) under \( \langle \cdot, \cdot \rangle \), in the sense of [1]. In particular, \( \langle a_u \cdot \ell, c_v \rangle = \delta_{uv} \) for \( u, v = 1, 2, \ldots, h_i \). By definition of the bilinear form, \( \delta_{uv} x = a_u \cdot \ell \cdot c_v = a_u \cdot c_v \cdot \ell \) and thus \( \langle a_u, c_v \cdot \ell \rangle = \delta_{uv} \). By nondegeneracy, \( \{c_v \cdot \ell\}_{v=1,2,\ldots,h_{d-i}} \) is a basis for \( A_{d-i} \) by Lemma 2.3. This gives that \( A_{d-i-1} \xrightarrow{\ell} A_{d-i} \) is surjective.

Conversely, assume that \( A_{d-i-1} \xrightarrow{\ell} A_{d-i} \) is surjective. This implies that \( h_{d-i-1} \geq h_{d-i} \). Thus we can let \( \{c_u\}_{u=1,2,\ldots,h_{d-i}} \) in \( A_{d-i} \) be such that \( \{c_u \cdot \ell\}_{u=1,2,\ldots,h_{d-i}} \) is a basis for \( A_{d-i} \). We set \( \{c_v\}_{v=1,2,\ldots,h_{d-i}} \) as a basis of \( A_{i-1} \) dual to \( \{c_u \cdot \ell\}_{u=1,2,\ldots,h_{d-i}} \), again in the sense of [1]. Then \( \langle a_u, c_v \cdot \ell \rangle = \delta_{uv} \) for \( u, v = 1, 2, \ldots, h_{d-i} \). So \( \delta_{uv} x = a_v \cdot c_u \cdot \ell = a_v \cdot \ell \cdot c_u \). And thus \( \langle a_v \cdot \ell, c_u \rangle = \delta_{uv} \). for \( u, v = 1, 2, \ldots, h_{d-i} \).
Notice that \( \{c_u\}_{u=1,2,...,h_{d-1}} \) is linearly independent in \( A_{d-i-1} \) because \( \{c_u \cdot \ell\}_{u=1,2,...,h_{d-1}} \) is linearly independent in \( A_{d-i} \). We wish to show that \( \{a_v \cdot \ell\}_{v=1,2,...,h_{d-1}} \) is linearly independent in \( A_{i+1} \), such that \( A_i \xrightarrow{\ell} A_{i+1} \) is injective. Suppose \( \beta_1 a_1 \cdot \ell + \cdots + \beta_{h_{d-1}} a_{h_{d-1}} \cdot \ell = 0 \) for coefficients \( \beta_i \) in our field \( K \). We then have that \( \langle \beta_1 a_1 \cdot \ell + \cdots + \beta_{h_{d-1}} a_{h_{d-1}} \cdot \ell, c_u \rangle = 0 \) for all \( u = 1,\ldots, h_{d-1} \), which implies that
\[
\beta_1 (a_1 \cdot \ell, c_u) + \cdots + \beta_{h_{d-1}} (a_{h_{d-1}} \cdot \ell, c_u) = 0
\]
for all \( u = 1,\ldots, h_{d-1} \). We showed above that \( \langle a_v \cdot \ell, c_u \rangle = \delta_{uv} \), and thus in order for the left hand side of \( \boxed{2} \) to be zero, we must have that the bilinear form \( \beta_u \langle a_u \cdot \ell, c_u \rangle \) is equal to zero for all \( u \). But this is true if and only if \( \beta_u = 0 \) for each \( u \), as desired.

We then make the following observation, which, combined with the preceding proposition, reduces the problem to studying the full-rank property of maps in the “middle.”

**Proposition 3.2.** For any \( 0 \leq i \leq (\lfloor \frac{d}{2} \rfloor - 1) \), if \( A_{i+1} \xrightarrow{\ell} A_{i+2} \) is injective, then \( A_i \xrightarrow{\ell} A_{i+1} \) is injective.

**Proof.** We first show that for any \( x \in A \), there exists some \( \ell' \in A_1 \) such that \( x \cdot \ell' \neq 0 \). Indeed, if \( x \cdot a = 0 \) for all \( a \) in the basis of \( A_{d-i} \), we get that \( x \cdot a = 0 \) for all \( a \) in this basis. This contradicts the fact that \( A \) is Gorenstein.

We now show the contrapositive of the statement. Suppose that \( A_i \xrightarrow{\ell} A_{i+1} \) is not injective, and let \( x \neq 0 \in A_i \) be such that \( x \cdot \ell = 0 \). Pick some \( \ell' \) so that \( x \cdot \ell' \neq 0 \). But \( (x \cdot \ell') \cdot \ell = x \cdot \ell \cdot \ell' = 0 \). Hence \( A_{i+1} \xrightarrow{\ell} A_{i+2} \) is not injective.

This yields the following corollary.

**Corollary 3.3.** An element \( \ell \in A_1 \) is a weak Lefschetz element of \( A \) if and only if \( A_{i+2} \xrightarrow{\ell} A_{i+1} \) is injective.

**Proof.** The claim follows directly from Proposition 3.1 and Proposition 3.2.

Corollary 3.3 greatly reduces the computation required to verify whether a given \( \ell \in A_1 \) forms a weak Lefschetz element for \( A \). We make a further reduction before computing examples of boundaries of simplicial polytopes by showing that the problem only requires us to study even dimensional polytopes. This is done via the bipyramid construction in the next subsection.

### 3.2 Reduction to Even Dimensional Polytopes

**Definition.** For a given polytope \( P \) such that the origin is in \( P \), define its bipyramid \( P' \) as the polytope with vertex set \( P \cup \{N, S\} \), where \( N \) and \( S \) are not in \( P \) and the line \( NS \) through the north pole and the south pole goes through the origin.

Let \( \Delta \) be the simplicial complex given by the boundary of a \( d \)-dimensional convex polytope \( P \), and let \( \Delta' \) be the boundary of the \( (d + 1) \)-dimensional bipyramid \( P' \) of \( P \). We have that for any \( F \subset \{x_1, x_2, \ldots, x_n\} \), the sets \( \{x_{n+1}\} \cup F \) and \( \{x_{n+2}\} \cup F \) are faces of \( \Delta' \) if and only if \( F \) is a face of \( \Delta \). Note \( \{x_{n+1}, x_{n+2}\} \) is not a face of \( \Delta' \).

Finally, let \( A = K[\Delta]/(\theta) \) and \( A' = K[\Delta]/(\theta') \), where \( \theta \) and \( \theta' \) are the geometric lsop’s of \( P \) and \( P' \).

**Proposition 3.4.**
\[
A' = A[x_{n+1}, x_{n+2}]/(x_{n+1}x_{n+2}, x_{n+1} - x_{n+2}) \simeq A[x_{n+1}]/(x_{n+1}^2)
\]
Also, \( A_i' = A_i \oplus x_{n+1}A_{i-1} \).

**Proof.** In \( P' \), we added two more vertices \( x_{n+1} \) and \( x_{n+2} \). Notice that the set of non-faces of \( \Delta' \) not containing \( x_{n+1} \) or \( x_{n+2} \) is exactly the set of non-faces of \( \Delta \). If a non-face of \( \Delta' \) contains \( x_{n+1} \) or \( x_{n+2} \), removing \( x_{n+1} \) or \( x_{n+2} \) gives a non-face of \( \Delta \). And \( \{x_{n+1}, x_{n+2}\} \) is a non-face of \( \Delta' \). So the set of minimal non-faces of \( \Delta' \) is exactly the set of minimal non-faces of \( \Delta \), union \( \{x_{n+1}, x_{n+2}\} \). This shows that \( K[\Delta'] = K[\Delta][x_{n+1}, x_{n+2}]/(x_{n+1}x_{n+2}) \).
Proof. Let $\alpha$ be a weak Lefschetz element of $A$ viewed in $A^{d+1}$ resulting in $\beta = A^{d+1}(\alpha)$. For any polytope of odd dimension, we can construct its bipyramid of one dimension higher. Then, $\beta$ is nonzero for any $\alpha$ such that $\alpha$ is injective, which implies that the map $A^{d+1}_\alpha \to A^{d+1}_\beta$ is injective. Thus $\ell'$ is a weak Lefschetz element of $A'$, as desired.

The converse argument follows similarly.

Remark. For any polytope of odd dimension, we can construct its bipyramid of one dimension higher. Then, setting $\alpha_n = 0$ in the conditions for $\ell'$ to be a weak Lefschetz element for the bipyramid gives us the conditions for $\ell$ to be a weak Lefschetz element in the original polytope.

Our problem is now reduced to studying even dimensional polytopes which may answer the following (refined) question:

**Question.** Let $\Delta$ represent the boundary of an even-dimensional simplicial polytope. For which $\Delta$ and isop $\theta$ is the weak Lefschetz property for an element $\ell$ dependent on the minors of $M(\theta, \ell)$?

We now explore specific classes of $\Delta$ for which the weak Lefschetz property of $\ell$ is matroidal.

### 4 Specific Classes

In this section we discuss the weak Lefschetz elements for stacked polytopes, cyclic polytopes, and cross polytopes. The first two are of interest, as they are “extremal” in the sense that for a given dimension, stacked polytopes have the fewest number of “large-dimensional” faces, while cyclic polytopes have the most (cf. [5]). Cross polytopes are another special family of interest, as we will see these polytopes have some intriguing properties.

#### 4.1 Stacked Polytopes

Stacking is a common operation that gives a new polytope $P_F$ from an old polytope $P$, where $F$ is a facet of $P$. It can be thought of as placing a new vertex an infinitesimal distance from $F$, outside of $P$, and taking the convex hull of $P$ and this new point. We now state a formal definition.

**Definition.** Let $P$ be a polytope with a set of facets $F$. Fixing some facet $F$, define the *stacked polytope* $P_F$ as the polytope with vertex set $P \cup \{x_n+1\}$ and facets $(F \setminus i) \cup (n+1)$.

We now use this construction to prove a property of weak Lefschetz elements in the bipyramids of simplicial polytopes that allows

**Proposition 3.5.** Let $d$ be odd. Let $\ell = \sum_{i=1}^{n} \alpha_i x_i \in A_1$ and let $\ell' = \sum_{i=1}^{n} \alpha_i x_i \in A'_1$ be the same element, but viewed in $A'_1$. Then $\ell$ is a weak Lefschetz element of $A$ if and only if $\ell'$ is a weak Lefschetz element of $A'$.

**Proof.** By Corollary 3.3, $\ell$ is a weak Lefschetz element of $A$ if and only if $A_{\frac{d+1}{2}} \to A_{\frac{d+1}{2}-1}$ is injective and $\ell'$ is a weak Lefschetz element of $A'$ if and only if $A_{\frac{d+1}{2}} \to A_{\frac{d+1}{2}-1}$ is injective.

Let $a + x_n b \in A_{\frac{d+1}{2}}$ be a generic element in $A_{\frac{d+1}{2}} = A_{\frac{d+1}{2}} \oplus x_n A_{\frac{d+1}{2}-1}$, where $a \in A_{\frac{d+1}{2}}$ and $b \in A_{\frac{d+1}{2}-1}$. Then $(a + x_n b) \cdot \ell' = a \cdot \ell' + x_n (b \cdot \ell')$. Assume that $\ell$ is a weak Lefschetz element. Then $a \cdot \ell \in A_{\frac{d+1}{2}}$ is nonzero for any $a \in A_{\frac{d+1}{2}}$ nonzero. Also, $b \cdot \ell \in A_{\frac{d+1}{2}-1}$ is nonzero for any $b \in A_{\frac{d+1}{2}-1}$ nonzero. So if $a + x_n b \in A_{\frac{d+1}{2}} \oplus x_n A_{\frac{d+1}{2}-1}$ is nonzero, $a$ or $b$ is nonzero, which implies that $a \cdot \ell \in A_{\frac{d+1}{2}}$ or $x_n b \cdot \ell \in x_n A_{\frac{d+1}{2}}$ is nonzero. This means that their sum $(a + x_n b) \cdot \ell'$ is nonzero, implying that the map $A_{\frac{d+1}{2}} \to A_{\frac{d+1}{2}-1}$ is injective. Thus $\ell'$ is a weak Lefschetz element of $A'$, as desired.

The converse argument follows similarly.
Proposition 4.1. Let $\Delta$ (and resp., $\Delta'$) be the simplicial complex given by the boundary of a polytope $P$ (and resp., its stacked polytope $P_F$ for some facet $F$). Fix the isos of $P$ and $P_F$ to be the geometric isos of their vertices.

If $A = K[\Delta]/(\theta')$, then $A' = K[\Delta']/(\theta') \cong A[x_{n+1}]/I$, where $I$ is generated by the additional minimal non-faces $x_F = \prod_{x \in F} x_i$ and $x_i x_{n+1}$ for $i \notin F$.

Furthermore, we may write any $\ell' \in A_1'$ as $\ell' = \sum_{j \notin F} a_i x_i$. Then $\ell'$ is a weak Lefschetz element for $A'$ if and only if $\ell$ is a weak Lefschetz element for $A$ and $a_{n+1} = 0$.

To prove Proposition 4.1, we need two lemmas. The first allows us to substitute monomials with smaller powers of $x_{n+1}$ for monomials with larger ones.

Lemma 4.2. $x_{n+1} x_i = d_i x_{n+1}^2$ for any $i \in F$, where $d_i$ is some constant.

Proof. Recall that by definition 4.1, since $F$ is a facet of $P$ then $(F \setminus \{i\}) \cup \{n+1\}$ is a facet of $P_F$. Thus the columns corresponding to $F$ in $M(\theta)$ are linearly independent, and also the columns corresponding to $(F \setminus \{i\}) \cup \{n+1\}$ in $M(\theta')$ are linearly independent. Hence, we may row reduce the isop (via Lemma 2.1) and assume one of the $\theta$ rows takes the following form, for constants $c_k$ with $c_i \neq 0$:

$$x_{n+1} + c_i x_i + \sum_{j \notin F} c_j x_j = 0.$$ 

Since $\{j, n+1\}$ is a non-face for $j \notin F$, we see that $\frac{x_{n+1}}{x_j} = \bar{0} \in A$. Thus, multiplying both sides of our equation by $x_{n+1}$, we get $x_{n+1}^2 = -c_i x_i x_{n+1}$. Take $d_i = -\frac{1}{c_i}$ to complete our proof. \hfill $\square$

The second lemma shows how the graded decomposition of $A'$ depends on that of $A$.

Lemma 4.3. $A'_k = A_k \oplus \text{span} \{x_{n+1}^k\}$.

In order to prove Lemma 4.3, we need to first introduce the concept of a shelling order on the facets of a pure simplicial complex $P$.

Definition. A shelling order for a pure simplicial complex $\Delta$ is a linear ordering on its facets $F_1, \ldots, F_t$ such that for each pair of facets $F_i, F_j$ with $i < j$, there is a facet $T_k$ with $k < j$ and an element $v \in T_j$ such that $T_i \cap T_j \subseteq T_k \cap T_j = T_j \setminus \{v\}$.

Remark. Such a shelling order does not always exist; if it exists we say that the simplicial complex is shellable.

We now provide the proof of Lemma 4.3.

Proof. Choose a shelling for $P$ where we shell facet $F$ last. This lifts to a shelling for $P_F$ where we shell the facets $(F \setminus \{i\}) \cup \{n+1\}$ in some order $i_1, i_2, \ldots, i_d$. Thus, a basis for $A'$ is given by concatenating a basis for $A$ with $x_{n+1}, x_{n+1} x_i, x_{n+1} x_{i_2}, \ldots, x_{n+1} x_{i_1}, \ldots x_{i_d}$. By repeated application of Lemma 4.2, we may use alternative basis elements $x_{n+1}^k$ where $1 \leq k \leq d$. \hfill $\square$

We are now ready to show the proposition.

Proof of Proposition 4.1. It’s clear that $A' = K[\Delta']/(\theta') \cong A[x_{n+1}]/I$. Since the columns of $F$ are independent, we may use the rows to set $\alpha_i = 0$ for $i \in F$.

We want to show the map $A'_k \xrightarrow{\ell} A'_k x_{n+1}$ is injective if and only if the map $A_k \xrightarrow{\ell} A_{k+1}$ is injective and $a_{n+1} \neq 0$. Let $a + bx_{n+1}^k \in A'_k$, where $a \in A_k$ and $b \in A_{k-1}$. By expressing $x_i$ for $i \in F$ as a linear combination of $x_j$ for $j \notin F$, we may assume $a$ has no monomials in $F$. Then

$$(a + bx_{n+1}^k)(\ell + a_{n+1} x_{n+1}) = a\ell + b a_{n+1} x_{n+1}^{k+1},$$

since $x_{n+1}^k \ell = ax_{n+1} = 0$. If $\ell$ is WL and $a_{n+1} \neq 0$, then this is not zero. If $\ell$ is not WL, then take $b = 0$ and $a \neq 0$ such that $a\ell = 0$. If $a_{n+1} = 0$, then take $a = 0$ and $b \neq 0$. \hfill $\square$
4.2 Cyclic Polytopes

The cyclic polytope $C(n,d)$ in $\mathbb{R}^d$ is the convex hull of substituting $n$ points $t_1 < t_2 < \cdots < t_n$ into the moment curve $x: \mathbb{R} \to \mathbb{R}^d$ given via the mapping $t \mapsto [t, t^2, \ldots, t^d]^\top$. We will use $[n]$ to label the set of vertices, where vertex $k$ corresponds to the point $x(t_k)$.

This next two lemmas is are well-known facts about cyclic polytopes.

**Lemma 4.4 (Gale’s evenness condition).** A size $d$ subset $S \subset [n]$ of $C(n,d)$ is a facet if and only if the following condition is satisfied:

$$\text{If } i < j \text{ are not in } S, \text{ the number of } k \in S \text{ between } i \text{ and } j \text{ is even.}$$

For the next lemma, recall that a polytope $P$ is $l$-neighborly if any subset $S$ of its vertices with cardinality $l$ form a facet.

**Lemma 4.5.** Let $P$ be a simplicial $d$-polytope on $f_0 = n$ vertices. Then for $0 \leq k \leq d$, the $h$-vector of $P$ satisfies

$$h_k \leq \binom{n-d-1+k}{k}.$$ 

Equality holds for all $k$ with $0 \leq k \leq l$ if and only if $l \leq \lfloor \frac{d}{2} \rfloor$ and $P$ is $l$-neighborly.

This next proposition constructs a basis for $A_k$ in the case of $C(n,d)$.

**Proposition 4.6.** Let $C(n,d)$ be a cyclic polytope with vertices given by $[n]$, and fix some facet $F$. Then for any $k \leq \lfloor \frac{d}{2} \rfloor$, the following set is a basis for $A_k$:

$$B := \{x_{i_1}x_{i_2}\cdots x_{i_k} : i_j \in [n] \setminus F \text{ for all } 1 \leq j \leq k\}.$$  

**Proof.** Without loss of generality, let $F$ be $\{1,2,\ldots,d\}$; the general case is entirely analogous ($F$ is a facet by Lemma 4.4). We first show that $B$ spans $A_k$. Consider any monomial of degree $k$ in $A_k$ given by $x_{i_1}x_{i_2}\cdots x_{i_k}$. If $i_j \not\in \{d+1,d+2,\ldots,n\}$ for some $j$, then it must be contained in $F$. As $\theta$ is an isop, there exists some linear combination of $\theta_1, \theta_2, \ldots, \theta_d$ that equals $\eta = x_{i_j} - \sum_{m=d+1}^{n} c_m x_m$. In $A$ we have $x_{i_j} = x_{i_j} - \eta$, which expresses $x_{i_j}$ as a linear combination of elements $x_m$, where $m \in [n] \setminus F$. Repeating this process allows us to express $x_{i_1}x_{i_2}\cdots x_{i_k}$ as a linear combination of elements in $B$.

It now suffices to show that the dimension of $A_k$ matches the number of elements in $B$. By a stars and bars argument, $B$ contains $(n-d-1+k)$ elements, which is precisely the dimension of $A_k$ given by Lemma 4.5 as cyclic polytopes are $[\frac{d}{2}]$-neighborly.

Using what we have proven, we can classify all weak Lefschetz elements for cyclic polytopes when $d$ is even.

**Theorem 4.7.** Let $C(n,d)$ be a cyclic polytope where $d$ is even. Then any nonzero $\ell$ is a weak Lefschetz element.

**Proof.** It suffices from Corollary 3.3 to show that the map $A_{d/2-1} \xrightarrow{\ell} A_{d/2}$ is injective, given via multiplication by some nonzero $\ell = a_{d+1}x_{d+1} + a_{d+2}x_{d+2} + \cdots + a_n d_n$. Let $M$ be the matrix for the map, where the bases of the domain and image are given by those in Proposition 4.6. From Lemma 2.4, we know that $M$ has at least as many rows as there are columns.

We claim that there exists a collection of $k \times k$ minors of $M$ with values $a_{d+1}^k, a_{d+2}^k, \ldots, a_n^k$, where $k$ is the cardinality of $A_{d/2-1}$, or the number of columns in $M$. This implies the theorem as any nonzero $\ell$ has some nonzero coefficient $a_j$, and thus $M$ would have a $k \times k$ minor $a_j^k$, implying that $M$ is full rank.

To prove the claim, pick any $j \in \{d+1,d+2,\ldots,n\}$ and order the bases of both $A_{d/2-1}$ and $A_{d/2}$ in lexicographic order based on the ordering of $A_1$ given by

$$x_j > x_{d+1} > \cdots > x_{j-1} > x_{j+1} > \cdots > x_n.$$  

One easily verifies that if we reorder the rows and columns of $M$ in such a manner, the first $k \times k$ minor is lower triangular with determinant $a_j^k$.\]
We now move onto the case where \( d \) is odd, where begin with two important lemmas.

**Lemma 4.8.** Let \( F \) be a facet of \( C(n,d) \) that does not contain \( n \). Then any non-face of cardinality \( \frac{d+1}{2} \) must contain some \( k \), where \( k \not\in F \) and \( k \neq n \).

**Proof.** Once again, we can assume that \( F \) is the facet \( \{1,2,\ldots,d\} \), as the argument for general \( F \) is completely analogous. Indeed, suppose for contradiction that we have a non-face of cardinality \( \frac{d+1}{2} \) that contains \( n \). The remaining \( (d-1)/2 \) vertices form a face by Lemma 4.5, as cyclic polytopes of odd dimension.

**Theorem 4.11.** For odd \( d \), polytopes of odd dimension.

**Proof.** Following the notation of Lemma 2.1, the ideal \( I_\Delta \) doesn’t contain any degree \( \frac{d+1}{2} \) polynomial with nonzero \( x_n^{\frac{d+1}{2}} \) coefficient.

**Proof.** Let \( G \) be a non-face of \( \Delta \) of size \( \frac{d+1}{2} \). Then by Lemma 4.8, there exists \( k \in G, k \not\in F, k \neq n \). Thus, \( x_G \) will be divisible by \( x_k \), so if it is of degree \( \frac{d+1}{2} \), then it can’t contain a term \( x_n^{\frac{d+1}{2}} \). This holds for any non-face \( G \), so it also holds for any polynomial in the ideal.

The following is an immediate corollary of Lemma 4.9.

**Lemma 4.10.** The element \( x_n^{\frac{d+1}{2}} \) represents a nonzero element in \( A_{(d+1)/2} \).

With this lemma, we can proceed to the proof of delineating the weak Lefschetz elements for cyclic polytopes of odd dimension.

**Theorem 4.11.** For odd \( d \), a nonzero \( \ell \) is weak Lefschetz if and only if all minors of \( M(\theta, \ell) \) with columns given by \( \{x_1, x_{i_1}, x_{i_2}, \ldots, x_{i_{d-1}}, x_n\} \) are nonzero, for any facet of \( P \{x_1, x_{i_1}, \ldots, x_{i_{d-1}}\} \) that does not contain \( x_n \).

**Proof.** We will prove that the determinant of the map \( A_{(d-1)/2} \xrightarrow{\ell} A_{(d+1)/2} \) factors precisely into a product of linear terms corresponding to the declared facets.

Take any facet \( F \) not containing \( x_n \). We now view \( A \) as its isomorphic image \( K[x_i : i \not\in F]/\phi_F(I_\Delta) \), adopting the notation from Lemma 2.1. Recall that this essentially means that we are expressing all variables corresponding to elements in \( F \) as a linear combination of variables corresponding to elements not in \( F \). Let \( \sum_{i \not\in F} \alpha_i x_i \) be any representative of \( \ell \) in \( K[x_i : i \not\in F]/\phi_F(I_\Delta) \).

By Proposition 4.6, we have the following explicit basis for \( A_{(d-1)/2} / G \) given by

\[
B := \{x_{i_1} \cdot \ldots \cdot x_{i_{(d-1)/2}} : i_j \in \{d+1, d+2, \ldots, n\} \text{ for all } 1 \leq j \leq k\}.
\]

By Lemma 4.10, \( x_n^{\frac{d+1}{2}} \) represents a nonzero element in \( A_{(d+1)/2} \). Thus, we can complete it to a basis \( B' \) for \( A_{(d+1)/2} \).

Let \( M \) be the matrix for the linear transformation \( A_{(d-1)/2} \xrightarrow{\ell} A_{(d+1)/2} \) with respect to \( B \) and \( B' \). We claim that if \( \alpha_n = 0 \), the row of \( M \) corresponding to the basis element represented by \( x_n^{\frac{d+1}{2}} \) are all 0’s. That is, we claim that the element represented by \( x_n^{\frac{d+1}{2}} \) is not in the image.

Take any representative of any element in \( A_{(d-1)/2} \), it is a polynomial \( f \) of degree \( \frac{d-1}{2} \). Multiply this polynomial by a linear element with 0 coefficient in front of \( x_n \) gives a polynomial \( g \) that has 0 coefficient in front of the monomial \( x_n^{\frac{d+1}{2}} \). But notice that by Lemma 4.9, the ideal \( \phi_F(I_\Delta) \) doesn’t contain any degree \( \frac{d+1}{2} \) polynomial with nonzero \( x_n^{\frac{d+1}{2}} \) coefficient. Thus, the element represented by \( g \) cannot be represented by \( x_n^{\frac{d+1}{2}} \), because their difference has nonzero coefficient in front of \( x_n^{\frac{d+1}{2}} \) and thus not in the ideal \( \phi_F(I_\Delta) \).

Claim proved.

The above claim implies that \( \ell \) is not weak Lefschetz if for some facet \( F \) not containing \( x_n \), \( \phi_F(l) \) has 0 coefficient in front of \( x_n \). We now show that \( \ell \) is weak Lefschetz if for any facet \( F \) not containing \( x_n \), \( \phi_F(l) \) has nonzero coefficient in front of \( x_n \).
Choose one specific facet $F_0$ not containing $x_n$. Let $M_0$ be the matrix for the linear transformation $A_{(d-1)/2} \xrightarrow{\ell} A_{(d+1)/2}$ described above, for this chosen facet $F_0$. Take any $l \in A_1$. Let $\phi_F(l)$ be represented by $\sum_{i \in F_0} \alpha_i x_i$. Then for each facet $F$ not containing $x_n$, $\phi_F(l)$ is represented by $\sum_{j \notin F} \beta_j x_j$, where $\beta_j$ is a linear expression of $\{\alpha_i : i \notin F_0\}$. This is because the maps $\phi_F$ are essentially expressing elements corresponding to vertices in $F$ as linear expressions of vertices corresponding to vertices not in $F$. In particular, $\beta_n$ is a linear expression of the $\alpha_i$'s. For example, $\beta_n = \alpha_n$. We claim that $\beta_n$'s will be distinct up to multiplication by a scalar for distinct $F$'s.

Assume otherwise, then we have $\beta_n = 0$ if and only if $\beta_n = 0$. By Lemma 2.2, the above statement is equivalent to $M(\theta, \ell)_{F_1, x_n}$ is singular if and only if $M(\theta, \ell)_{F_2, x_n}$ is singular for any fixed $\ell$. But by choosing $\ell = x$ for some $i \in F_2 \setminus F_1$, we see that $M(\theta, \ell)_{F_1, x_n} \neq 0$ because the complementary minor of $x_i$ has nonzero coefficient. (In fact, all minors of $M(\theta)$ does.) But $M(\theta, \ell)_{F_1, x_n} = 0$. Contradiction. Thus, the $\beta_n$'s are distinct up to multiplication by a scalar for distinct $F$'s.

Recall that $\det M_0 = 0$ if $\phi_F(l)$ has 0 coefficient in front of $x_n$. Thus, $\det M_0 = 0$ if $\beta_F = 0$. Thus, $\beta_F$ divides $\det M_0$, viewed as polynomials in variables $\{\alpha_i : i \notin F_0\}$. Since $\det M_0$ has degree at most the dimension of $A_{(d-1)/2}$, which equals $(n-d/2-3/2)$ by Lemma 1.5. We now show that this is the exact number of facets $F$ not containing $x_n$. First, we notice that given a facet $F$ that does not contain $x_n$, it has to contain $x_1$. This is because otherwise, there are $d$ and thus an odd number of, vertices in the facet between $x_1$ and $x_n$, which contradicts Lemma 4.4. Furthermore, by the evenness condition, it is easy to see that all the other $d-1$ elements after $x_1$ in the facets come in consecutive pairs. Conversely, all choices of consecutive pairs from $x_2 \cdots x_{n-1}$ union $x_1$ is a facet. Therefore, the number of facets not containing $x_n$ equals the number of ways to choose $(d-1)/2$ consecutive pairs from $x_2 \cdots x_{n-1}$, which equals $(n-d/2-3/2)$ by a simple combinatorial argument.

Since each $\beta_n$ is distinct and they all divide $\det M_0$, $\det M_0$ is a multiple of $\prod_F \beta_n$. Since the degrees match, it’s a scalar multiple.

We now show that the determinant is not the zero polynomial. This follows from the fact that Stanley-McMullen have shown the existence of a weak Lefschetz element for the boundary complex of a simplicial polytope, and hence in this case there is at least some choice of $a_i$ that yields a nonzero value.

And thus $\det M_0$ is a nonzero multiple of $\prod_F \beta_n$. So $\ell$ is weak Lefschetz if all $\beta_n$ non-zero, which is equivalent to $M(\theta, \ell)_{F, x_n}$ singular by Lemma 2.2.

### 4.3 Cross Polytopes

Define the dimension $r$ cross polytope to the be convex hull of

$$\{\pm e_i|1 \leq i \leq r\} \subset \mathbb{R}^r$$

Then, let $\Delta$ be the boundary of the $r$-dimensional cross-polytope, and label the vertex $e_i$ by $x_i$ and $-e_i$ by $y_i$. The minimal non-faces of this polytope are $\{x_i, y_i\}$, so we get that

$$K[\Delta] = K[x_1, y_1, x_2, y_2, \ldots, x_r, y_r]/(x_1 y_1, x_2 y_2, \ldots, x_r y_r)$$

We consider the geometric isop $\theta = (\theta_1, \ldots, \theta_r)$, where $\theta_i = x_i - y_i$. Then, we have that

$$A = K[\Delta]/(\theta) \cong K[x_1, \ldots, x_r]/(x_1^2, x_2^2, \ldots, x_r^2)$$

We see that $A$ is a graded ring over $K$ and that a basis for each component $A_d$ is given by

$$\{x_I | I \in \binom{[n]}{d}\}$$

where $x_I := \prod_{i \in I} x_i$. Further, given $\ell = \sum_{i=1}^r a_i x_i + b_i y_i$, we can write

$$\ell = \sum_{i=1}^r c_i x_i$$


for $c_i := a_i - b_i$. Since any monomial in $A$ with a square term is 0, we get the following multiplication relation:

$$x_i x_I = \begin{cases} x_i J & i \notin I \\ 0 & i \in I \end{cases}$$

Now, consider $\ell : A_d \to A_{d+1}$ as a linear operator, and let $M(r, d)$ the matrix representing $\ell$ matrix under this basis. We index entries of $M(r, d)$ by $M(r, d)_{J, I}$, where $I \in \binom{[r]}{d}$, $J \in \binom{[r]}{d+1}$. Then, our multiplication relation implies that

$$M(r, d)_{J, I} = \begin{cases} 0 & I \notin J \\ c_i & J \setminus I = \{i\} \end{cases}$$

**Example.** Consider $A$ when $r = 4$ we have in this case that the graded components of $A$ are

$$A_0 = K$$
$$A_1 = K(x_1, x_2, x_3, x_4)$$
$$A_2 = K(x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, x_3 x_4)$$
$$A_3 = K(x_2 x_3 x_4, x_1 x_3 x_4, x_1 x_2 x_4, x_1 x_2 x_3)$$
$$A_4 = K(x_1 x_2 x_3 x_4)$$

Further, consider $\ell = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4$ as a linear operator from $A_1 \to A_2$. The matrix under this basis is

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 x_2 & c_2 & c_1 & 0 & 0 \\ x_1 x_3 & c_3 & 0 & c_1 & 0 \\ x_1 x_4 & c_4 & 0 & 0 & c_1 \\ x_2 x_3 & 0 & c_3 & c_2 & 0 \\ x_2 x_4 & 0 & c_4 & 0 & c_2 \\ x_3 x_4 & 0 & 0 & c_4 & c_3 \end{pmatrix}$$

For the remainder of this subsection, let $G(r, d)$ be the graph with vertices subsets of $[r]$ of size $d$ and $d + 1$, and where there is an edge of weight $c_i$ between $I \in \binom{[r]}{d}$ and $J \in \binom{[r]}{d+1}$ whenever $J = I \cup \{i\}$

**Proposition.** Let $\Delta$ be the boundary of the $d$-dimensional cross polytope. Consider $\ell = \sum_{i=1}^{r} c_i x_i \in K[\Delta]/(\emptyset)$

1. If $r$ is odd, then $\ell$ is weak Lefschetz if and only if $c_i \neq 0$ for all $i$.
2. If $r$ is even, then $\ell$ is weak Lefschetz if and only if $c_i = 0$ for at most one $i$.

Before we prove this proposition, we show some helpful lemmas.

**Lemma 4.12.** Every maximal minor of $M(r, d)$ has form $\alpha c_1^{e_1} \ldots c_r^{e_r}$, where $\alpha \in \mathbb{Z}$ and

$$e_i = \text{number of occurrences of } i \text{ in the rows indexing the minor} - \binom{r-1}{d-1}$$

**Proof.** A maximal minor in $M(r, d)$ can be identified by the family $\mathcal{J} \subset \binom{[r]}{d+1}$ of sets indexing the rows. Let $G'$ denote subgraph of $G(r, d)$ induced by restricting to $\binom{[r]}{d} \cup \mathcal{J}$. Looking at the Leibniz determinant formula,
one can observe that a nonzero term in the sum over permutations corresponds to a perfect matching \( \mathcal{M} \) in \( G' \). In particular, the term is equal to the product of edge weights of edges in the matching, ie.

\[
\prod_{e \in \mathcal{M}} \text{wt}(e) = \prod_{J \in \mathcal{J}} c_{J - \sigma(J)} = \frac{\prod_{J \in \mathcal{J}} \prod_{i \in J} c_i}{\prod_{J \in \binom{[r]}{d+1}} \prod_{i \in J} c_i}.
\]

The number of occurrences of \( c_i \) in the numerator is just the number of occurrences of \( i \) in \( \mathcal{J} \), and the number of occurrences in the denominator is \( \binom{d-1}{r-1} \).

Note that in the odd case, this tells us that the determinant of the middle map from \( A_{\frac{r}{2}-1} \) to \( A_{\frac{r}{2}+1} \) has form \( \alpha(c_1 \ldots c_r)^{\frac{1}{2}}(\frac{r}{2})_1 \) and \( \alpha \in \mathbb{Z} \).

**Lemma 4.13.** When \( r \) is even, any maximal minor of \( M(r, \frac{r-1}{2}) \) has at most one \( c_i \) = 0.

**Proof.** Let \( d = r/2 - 1 \). Using the same bipartite matching setup as in the proof of lemma 4.12, we see that in order to find a minor such that \( e_{r-1} = e_r = 0 \), we need to find a maximal bipartite matching in \( G(r, d) \) so that all \( J \in \binom{[r]}{d+1} \) such that \( r \in J \) must be matched to a subset \( I \) containing \( r \), and similarly for \( r - 1 \), as otherwise a factor of \( c_r \) would appear in the minor. There are \( \binom{d+1}{r} - \binom{r-2}{d+1} \) such \( J \), but only \( \binom{r}{d} - \binom{r-2}{d} \) such \( I \). The remaining \( \binom{n-2}{d+1} \) subsets \( I \) of size \( d \) containing neither \( r \) nor \( r - 1 \) must be matched to the remaining \( \binom{n-2}{d+1} \) subsets of size \( d + 1 \) containing neither \( r \) nor \( r - 1 \), but when \( d = r/2 - 1 \), \( \binom{n-2}{d+1} < \binom{n-2}{d} \), so this is not possible.

**Proof.** Proof of proposition 4.3

1. \( r \) is odd. It suffices to consider the multiplication by \( \ell \) map from \( A_{\frac{r}{2}-1} \) to \( A_{\frac{r}{2}+1} \). As mentioned above, the determinant of this matrix is \( \alpha(c_1 \ldots c_r)^{\frac{1}{2}}(\frac{r}{2})_1 \) for some \( \alpha \in \mathbb{Z} \). To show \( \alpha \neq 0 \), it suffices to make the substitution \( c_i = 1 \) for all \( i \). The resulting matrix is the incidence matrix for the two middle ranks of the the Boolean poset on an odd number of elements, which is well known to be invertible [1].

2. \( r \) is even. It suffices to consider the multiplication by \( \ell \) map from \( A_{\frac{r}{2}-1} \) to \( A_{\frac{r}{2}} \). We will reduce this case to the odd case to show that there exists a maximal minor of \( M(r, \frac{r}{2}) \) of form \( \alpha c_1 c_2 \ldots c_{r-1} \), with \( \alpha \neq 0 \). Then we are done, since by symmetry, there exists a maximal minor with nonzero determinant and \( e_i = 0 \) for any \( i \), and by lemma 4.13 we cannot have \( e_i = e_j = 0 \) for \( i \neq j \).

First, assume \( c_1, \ldots, c_{r-1} \neq 0 \). Thus \( M(r-1, r/2) \) is invertible and \( M(r-1, r/2-1) \) contains a nonzero maximal minor. Say this minor is corresponds to rows \( \mathcal{J} \subset \binom{[r-1]}{r/2-1} \). Let \( \mathcal{J}' = \{ J \cup \{ r \} | J \in \mathcal{J} \} \).

Then the desired minor in \( M(r, r/2-1) \) corresponds to rows \( \binom{[r-1]}{r/2} \cup \mathcal{J}' \subset \binom{[r]}{r/2} \). Let \( H \) be the subgraph of \( G(r, r/2-1) \) obtained by restricting to vertices \( \mathcal{J}' \). Note that \( H \) decomposes into two disjoint bipartite graphs \( H_A \) and \( H_B \) on vertices containing and not containing \( r \) respectively, so a matching in \( H \) can be thought of as a pair of matchings, one in \( H_A \) and one in \( H_B \). Say that \( A_+ \) and \( A_- \) are the number of even and odd matchings respectively of \( H_A \), and that \( B_+ \) and \( B_- \) are the number of even and odd matchings respectively of \( H_B \). Without loss of generality, say that \( A_+ > A_- \) and \( B_+ > B_- \) (otherwise we could just swap the ordering of two vertices/rows). But then the number of even and odd matchings of \( H \) is \( A_+ B_+ + A_- B_- > A_+ B_- + A_- B_+ \) respectively, and in particular these two quantities are unequal, so we conclude that \( \alpha \neq 0 \).

Finally, since all \( I \in \binom{[r]}{\frac{r-1}{2}} \) such that \( r \in I \) must be matched to \( J \in \binom{[r]}{\frac{r}{2}} \) such that \( r \in J \), we know that \( e_r = 0 \).
4.4 Investigating when a minor is 0

Here, we investigate conditions for when a minor of $M(r, d)$ is zero. In light of lemma 4.12, we can specialize each $c_i$ to be 1. Call this new matrix $M^*(r, d)$. Then, we are working with a matrix with rows and columns indexed by $\binom{r}{d}$ and $\binom{r}{d+1}$, respectively, where

$$M^*(r, d)_{J,I} = \begin{cases} 1 & I \subseteq J \\ 0 & \text{otherwise} \end{cases}$$

Now, let $\mathcal{M}$ be the linear matroid over $\mathbb{R}$ on the rows of this matrix. Let $R_J \in \mathbb{R}^{(d)}$ be the $J$th row of $M^*$ and let $V = \binom{r}{d}$ be the vertex set of $\mathcal{M}$.

In this section, assume we are working in a field of characteristic 0.

4.4.1 The relations simplex

Here, we discuss how to associate a simplex $\Phi$ to a set $J \subseteq 2^V$ and state some basic results about how they relate to dependencies between the $R_J$s.

Definition. The relations simplex $\Phi$ of $J$ is the simplex on $[r]$ generated by $J \in J$.

Corollary 4.14. $\Phi$ is pure and the facets of $\Phi$ are $J$.

The primary motivation for the preceding definition is that assigning weights to the facets of $\Phi$ is in correspondence with dependencies between the $R_J$s.

Definition. Let a weighting of $\Phi$ be a function $\text{wt}$ from facets of $\Phi$ ($= J$) to $K$. We extend $\text{wt}$ to all of $\Phi$ via

$$\text{wt}(I) = \sum_{J \in J : J \supseteq I} \text{wt}(J)$$

Definition. If $\text{wt}(I) = 0$ for all $|I| = d$, call $\text{wt}$ a dependent weighting.

Proposition 4.15. Given $J \subseteq 2^V$, there is a linear dependency between between the $R_J$s if and only if there exists a non-zero dependent weighting of $\Phi$.

Proof. First, we prove

Lemma 4.16. For $\beta_J \in K$,

$$\sum_{J \in J} \beta_J R_J = 0 \iff \text{wt}(J) = \beta_J, \text{ defines a dependent weighting}$$

Proof. We have $\sum_{J \in J} \beta_J R_J = 0$ if and only if $\sum_{J \supseteq I} \beta_J = 0$ for all $|I| = d$ if and only if $\text{wt}(J) = \beta_J$ defines a dependent weighting.

The claim follows from the lemma.

Corollary 4.17. If $J$ is a circuit in $\mathcal{M}$, then $\Phi$ is connected.

Proof. If $\Phi$ were not connected, one of the connected components of $\Phi$ must have a non-zero dependent weighting. This component describes a dependent subset of $J$, thus $J$ is not a circuit.
4.4.2 The case \( d = 1 \)

First, we investigate \( \mathcal{M} \) when \( d = 1 \) but for general \( r > 2 \). We characterize the independent sets of \( \mathcal{M} \), and, as a corollary, say exactly when \( M(r,d) \) is non-zero.

In the \( d = 1 \) case, \( \Phi \) is a graph. Then, let \( E(\Phi) \) be the set of edges in \( \Phi \) and \( V(\Phi) \) the set of vertices.

**Proposition 4.18.** \( \mathcal{J} \) is an independent set if and only if for each connected component \( \Psi \subset \Phi \), \( |E(\Psi)| \leq |V(\Psi)| \) and \( \Phi \) contains no even cycles.

**Proof.** We prove the contrapositive, that \( \mathcal{J} \) is a dependent set if and only if one of the following holds:

1. There is a connected component \( \Psi \subset \Phi \) with \( |E(\Psi)| > |V(\Psi)| \)
2. \( \Phi \) contains an even cycle

First, let \( \mathcal{J} \) be a set with a component \( \Phi \subset \Phi \) with \( |E(\Psi)| > |V(\Psi)| \). Then, for \( \mathcal{J} \subset E(\Psi) \), the row vectors \( R_j \) have only \( |V(\Psi)| \) columns in which they have non-zero values. Then, the span of the \( R_j \) is at most of dimension \( |V(\Psi)| \) but we have \( |E(\Psi)| > |V(\Psi)| \) vectors, meaning there is a dependency between them.

Now, say \( \Phi \) contains an even cycle. Consider the weighting \( w \) which assigns weight 0 to edges not in the cycle, and alternates between weights 1 and \(-1\) for edges in the cycle. We see this is a dependent weighting.

Now, let \( \mathcal{J} \) be a dependent set. If condition (1) holds, we are done. Moving forward, assume condition (1) fails. Let \( \mathcal{C} \subset \mathcal{J} \) be a circuit, which exists since \( \mathcal{J} \) is dependent. Let \( \Psi \) be the relations simplex to associated \( \mathcal{C} \). From corollary \( \ref{cor:4.17} \) we know \( \Psi \) is connected.

Let \( L \) be a matrix with row vectors \( R_j \) for \( j \in \mathcal{J} \). Let \( q \) be the number of 1s in \( L \), which we know is equal to \( 2|E(\Psi)| \). If such a matrix has a column with only 1 one in it, then that row must have coefficient 0 in a linear dependency. This would contradict \( C \) being a circuit, so every column with at least one 1 must have at least two. Then, since there are \( |V(\Psi)| \) columns which are not all zeroes, \( q \geq 2|V(\Psi)| \), so in particular \( |E(\Psi)| \geq |V(\Psi)| \). Since we are not in case 1, \( |E(\Psi)| = |V(\Psi)| \).

Then, there are exactly 2 ones in each column of \( L \). Then, this says each vertex of \( \Psi \) has degree 2. Since \( \Psi \) is connected, \( \Psi \) is a cycle.

Now we must show \( \Psi \) is an even cycle. Say the linear dependency between the \( R_j \), \( j \in E(\Psi) \) is

\[
\sum_{j \in \mathcal{J}} \beta_j R_j = 0.
\]

Then, if \( R_j, R_{j'} \) have a one in the same column, we have \( \beta_j = -\beta_j \). Since \( \Phi \) is connected, we can take all \( \beta_j \)s to be \( \pm 1 \). Now, since the weight at each vertex in \( V(\Phi) \) is 0, the sum of the weights of all vertices is 0, so

\[
2 \sum_{j \in \mathcal{J}} \beta_j = 0.
\]

Thus, since \( \beta_j = \pm 1 \), the number of \( \beta_j \)s is even, so \( |\mathcal{J}| = |V(\mathcal{J})| \) is even. Thus, \( \Phi \) contains an even cycle. \( \square \)

**Corollary 4.19.** A minor with rows \( \mathcal{J} \subset 2^V \) is non-zero if and only if each component of \( \Phi \) is odd unicyclic.

**Proof.** A minor with rows \( \mathcal{J} \subset 2^V \) has the constraint that \( |\mathcal{J}| = r \). Then, since \( V(\mathcal{J}) = r \), this says \( |E(\Phi)| = |\mathcal{J}| = r = |V(\mathcal{J})| \). We know a minor is non-zero if and only if \( \mathcal{J} \) is an independent set. If every component of \( \Phi \) is odd unicyclic, we see this satisfies the conditions of proposition \( \ref{prop:4.18} \) so the minor is non-zero. If the minor is non-zero, then \( |E(\Phi)| = |V(\Phi)| \) in combination with condition (1) of proposition \( \ref{prop:4.18} \) says that each connected component of \( \Phi \) is unicyclic. Further, as to not violate condition (2) each connected component is odd unicyclic. \( \square \)

5 Two non-matroidal counterexamples

We know that the weak Lefschetz property will not hold for a general simplicial polytope and geometric lsop \( \theta \), as seen in the following counterexamples.
5.1 Tetrahedron Boundary Counterexample

Let \( x_1, \ldots, x_8 \) be vertices in \( \mathbb{R}^3 \), given by the coordinates \((1,0,0), (0,1,0), (0,0,1), (0,1,1), (1,0,1), (1,1,0), (1,1,1), (-1,-1,-1)\), respectively. Consider the boundary of the tetrahedron \( \Delta_0 \) given by the following facets: \{348\}, \{358\}, \{158\}, \{168\}, \{268\}, \{247\}, \{267\}, \{347\}, \{357\}, \{157\}, \{167\}. This gives rise to a geometric interpretation of \( \Delta_0 \) as

![Diagram of a tetrahedron with vertices and edges labeled]

We let the linear system of parameters \((\theta)\) be given by the coordinates of our vertices, such that \( \theta_i \) is a vector corresponding to the \( i^{th} \) coordinate of each \( v_j \), in order from \( v_1 \) to \( v_8 \). In order for an element \( \ell = a_1x_1 + \cdots + a_nx_n \) to be Weak Lefschetz, in particular the middle map from \( A_1 \rightarrow A_2 \) given by multiplication by \( \ell \) must be invertible, as both \( A_1 \) and \( A_2 \) have dimension five. After establishing bases for the graded components \( A_1 \) and \( A_2 \), a simple computation gives the determinant of our middle map as \((-1)(a_1+a_3)(a_1^2+a_2^2+a_6^2+a_7a_8)\). In order for our desired map to be invertible, we need this determinant to be nonzero.

To see that the invertibility of the desired map is not determined by our matrix \( \hat{M}(\ell) \), and hence the Weak Lefschetz property is not in general matroidal for a simplicial complex, it is sufficient to find two evaluations of the coefficients of \( \ell \) such that both evaluations produce the same set of minors of \( \hat{M}(\ell) \) being zero or nonzero, while one evaluation makes the \( \ell \)-map determinant zero and the other makes this determinant nonzero. Indeed, it is not difficult to check that if we set \((a_1, \ldots, a_8) = (0,0,2,0,0,1,4)\), we get a zero determinant for our \( \ell \)-map, whereas when we set \((a_1, \ldots, a_8) = (0,0,0,6,0,0,3,5)\), we get that our desired map is invertible. However, both matrices corresponding to the stated linear system of parameters and the respective coefficients of \( \ell \) produce the same sets of zero and nonzero minors.

5.2 A Remark on Cross-polytopes

In fact, depending on \( \theta \), the weak Lefschetz property for \( \ell \) can be arbitrarily “far” from being matroidal. In particular, take our simplicial complex \( \Delta \) to be the boundary of the cross polytope in \( n \) dimensions, which has \( 2n \) vertices. We can pick a valid lsop given by choosing \( 2n \) generic points in the ambient \( n \)-dimensional space. Running code in Sage reveals that the weak Lefschetz property for \( \ell = \sum_{i=1}^{2n} a_ix_i \) is controlled by a polynomial that seems to scale at least linearly with the dimension \( n \). For example, letting \( n = 3 \) yields a middle map whose determinant is an irreducible polynomial of the form

\[
c_1a_7^2a_5 + c_2a_4a_3^2 + c_3a_2^2a_6 + c_4a_4a_5a_6 + c_5a_2^2a_6 + c_6a_4a_5^2 + c_7a_5a_6^2,
\]

where all \( c_i \) are constants (which will depend on \( \theta \)). The \( n = 4 \) case requires \( a_8 \) to be nonzero, as well as the following relies on an irreducible degree 4 polynomial:

\[
c_1a_6^4 + c_2a_6^3a_8 + c_3a_2^2a_7a_8 + c_4a_5a_6a_7a_8 + c_5a_5a_7a_8 + c_6a_5a_6^2a_8 + c_7a_6a_7^2a_8 + c_8a_5a_8
+c_9a_2^2a_8^2 + c_{10}a_5a_6a_8^2 + c_{11}a_2^2a_8^2 + c_{12}a_5a_7a_8^2 + c_{13}a_7a_8^2 + c_{14}a_5a_8^3 + c_{15}a_6a_8^3 + c_{16}a_7a_8^3.
\]

Following this pattern, the \( n = 5 \) case is even more complicated, depending on the vanishing of an irreducible degree 10 polynomial. The particular polynomials themselves are not as relevant for this discussion, and details will be spared.

These calculations imply that the choice of \( \theta \) has important ramifications in simplifying the condition for \( \ell \) to be weak Lefschetz. Indeed, we saw in Proposition 4.3 that if we pick \( \theta \) to be the geometric lsop,
we do obtain a matroidal condition for the weak Lefschetz property to hold. The core difference between the generic and geometric $\theta$ are the linear dependencies. In the latter case, there are $n$ pairs of vectors that are multiples of one another, and as we saw in the case of $n = 3$, breaking all such dependencies led to a weak Lefschetz condition that depended on an irreducible polynomial of degree 3. Would breaking fewer dependencies lead to polynomials of lower degree? The answer is in the affirmative. Recall that with no dependencies broken, the determinant controlling $\ell$ being weak Lefschetz is precisely the product $a_4 \cdot a_5 \cdot a_6$. Breaking one dependency by moving the vertex associated to $a_6$, the determinant of the middle map takes the form

$$a_5 \cdot a_4 \cdot (c_1 a_4 + c_2 a_6).$$

Curiously, we see that the $a_6$ term in the original determinant has been replaced with a more complicated linear term. If we shift the vertex associated to $a_5$, we obtain

$$a_4 \cdot (c_1 a_4 a_5 + c_2 a_4 a_6 + c_3 a_5 a_6),$$

which follows a similar logic as before. It appears that the more generic our $\theta$, the more unwieldy our determinant. We do not understand this phenomenon very well at all, so needless to say, there is more to uncover in this direction.

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