Weak Lefschetz Property for Boundaries of Simplicial Polytopes

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Abstract

We investigate the weak Lefschetz property for Artinian reductions $A = K[\Delta]/(\underline{\theta})$ of the Stanley-Reisner ring $K[\Delta]$ of a simplicial complex Δ by a linear system of parameters $\underline{\theta} = (\theta_1, \ldots, \theta_d)$. It is known that deciding whether or not the ring A is Artinian is *matroidal*, in the sense that it is controlled only by the matroid $M(\underline{\theta})$ for the coefficient matrix specifying $\underline{\theta}$. We consider the *weak Lefschetz property* for a degree one element ℓ in $K[\Delta]$, meaning that multiplication by ℓ from $A_i \to A_{i+1}$ has full rank for all i, and ask when it is *matroidal* in the sense of of being controlled only by the lifted matroid $\hat{M}(\underline{\theta}, \ell)$ for the coefficient matrix of $\underline{\theta}$ and ℓ.

It is known from work of Stanley [3], McMullen and others that when $K = \mathbb{R}$ and Δ is the boundary of a simplicial polytope, there exists a choice $\underline{\theta}$ and degree one elements ℓ having the weak Lefschetz property. In this context, we find some examples where the weak Lefschetz property **is not** matroidal, and others where it **is** matroidal.

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1 Introduction

Let Δ be a simplicial complex on n vertices. Its Stanley-Reisner ring is the quotient

$$K[\Delta] := K[x_1, \ldots, x_n]/I_\Delta$$

where the ideal I_Δ is generated by all squarefree monomials $x^A := \prod_{i \in A} x_i$ for which A is *not* a face of Δ. The early history and applications of $K[\Delta]$ in the theory of face numbers for convex polytopes make use of quotients $A = K[\Delta]/(\underline{\theta})$ in which $\underline{\theta} = (\theta_1, \ldots, \theta_d)$ is a *linear system of parameters (lsop)*, meaning that

- each θ_i is an element of $K[\Delta]$ of degree one,
- the quotient A is Artinian (i.e., of Krull dimension zero; equivalently, finite-dimensional over K), and
- d is the smallest cardinality possible (which is one more than the dimension of Δ).

See Stanley [4, Chap. II,III] for history, background, and terms not defined here.

The condition under which a candidate set of degree one elements $\theta = (\theta_1, \ldots, \theta_d)$ form an lsop is due to Kind and Kleinschmidt [4, Lemma III.2.4(a)]: the coefficient matrix (c_{ij}) in $K^{d\times n}$ where $\theta_i = \sum_{j=1}^n c_{ij}x_j$ should have the property that every submatrix obtained by restricting to a set of columns F indexing a face F of Δ is of maximal rank $\#F$. This condition is *matroidal* in the sense that it depends only on the matroid $M(\underline{\theta})$ associated with the matrix (c_{ij}), meaning the specification of which subsets of its columns are linear independent or dependent.

The interest in the Artinian reduction $A = K[\Delta]/(\underline{\theta})$ has often centered on deciding whether it contains an element of degree one $\ell = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ for which the maps $A_i \xrightarrow{\cdot\ell} A_{i+1}$ given by multiplication by ℓ have maximal rank for all i, where A_i is the i^{th} graded component. In this case, one says that that ℓ is a *weak Lefschetz element*; see Migliore and Nagel [2] for more on the weak Lefschetz property. Having fixed $\underline{\theta}$, one might ask whether this property depends solely on the matroid $M(\underline{\theta})$ and on the *lifted matroid* $M(\underline{\theta}, \ell)$ corresponding to the columns of the matrix in $K^{(d+1)\times n}$ having the vectors of $(\theta_1, \ldots, \theta_d, \ell)$ as its rows; in this case we will say that the weak Lefschetz property for ℓ is *matroidal.* This is the question we examine here.

Question. *For which simplicial complexes Δ and lsop $\underline{\theta}$ is the weak Lefschetz property for ℓ matroidal?*

From now on we will assume that Δ is the boundary of a d-dimensional, convex, simplicial polytope P in $\mathbb{R}^d$, containing the origin in its interior. It is not hard to see that in this setting, one can form an lsop $\underline{\theta}$ by taking the columns of the matrix (c_{ij}) to be the vectors $v_1, \ldots, v_n$ which are the vertices of P; we will call this the *geometric* choice of lsop. Unless specified, we will always work with the geometric lsop. Work of Stanley and McMullen (see [4, §III.1]) shows that there always exist weak Lefschetz elements ℓ in this context.[1]

2 Preliminaries

Definition. Recall that $M(\underline{\theta})$ is the $d \times n$ coefficient matrix of θ. Given any facet F of Δ, we use $M(\underline{\theta})_F$ to denote the $d \times d$ minor of $M(\underline{\theta})$ with columns indexed by vertices in F. Furthermore, given any vertex $x \notin F$, we use $M(\underline{\theta}, \ell)_{F,x}$ to denote the $(d+1) \times (d+1)$ minor of $M(\underline{\theta}, \ell)$ with columns indexed by vertices in $F \bigcup \{x\}$

Lemma 2.1. *Let F be a facet of Δ. We have a canonical graded algebra isomorphism*

$$\phi_F : K[x_1, x_2, \cdots, x_n]/(\underline{\theta}) \to K[x_i : i \notin F]$$

given by $\bar{x_i} \mapsto x_i$ for all $i \notin F$. This gives rises to a graded algebra isomorphism

$$\psi_F : A \to K[x_i : i \notin F]/\phi_F(I_\Delta)$$

Proof. Notice that for any facet F of Δ, $M(\underline{\theta})_F$ is invertible. Thus, we can perform row reduction to express all elements in $\{x_i : i \in F\}$ as a linear combination of $\{x_i : i \notin F\}$. This gives us the isomorphism ϕ_F.

Furthermore, notice that $A = K[\Delta]/(\underline{\theta}) = K[x_1, x_2, \cdots, x_n]/(I_\Delta, \underline{\theta}) \cong K[x_i : i \notin F]/\phi_F(I_\Delta)$ via ϕ. ∎

Lemma 2.2. *Let $\ell = \sum_{i\in[n]} \alpha_i x_i$ (and resp., $\sum_{i\notin F} \beta_i x_i$) be any representative for some generic element in A_1 (and resp., its image under ψ_F). Then for any $i_0 \notin F$, $\beta_{i_0} = 0$ if and only if $\hat{M}(\underline{\theta}, \ell)_{F,x_{i_0}}$ is invertible.*

[1] In fact, they showed that there are always *strong* Lefschetz elements in this context, that is, those for which the maps $A_i \to A_{i+m}$ which multiply by ℓ^m all have maximal rank.

Proof. $\beta_{i_0} = 0$ if and only if applying row reduction to $\hat{M}(\underline{\theta}, \ell)_{F,x_{i_0}}$ to zero out the last row with columns except x_{i_0} automatically zeros out x_{i_0}. And this happens if and only if $\hat{M}(\underline{\theta}, \ell)_{F,x_{i_0}}$. ☐

The following well-known theorem is crucial for future calculations; a proof can be found in [4, II.6].

Lemma 2.3 (Dehn-Sommerville equations)**.** *The h-vector of the boundary of a simplicial d-polytope satisfies*

$$h_k = h_{d-k}, \quad \text{for } k = 0, 1, \ldots, d.$$

Remark. Since the h-vector enumerates the dimensions of the graded components of A as a K-vector space, we also have that $\dim_{\mathbb{C}} A_i = \dim_{\mathbb{C}} A_{d-i}$.

A final lemma further characterizes the h-vector. This is Theorem 1.1(b) in [4, III.1].

Lemma 2.4. *The h-vector for a d-dimensional simplicial polytope P is unimodal, and in particular*

$$h_1 \le h_2 \le \cdots \le h_{\lfloor \frac{d}{2} \rfloor} \quad \text{and} \quad h_{\lceil \frac{d}{2} \rceil} \ge h_{\lceil \frac{d}{2} \rceil + 1} \ge \cdots \ge h_d.$$

3 Reducing the Weak Lefschetz Property

A priori, checking that our candidate choice of a degree one element $\ell \in A_1$ is weak Lefschetz requires verifying that *every* possible multiplication by ℓ map $A_i \xrightarrow{\cdot \ell} A_{i+1}$ is full rank. It would be ideal if we could simplify the criterion. In this section, we first show that it suffices to show that the middle map $A_{\lfloor \frac{d-1}{2} \rfloor} \xrightarrow{\cdot \ell} A_{\lfloor \frac{d+1}{2} \rfloor}$ is full rank. We then apply the bipyramid construction in order to reduce the odd case to the even case.

3.1 The Gorenstein Property

It is known that $A = K[\Delta]/(\underline{\theta})$ is a *Gorenstein ring*. Since A is a graded algebra over the field K, this is equivalent to the product map $\langle \cdot, \cdot \rangle \colon A_i \times A_{d-i} \to A_d$ being a nondegenerate bilinear form. In particular, given some basis $\{a_j\}$ of A_i and basis $\{b_j\}$ for A_{d-1}, we have that

$$\langle a_j, b_k \rangle = \delta_{jk}. \tag{1}$$

Details can be found in Stanley [4, §II.5].

Using this fact, we can prove the following proposition that exploits this "duality." In particular, for the weak Lefschetz property it will suffice to check that only half of the maps are full rank.

Proposition 3.1. *Let $A = K[\Delta]/(\underline{\theta})$ For any $i \le \lfloor \frac{d}{2} \rfloor$, the map $A_i \xrightarrow{\cdot \ell} A_{i+1}$ is injective if and only if the map $A_{d-i-1} \xrightarrow{\cdot \ell} A_{d-i}$ is surjective.*

Proof. Let $\{x\}$ be a basis for A_d. Assume first that $A_i \xrightarrow{\cdot \ell} A_{i+1}$ is injective. Fix a basis $\{a_u\}_{u=1,2,\ldots,h_i}$ for A_i. By the injectivity assumption we know that $\cdot \ell$ maps A_i to an h_i dimensional subspace of A_{i+1}, so $\{a_u \cdot \ell\}_{u=1,2,\ldots,h_i}$ is a linearly independent set in A_{i+1}. Extend it to

$$(\{a_u \cdot \ell\}_{u=1,2,\cdots h_i}) \cup \left(\{b_u\}_{u=h_i+1,h_i+2,\cdots h_{i+1}}\right)$$

a basis of A_{i+1}. There exists a basis $\{c_v\}_{v=1,2,\cdots h_{d-i-1}}$ for A_{d-i-1} that is dual to $(\{a_u \cdot \ell\}_{u=1,2,\ldots h_i}) \cup (\{b_u\}_{u=h_i+1,h_i+2,\ldots h_{i+1}})$ under $\langle \cdot, \cdot \rangle$, in the sense of (1). In particular, $\langle a_u \cdot \ell, c_v \rangle = \delta_{uv}$ for $u, v = 1, 2, \ldots, h_i$. By definition of the bilinear form, $\delta_{uv} x = a_u \cdot \ell \cdot c_v = a_u \cdot c_v \cdot \ell$, and thus $\langle a_u, c_v \cdot \ell \rangle = \delta_{uv}$. By nondegeneracy, $\{c_v \cdot \ell\}_{v=1,2,\ldots,h_i}$ is a basis for A_{d-i} by Lemma 2.3. This gives that $A_{d-i-1} \xrightarrow{\cdot \ell} A_{d-i}$ is surjective.

Conversely, assume that $A_{d-i-1} \xrightarrow{\cdot \ell} A_{d-i}$ is surjective. This implies that $h_{d-i-1} \ge h_{d-i}$. Thus we can let $\{c_u\}_{u=1,2,\ldots h_{d-i}}$ in A_{d-i-1} be such that $\{c_u \cdot \ell\}_{u=1,2,\ldots,h_{d-i}}$ is a basis for A_{d-i}. We set $\{a_v\}_{v=1,2,\ldots h_i}$ as a basis of A_i dual to $\{c_u \cdot \ell\}_{u=1,2,\ldots,h_{d-i}}$, again in the sense of (1). Then $\langle a_v, c_u \cdot \ell \rangle = \delta_{uv}$ for $u, v = 1, 2, \ldots, h_{d-i}$. So $\delta_{uv} x = a_v \cdot c_u \cdot \ell = a_v \cdot \ell \cdot c_u$. And thus $\langle a_v \cdot \ell, c_u \rangle = \delta_{uv}$ for $u, v = 1, 2, \ldots, h_{d-i}$.

Notice that $\{c_u\}_{u=1,2,\ldots,h_{d-i}}$ is linearly independent in A_{d-i-1} because $\{c_u \cdot \ell\}_{u=1,2,\cdots h_{d-i}}$ is linearly independent in A_{d-i}. We wish to show that $\{a_v \cdot \ell\}_{v=1,2,\cdots h_{d-i}}$ is linearly independent in A_{i+1}, such that $A_i \xrightarrow{\cdot \ell} A_{i+1}$ is injective. Suppose $\beta_1 a_1 \cdot \ell + \cdots + \beta_{h_{d-i}} a_{h_{d-i}} \cdot \ell = 0$ for coefficients β_i in our field K. We then have that $\langle \beta_1 a_1 \cdot \ell + \cdots + \beta_{h_{d-i}} a_{h_{d-i}} \cdot \ell, c_u \rangle = 0$ for all $u = 1, \ldots, h_{d-i}$, which implies that

$$\beta_1 \langle a_1 \cdot \ell, c_u \rangle + \cdots + \beta_{h_{d-i}} \langle a_{h_{d-i}} \cdot \ell, c_u \rangle = 0 \tag{2}$$

for all $u = 1, \ldots, h_{d-i}$. We showed above that $\langle a_v \cdot \ell, c_u \rangle = \delta_{uv}$, and thus in order for the left hand side of (2) to be zero, we must have that the billinear form $\beta_u \langle a_u \cdot \ell, c_u \rangle$ is equal to zero for all u. But this is true if and only if $\beta_u = 0$ for each u, as desired. $\square$

We then make the following observation, which, combined with the preceding proposition, reduces the problem to studying the full-rank property of maps in the "middle."

Proposition 3.2. *For any $0 \le i \le (\lfloor \frac{d}{2} \rfloor - 1)$, if $A_{i+1} \xrightarrow{\cdot \ell} A_{i+2}$ is injective, then $A_i \xrightarrow{\cdot \ell} A_{i+1}$ is injective.*

Proof. We first show that for any $x \in A$, there exists some $\ell' \in A_1$ such that $x \cdot \ell' \neq 0$. Indeed, if $x \cdot x_i = 0$ for all $i = 1, 2, \ldots, n$ then x times any monomial in the x_i's is 0. In particular, if we focus on the monomials in the basis of A_{d-i}, we get that $x \cdot a = 0$ for all a in this basis. This contradicts the fact that A is Gorenstein.

We now show the contrapositive of the statement. Suppose that $A_i \xrightarrow{\cdot \ell} A_{i+1}$ is not injective, and let $x \neq 0 \in A_i$ be such that $x \cdot \ell = 0$. Pick some ℓ' so that $x \cdot \ell' \in A_{i+1}$ is nonzero. But $(x \cdot \ell') \cdot \ell = x \cdot \ell \cdot \ell' = 0$. Hence $A_{i+1} \xrightarrow{\cdot \ell} A_{i+2}$ is not injective. $\square$

This yields the following corollary.

Corollary 3.3. *An element $\ell \in A_1$ is a weak Lefschetz element of A if and only if $A_{\lfloor \frac{d-1}{2} \rfloor} \xrightarrow{\cdot \ell} A_{\lfloor \frac{d+1}{2} \rfloor}$ is injective.*

Proof. The claim follows directly from Proposition 3.1 and Proposition 3.2. $\square$

Corollary 3.3 greatly reduces the computation required to verify whether a given $\ell \in A_1$ forms a weak Lefschetz element for A. We make a further reduction before computing examples of boundaries of simplicial polytopes by showing that the problem only requires us to study even dimensional polytopes. This is done via the bipyramid construction in the next subsection.

3.2 Reduction to Even Dimensional Polytopes

Definition. For a given polytope P such that the origin is in P, define its *bipyramid* P' as the polytope with vertex set $P \cup \{N, S\}$, where N and S are not in $\operatorname{span} P$ and the line NS through the north pole and the south pole goes through the origin.

Let Δ be the simplicial complex given by the boundary of a d-dimensional convex polytope P, and let Δ' be the boundary of the $(d+1)$-dimensional bipyramid P' of P. We have that for any $F \subset \{x_1, x_2, \cdots, x_n\}$, the sets $\{x_{n+1}\} \cup F$ and $\{x_{n+2}\} \cup F$ are faces of Δ' if and only if F is a face of Δ. Note $\{x_{n+1}, x_{n+2}\}$ is not a face of Δ'.

Finally, let $A = K[\Delta]/(\underline{\theta})$ and $A' = K[\Delta']/(\underline{\theta'})$, where $\underline{\theta}$ and $\underline{\theta}'$ are the geometric lsop's of P and P'.

Proposition 3.4.

$$A' = A[x_{n+1}, x_{n+2}]/(x_{n+1}x_{n+2}, x_{n+1} - x_{n+2}) \simeq A[x_{n+1}]/(x_{n+1}^2)$$

Also, $A'_i = A_i \bigoplus x_{n+1} A_{i-1}$.

Proof. In P', we added two more vertices x_{n+1} and x_{n+2}. Notice that the set of non-faces of Δ' not containing x_{n+1} or x_{n+2} is exactly the set of non-faces of Δ. If a non-face of Δ' contains x_{n+1} or x_{n+2}, removing x_{n+1} or x_{n+2} gives a non-face of Δ. And $\{x_{n+1}, x_{n+2}\}$ is a non-face of Δ'. So the set of minimal non-faces of Δ' is exactly the set of minimal non-faces of Δ, union $\{x_{n+1}, x_{n+2}\}$. This shows that $K[\Delta'] = K[\Delta][x_{n+1}, x_{n+2}]/(x_{n+1}x_{n+2})$.

Notice that $\underline{\theta}'$ is a $(d+1)$ by $(n+2)$ matrix, where the first n columns are obtained by adjoining one row of 0's to $\underline{\theta}$. The $(n+1)^{\text{st}}$ column consists of all 0's in the first d rows and a 1 in the last row, corresponding to the coordinates of x_{n+1}. Similarly, the $(n+2)^{\text{nd}}$ column consists of all 0's in the first d rows and a -1 in the last row, corresponding to the coordinates of x_{n+2}. This tells us that $\overline{x_{n+1}-x_{n+2}} = \overline{0}$ in $K[\Delta']/(\underline{\theta}')$, resulting in $A' = A[x_{n+1}, x_{n+2}]/(x_{n+1}x_{n+2}, x_{n+1}-x_{n+2}) \cong A[x_{n+1}]/(x_{n+1}^2)$.

Finally, since we can express $A' = A[x_{n+1}]/(x_{n+1}^2)$ as the direct sum $A \bigoplus x_{n+1}A$, it is not difficult to see that this direct sum extends to each of the graded components, that is, $A'_i = A_i \bigoplus x_{n+1}A_{i-1}$.

□

We now use this construction to prove a property of weak Lefschetz elements in the bipyramids of boundaries of simplicial polytopes that allows

Proposition 3.5. *Let d be odd. Let $\ell = \sum_{i=1}^{n} \alpha_i x_i \in A_1$ and let $\ell' = \sum_{i=1}^{n} \alpha_i x_i \in A'_1$ be the same element, but viewed in A'_1. Then ℓ is a weak Lefschetz element of A if and only if ℓ' is a weak Lefschetz element of A'.*

Proof. By Corollary 3.3, ℓ is a weak Lefschetz element of A if and only if $A_{\frac{d-1}{2}} \xrightarrow{\cdot\ell} A_{\frac{d+1}{2}}$ is injective and ℓ' is a weak Lefschetz element of A' if and only if $A'_{\frac{d-1}{2}} \xrightarrow{\cdot\ell'} A'_{\frac{d+1}{2}}$ is injective.

Let $a + x_{n+1}b$ be a generic element in $A'_{\frac{d-1}{2}} = A_{\frac{d-1}{2}} \bigoplus x_{n+1}A_{\frac{d-1}{2}-1}$, where $a \in A_{\frac{d-1}{2}}$ and $b \in A_{\frac{d-1}{2}-1}$. Then $(a + x_{n+1}b)\cdot\ell' = a\cdot\ell' + x_{n+1}(b\cdot\ell')$. Assume that ℓ is a weak Lefschetz element. Then $a\cdot\ell \in A_{\frac{d+1}{2}}$ is nonzero for any $a \in A_{\frac{d-1}{2}}$ nonzero. Also, $b\cdot\ell \in A_{\frac{d-1}{2}}$ is nonzero for any $b \in A_{\frac{d-1}{2}-1}$ nonzero. So if $a + x_{n+1}b \in A_{\frac{d-1}{2}} \bigoplus x_{n+1}A_{\frac{d-1}{2}-1}$ is nonzero, a or b is nonzero, which implies that $a\cdot\ell \in A_{\frac{d+1}{2}}$ or $x_{n+1}b\cdot\ell \in x_{n+1}A_{\frac{d-1}{2}}$ is nonzero. This means that their sum $(a + x_{n+1}b)\cdot\ell'$ is nonzero, implying that the map $A_{\frac{d-1}{2}} \xrightarrow{\cdot\ell'} A_{\frac{d+1}{2}}$ is injective. Thus ℓ' is a weak Lefschetz element of A', as desired.

The converse argument follows similarly. □

Remark. For any polytope of odd dimension, we can construct its bipyramid of one dimension higher. Then, setting $\alpha_{n+1} = \alpha_{n+2} = 0$ in the conditions for ℓ' to be a weak Lefschetz element for the bipyramid gives us the conditions for ℓ to be a weak Lefschetz element in the original polytope.

Our problem is now reduced to studying even dimensional polytopes which may answer the following (refined) question:

Question. *Let Δ represent the boundary of an even-dimensional simplicial polytope. For which Δ and lsop $\underline{\theta}$ is the weak Lefschetz property for an element ℓ dependent on the minors of $\hat{M}(\underline{\theta}, \ell)$?*

We now explore specific classes of Δ for which the weak Lefschetz property of ℓ *is* matroidal.

4 Specific Classes

In this section we discuss the weak Lefschetz elements for stacked polytopes, cyclic polytopes, and cross polytopes. The first two are of interest, as they are "extremal" in the sense that for a given dimension, stacked polytopes have the fewest number of "large-dimensional" faces, while cyclic polytopes have the most (cf. [5]). Cross polytopes are another special family of interest, as we will see these polytopes have some intriguing properties.

4.1 Stacked Polytopes

Stacking is a common operation that gives a new polytope P_F from an old polytope P, where F is a facet of P. It can be thought of as placing a new vertex an infinitesimal distance from F, outside of P, and taking the convex hull of P and this new point. We now state a formal definition.

Definition. Let P be a polytope with a set of facets $\mathcal{F}$. Fixing some facet F, define the *stacked polytope* P_F as the polytope with vertex set $P \cup \{x_{n+1}\}$ and facets $(\mathcal{F} \setminus F) \cup \bigcup_{i \in F}((F \setminus i) \cup \{n+1\})$.

Proposition 4.1. *Let Δ (and resp., Δ') be the simplicial complex given by the boundary of a polytope P (and resp., its stacked polytope P_F for some facet F). Fix the lsops of P and P_F to be the geometric lsops of their vertices.*

If $A = K[\Delta]/(\underline{\theta})$, then $A' = K[\Delta']/(\underline{\theta}') \cong A[x_{n+1}]/I$, where I is generated by the additional minimal non-faces $x_F = \prod_{i\in F} x_i$ and $x_i x_{n+1}$ for $i \notin F$.

Furthermore, we may write any $\ell' \in A'_1$ as $\ell' = \sum_{i\notin F} \alpha_i x_i$. Then ℓ' is a weak Lefschetz element for A' if and only if ℓ is a weak Lefschetz element for A and $\alpha_{n+1} \neq 0$.

To prove Proposition 4.1, we need two lemmas. The first allows us to substitute monomials with smaller powers of x_{n+1} for monomials with larger ones.

Lemma 4.2. *$x_{n+1}x_i = d_i x_{n+1}^2$ for any $i \in F$, where d_i is some constant.*

Proof. Recall that by definition 4.1, since F is a facet of P then $(F \setminus \{i\}) \cup \{n+1\}$ is a facet of P_F. Thus the columns corresponding to F in $M(\underline{\theta})$ are linearly independent, and also the columns corresponding to $(F \setminus \{i\}) \cup \{n+1\}$ in $M(\underline{\theta})$ are linearly independent. Hence, we may row reduce the lsop (via Lemma 2.1) and assume one of the θ rows takes the following form, for constants c_k with $c_i \neq 0$:

$$x_{n+1} + c_i x_i + \sum_{j\notin F} c_j x_j = 0.$$

Since $\{j, n+1\}$ is a non-face for $j \notin F$, we see that $\overline{x_j x_{n+1}} = \overline{0} \in A$. Thus, multiplying both sides of our equation by x_{n+1}, we get $x_{n+1}^2 = -c_i x_i x_{n+1}$. Take $d_i = -\frac{1}{c_i}$ to complete our proof. □

The second lemma shows how the graded decomposition of A' depends on that of A.

Lemma 4.3. *$A'_k = A_k \oplus \operatorname{span}\{x_{n+1}^k\}$.*

In order to prove Lemma 4.3, we need to first introduce the concept of a *shelling order* on the facets of a pure simplicial complex P.

Definition. A *shelling order* for a pure simplicial complex Δ is a linear ordering on its facets $F_1, \dots, F_t$ such that for each pair of facets F_i, F_j with $i < j$, there is a facet T_k with $k < j$ and an element $v \in T_j$ such that $T_i \cap T_j \subseteq T_k \cap T_j = T_j \setminus \{v\}$.

Remark. Such a shelling order does not always exist; if it exists we say that the simplicial complex is *shellable*.

We now provide the proof of Lemma 4.3.

Proof. Choose a shelling for P where we shell facet F last. This lifts to a shelling for P_F where we shell the facets $(F \setminus \{i\}) \cup \{n+1\}$ in some order $i_1, i_2, \dots, i_d$. Thus, a basis for A' is given by concatenating a basis for A with $x_{n+1}, x_{n+1}x_{i_1}, x_{n+1}x_{i_1}x_{i_2}, \cdots, x_{n+1}x_{i_1}\dots x_{i_d}$. By repeated application of Lemma 4.2, we may use alternative basis elements x_{n+1}^k where $1 \leq k \leq d$. □

We are now ready to show the proposition.

Proof of Proposition 4.1. It's clear that $A' = K[\Delta']/(\underline{\theta}') \simeq A[x_{n+1}]/I$. Since the columns of F are independent, we may use the rows to set $\alpha_i = 0$ for $i \in F$.

We want to show the map $A'_k \xrightarrow{\ell'} A'_{k+1}$ is injective if and only if the map $A_k \xrightarrow{\ell} A_{k+1}$ is injective and $\alpha_{n+1} \neq 0$. Let $a + bx_{n+1}^k \in A'_k$, where $a \in A_k$ and $b \in A_{k-1}$. By expressing x_i for $i \in F$ as a linear combination of x_j for $j \notin F$, we may assume a has no monomials in F. Then

$$(a + bx_{n+1}^k)(\ell + \alpha_{n+1}x_{n+1}) = a\ell + b\alpha_{n+1}x_{n+1}^{k+1}$$

since $x_{n+1}^k \ell = a x_{n+1} = 0$. If ℓ is WL and $\alpha_{n+1} \neq 0$, then this is not zero. If ℓ is not WL, then take $b = 0$ and $a \neq 0$ such that $a\ell = 0$. If $\alpha_{n+1} = 0$, then take $a = 0$ and $b \neq 0$. □

4.2 Cyclic Polytopes

The cyclic polytope $C(n, d)$ in $\mathbb{R}^d$ is the convex hull of substituting n points $t_1 < t_2 < \cdots < t_n$ into the moment curve $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^d$ given via the mapping $t \mapsto [t, t^2, \ldots, t^d]^\top$. We will use $[n]$ to label the set of vertices, where vertex k corresponds to the point $\mathbf{x}(t_k)$.

This next two lemmas is are well-known facts about cyclic polytopes.

Lemma 4.4 (Gale's evenness condition). *A size d subset $S \subset [n]$ of $C(n, d)$ is a facet if and only if the following condition is satisfied:*

If $i < j$ are not in S, the number of $k \in S$ between i and j is even.

For the next lemma, recall that a polytope P is l-neighborly if any subset S of its vertices with cardinality l form a facet.

Lemma 4.5. *Let P be a simplicial d-polytope on $f_0 = n$ vertices. Then for $0 \le k \le d$, the h-vector of P satisfies*

$$h_k \le \binom{n-d-1+k}{k}.$$

Equality holds for all k with $0 \le k \le l$ if and only if $l \le \lfloor \frac{d}{2} \rfloor$ and P is l-neighborly.

This next proposition constructs a basis for A_k in the case of $C(n, d)$.

Proposition 4.6. *Let $C(n, d)$ be a cyclic polytope with vertices given by $[n]$, and fix some facet F. Then for any $k \le \lfloor \frac{d}{2} \rfloor$, the following set is a basis for A_k:*

$$B \coloneqq \{x_{i_1} x_{i_2} \cdots x_{i_k} : i_j \in [n] \setminus F \text{ for all } 1 \le j \le k\}.$$

Proof. Without loss of generality, let F be $\{1, 2, \ldots, d\}$; the general case is entirely analogous (F is a facet by Lemma 4.4). We first show that B spans A_k. Consider any monomial of degree k in A_k given by $x_{i_1} x_{i_2} \cdots x_{i_k}$. If $i_j \notin \{d+1, d+2, \ldots, n\}$ for some j, then it must be contained in F. As $\underline{\theta}$ is an lsop, there exists some linear combination of $\theta_1, \theta_2, \ldots, \theta_d$ that equals $\eta = x_{i_j} + \sum_{m=d+1}^{n} c_m x_m$. In A we have $x_{i_j} = x_{i_j} - \eta$, which expresses x_{i_j} as a linear combination of elements x_m, where $m \in [n] \setminus F$. Repeating this process allows us to express $x_{i_1} x_{i_2} \cdots x_{i_k}$ as a linear combination of elements in B.

It now suffices to show that the dimension of A_k matches the number of elements in B. By a stars and bars argument, B contains $\binom{n-d-1+k}{k}$ elements, which is precisely the dimension of A_k given by Lemma 4.5, as cyclic polytopes are $\lfloor \frac{d}{2} \rfloor$-neighborly. ☐

Using what we have proven, we can classify all weak Lefschetz elements for cyclic polytopes when d is even.

Theorem 4.7. *Let $C(n, d)$ by a cyclic polytope where d is even. Then any nonzero ℓ is a weak Lefschetz element.*

Proof. It suffices from Corollary 3.3 to show that the map $A_{d/2-1} \xrightarrow{\cdot \ell} A_{d/2}$ is injective, given via multiplication by some nonzero $\ell = a_{d+1} x_{d+1} + a_{d+1} x_{d+2} + \cdots + a_n d_n$. Let M be the matrix for the map, where the bases of the domain and image are given by those in Proposition 4.6. From Lemma 2.4, we know that M has at least as many rows as there are columns.

We claim that there exists a collection of $k \times k$ minors of M with values $a_{d+1}^k, a_{d+2}^k, \ldots, a_n^k$, where k is the cardinality of $A_{d/2-1}$, or the number of columns in M. This implies the theorem as any nonzero ℓ has some nonzero coefficient a_j, and thus M would have a $k \times k$ minor a_j^k, implying that M is full rank.

To prove the claim, pick any $j \in \{d+1, d+2, \ldots, n\}$ and order the bases of both $A_{d/2-1}$ and $A_{d/2}$ in lexicographic order based on the ordering of A_1 given by

$$x_j > x_{d+1} > \cdots > x_{j-1} > x_{j+1} > \cdots > x_n.$$

One easily verifies that if we reorder the rows and columns of M in such a manner, the first $k \times k$ minor is lower triangular with determinant a_j^k. ☐

We now move onto the case where d is odd, where begin with two important lemmas.

Lemma 4.8. *Let F be a facet of $C(n, d)$ that does not contain n. Then any non-face of cardinality $\frac{d+1}{2}$ must contain some k, where $k \notin F$ and $k \neq n$.*

Proof. Once again, we can assume that F is the facet $\{1, 2, \ldots, d\}$, as the argument for general F is completely analogous. Indeed, suppose for contradiction that we have a non-face of cardinality $\frac{d+1}{2}$ that contains n. The remaining $(d-1)/2$ vertices form a face by Lemma 4.5, as cyclic polytopes are $\lfloor \frac{d}{2} \rfloor$-neighborly. Adding n at the end still satisfies the evenness condition given in Lemma 4.4, and hence we would have a facet, a contradiction. Next, such a nonface clearly cannot be contained in the facet $\{1, \ldots, d\}$. Hence every nonface of cardinality $(d+1)/2$ has at least one vertex in the set $[n] \setminus (F \cup \{n\})$. $\square$

Lemma 4.9. *Following the notation of Lemma 2.1, the ideal $\phi_F(I_\Delta)$ doesn't contain any degree $\frac{d+1}{2}$ polynomial with nonzero $x_n^{\frac{d+1}{2}}$ coefficient.*

Proof. Let G be a non-face of Δ of size $\frac{d+1}{2}$. Then by Lemma 4.8, there exists $k \in G$, $k \notin F$, $k \neq n$. Thus, x_G will be divisible by x_k, so if it is of degree $\frac{d+1}{2}$, then it can't contain a term $x_n^{\frac{d+1}{2}}$. This holds for any non-face G, so it also holds for any polynomial in the ideal. $\square$

The following is an immediate corollary of Lemma 4.9.

Lemma 4.10. *The element $x_n^{(d+1)/2}$ represents a nonzero element in $A_{(d+1)/2}$.*

With this lemma, we can proceed to the proof of delineating the weak Lefschetz elements for cyclic polytopes of odd dimension.

Theorem 4.11. *For odd d, a nonzero ℓ is weak Lefschetz if and only if all minors of $M(\underline{\theta}, \ell)$ with columns given by $\{x_1, x_{i_1}, x_{i_2}, \ldots, x_{i_{d-1}}, x_n\}$ are nonzero, for any facet of P $\{x_1, x_{i_1}, \ldots, x_{i_{d-1}}\}$ that does not contain x_n.*

Proof. We will prove that the determinant of the map $A_{(d-1)/2} \xrightarrow{\cdot \ell} A_{(d+1)/2}$ factors precisely into a product of linear terms corresponding to the declared facets.

Take any facet F not containing x_n. We now view A as its isomorphic image $K[x_i : i \notin F]/\phi_F(I_\Delta)$, adopting the notation from Lemma 2.1. Recall that this essentially means that we are expressing all variables corresponding to elements in F as a linear combination of variables corresponding to elements not in F. Let $\sum_{i \notin F} \alpha_i x_i$ be any representative of ℓ in $K[x_i : i \notin F]/\phi_F(I_\Delta)$.

By Proposition 4.6, we have the following explicit basis for $A_{(d-1)/2}$ given by

$$B := \{x_{i_1} x_{i_2} \cdots x_{i_{(d-1)/2}} : i_j \in \{d+1, d+2, \ldots, n\} \text{ for all } 1 \leq j \leq k\}.$$

By Lemma 4.10, $x_n^{\frac{d+1}{2}}$ represents a nonzero element in $A_{(d+1)/2}$. Thus, we can complete it to a basis B' for $A_{(d+1)/2}$.

Let M be the matrix for the linear transformation $A_{(d-1)/2} \xrightarrow{\cdot \ell} A_{(d+1)/2}$ with respect to B and B'. We claim that if $\alpha_n = 0$, the row of M corresponding to the basis element represented by $x_n^{\frac{d+1}{2}}$ are all 0's. That is, we claim that the element represented by $x_n^{\frac{d+1}{2}}$ is not in the image.

Take any representative of any element in $A_{(d-1)/2}$, it is a polynomial f of degree $\frac{d-1}{2}$. Multiply this polynomial by a linear element with 0 coefficient in front of x_n gives a polynomial g that has 0 coefficient in front of the monomial $x_n^{\frac{d+1}{2}}$. But notice that by Lemma 4.9, the ideal $\phi_F(I_\Delta)$ doesn't contain any degree $\frac{d+1}{2}$ polynomial with nonzero $x_n^{\frac{d+1}{2}}$ coefficient. Thus, the element represented by g cannot be represented by $x_n^{\frac{d+1}{2}}$, because their difference has nonzero coefficient in front of $x_n^{\frac{d+1}{2}}$ and thus not in the ideal $\phi_F(I_\Delta)$. Claim proved.

The above claim implies that $\cdot \ell$ is not weak Lefschetz if for some facet F not containing x_n, $\phi_F(l)$ has 0 coefficient in front of x_n. We now show that $\cdot \ell$ is weak Lefschetz if for any facet F not containing x_n, $\phi_F(l)$ has nonzero coefficient in front of x_n.

Choose one specific facet F_0 not containing x_n. Let M_0 be the matrix for the linear transformation $A_{(d-1)/2} \xrightarrow{\cdot \ell} A_{(d+1)/2}$ described above, for this chosen facet F_0. Take any $l \in A_1$. Let $\phi_{F_0}(l)$ be represented by $\sum_{i \notin F_0} \alpha_i x_i$. Then for each facet F not containing x_n, $\phi_F(l)$ is represented by $\sum_{j \notin F} \beta_j^F x_j$, where β_j^F is a linear expression of $\{\alpha_i : i \notin F_0\}$. This is because the maps ϕ_F are essentially expressing elements corresponding to vertices in F as linear expressions of vertices corresponding to vertices not in F. In particular, β_n^F is a linear expression of the α_i's. For example, $\beta_n^{F_0} = \alpha_n$. We claim that β_n^F's will be distinct up to multiplication by a scalar for distinct F's.

Assume otherwise, then we have $\beta_n^{F_1} = 0$ if and only if $\beta_n^{F_2} = 0$. By Lemma 2.2, the above statement is equivalent to $M(\underline{\theta}, \ell)_{F_1, x_n}$ is singular if and only if $M(\underline{\theta}, \ell)_{F_2, x_n}$ is singular for any fixed ℓ. But by choosing $\ell = x_i$ for some $i \in F_2 \setminus F_1$, we see that $M(\underline{\theta}, \ell)_{F_1, x_n} \neq 0$ because the complementary minor of x_i has nonzero coefficient. (In fact, all minors of $M(\underline{\theta})$ does.) But $M(\underline{\theta}, \ell)_{F_1, x_n} = 0$. Contradiction. Thus, the β_n^F's are distinct up to multiplication by a scalar for distinct F's.

Recall that $\det M_0 = 0$ if $\phi_F(l)$ has 0 coefficient in front of x_n. Thus, $\det M_0 = 0$ if $\beta_F = 0$. Thus, β_F divides $\det M_0$, viewed as polynomials in variables $\{\alpha_i : i \notin F_0\}$. Since $det M_0$ has degree at most the dimension of $A_{(d-1)/2}$, which equals $\binom{n-d/2-3/2}{(d-1)/2}$ by Lemma 4.5. We now show that this is the exact number of facets F not containing x_n. First, we notice that given a facet F that does not contain x_n, it has to contain x_1. This is because otherwise, there are d, and thus an odd number of, vertices in the facet between x_1 and x_n, which contradicts Lemma 4.4. Furthermore, by the evenness condition, it is easy to see that all the other $d-1$ elements after x_1 in the facets come in consecutive pairs. Conversely, all choices of consecutive pairs from $x_2 \cdots x_{n-1}$ union x_1 is a facet. Therefore, the number of facets not containing x_n equals the number of ways to choose $(d-1)/2$ consecutive pairs from $x_2 \cdots x_{n-1}$, which equals $\binom{n-d/2-3/2}{(d-1)/2}$ by a simple combinatorial argument.

Since each β_n^F is distinct and they all divide $\det M_0$, $\det M_0$ is a multiple of $\prod_F \beta_n^F$. Since the degrees match, it's a scalar multiple.

We now show that the determinant is not the zero polynomial. This follows from the fact that Stanley-McMullen have shown the existence of a weak Lefschetz element for the boundary complex of a simplicial polytope, and hence in this case there is at least some choice of a_i that yields a nonzero value.

And thus $\det M_0$ is a nonzero multiple of $\prod_F \beta_n^F$. So ℓ is weak Lefschetz if all β_n^F nonzero, which is equivalent to $M(\underline{\theta}, \ell)_{F, x_n}$ singular by Lemma 2.2. □

4.3 Cross Polytopes

Define the dimension r cross polytope to the be convex hull of

$$\{\pm e_i | 1 \leq i \leq r\} \subset \mathbb{R}^r$$

Then, let Δ be the boundary of the r-dimensional cross-polytope, and label the vertex e_i by x_i and $-e_i$ by y_i. The minimal non-faces of this polytope are $\{x_i, y_i\}$, so we get that

$$K[\Delta] = K[x_1, y_1, x_2, y_2, \ldots, x_r, y_r]/(x_1 y_1, x_2 y_2, \ldots, x_r y_r)$$

We consider the geometric lsop $\underline{\theta} = (\theta_1, ..., \theta_r)$, where $\theta_i = x_i - y_i$. Then, we have that

$$A = K[\Delta]/(\underline{\theta}) \cong K[x_1, \ldots, x_r]/(x_1^2, x_2^2, \ldots, x_r^2)$$

We see that A is a graded ring over K and that a basis for each component A_d is given by

$$\{x_I | I \in \binom{[n]}{d}\}$$

where $x_I := \prod_{i \in I} x_I$. Further, given $\ell = \sum_{i=1}^r a_i x_i + b_i y_i$, we can write

$$\ell = \sum_{i=1}^r c_i x_i$$

for $c_i := a_i - b_i$. Since any monomial in A with a square term is 0, we get the following multiplication relation:

$$x_i x_I = \begin{cases} x_{i\cup I} & i \notin I \\ 0 & i \in I \end{cases}$$

Now, consider $\ell : A_d \to A_{d+1}$ as a linear operator, and let $M(r,d)$ the matrix representing ℓ matrix under this basis. We index entries of $M(r,d)$ by $M(r,d)_{J,I}$, where $I \in \binom{[r]}{d}$, $J \in \binom{[r]}{d+1}$. Then, our multiplication relation implies that

$$M(r,d)_{J,I} = \begin{cases} 0 & I \not\subset J \\ c_i & J \setminus I = \{i\} \end{cases}$$

Example. Consider A when $r = 4$ we have in this case that the graded components of A are

$$\begin{aligned}
&A_0 = K \\
&A_1 = K\langle x_1, x_2, x_3, x_4\rangle \\
&A_2 = K\langle x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4\rangle \\
&A_3 = K\langle x_2x_3x_4, x_1x_3x_4, x_1x_2x_4, x_1x_2x_3\rangle \\
&A_4 = K\langle x_1x_2x_3x_4\rangle
\end{aligned}$$

Further, consider $\ell = c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4$ as a linear operator from $A_1 \to A_2$. The matrix under this basis is

$$\begin{array}{c} \\ x_1x_2 \\ x_1x_3 \\ x_1x_4 \\ x_2x_3 \\ x_2x_4 \\ x_3x_4 \end{array}
\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\
\begin{pmatrix} c_2 & c_1 & 0 & 0 \\ c_3 & 0 & c_1 & 0 \\ c_4 & 0 & 0 & c_1 \\ 0 & c_3 & c_2 & 0 \\ 0 & c_4 & 0 & c_2 \\ 0 & 0 & c_4 & c_3 \end{pmatrix} \end{array}$$

For the remainder of this subsection, let $G(r,d)$ be the graph with vertices subsets of $[r]$ of size d and $d+1$, and where there is an edge of weight c_i between $I \in \binom{[r]}{d}$ and $J \in \binom{[r]}{d+1}$ whenever $J = I \cup \{i\}$

Proposition. *Let Δ be the boundary of the d-dimensional cross polytope. Consider $\ell = \sum_{i=1}^r c_i x_i \in K[\Delta]/(\underline{\theta})$*

1. *If r is odd, then ℓ is weak Lefschetz if and only if $c_i \neq 0$ for all i.*
2. *If r is even, then ℓ is weak Lefschetz if and only if $c_i = 0$ for at most one i.*

Before we prove this proposition, we show some helpful lemmas.

Lemma 4.12. *Every maximal minor of $M(r,d)$ has form $\alpha c_1^{e_1} \dots c_r^{e_r}$, where alpha $\in \mathbb{Z}$ and*

$$e_i = \{\textit{number of occurrences of } i \textit{ in the rows indexing the minor}\} - \binom{r-1}{d-1}$$

Proof. A maximal minor in $M(r,d)$ can be identified by the family $\mathcal{J} \subset \binom{[r]}{d+1}$ of sets indexing the rows. Let G' denote subgraph of $G(r,d)$ induced by restricting to $\binom{[r]}{d} \cup \mathcal{J}$. Looking at the Leibniz determinant formula, one can observe that a nonzero term in the sum over permutations corresponds to a perfect matching $\mathcal{M}$ in G'. In particular, the term is equal to the product of edge weights of edges in the matching, ie.

$$\begin{aligned} x_{1,\sigma(1)} \cdots x_{\binom{r}{d},\sigma\binom{r}{d}} &= \prod_{e \in \mathcal{M}} wt(e) \\ &= \prod_{J \in \mathcal{J}} c_{J-\sigma(J)} \\ &= \frac{\prod_{J \in \mathcal{J}} [\prod_{i \in J} c_i]}{\prod_{I \in \binom{[r]}{d}} [\prod i \in I]} \end{aligned}$$

The number of occurrences of c_i in the numerator is just the number of occurrences of i in $\mathcal{J}$, and the number of occurrences in the denominator is $\binom{r-1}{d-1}$. $\square$

Note that in the odd case, this tells us that the determinant of the middle map from $A_{\frac{r-1}{2}}$ to $A_{\frac{r+1}{2}}$ has form $\alpha(c_1 \dots c_r)^{\frac{1}{r}\binom{r}{\frac{r-1}{2}}}$ and $\alpha \in \mathbb{Z}$.

Lemma 4.13. *When r is even, any maximal minor of $M(r, \frac{r-1}{2})$ has at most one $e_i = 0$.*

Proof. Let $d = r/2 - 1$. Using the same bipartite matching setup as in the proof of lemma 4.12, we see that in order to find a minor such that $e_{r-1} = e_r = 0$, we need to find a maximal bipartite matching in $G(r, d)$ so that all $J \in \binom{[r]}{d+1}$ such that $r \in J$ must be matched to a subset I containing r, and similarly for $r-1$, as otherwise a factor of c_r would appear in the minor. There are $\binom{r}{d+1} - \binom{r-2}{d+1}$ such J, but only $\binom{r}{d} - \binom{r-2}{d}$ such I. The remaining $\binom{n-2}{d+1}$ subsets I of size d containing neither r nor $r-1$ must be matched to the remaining $\binom{n-2}{d+1}$ subsets of size $d+1$ containing neither r nor $r-1$, but when $d = r/2 - 1$, $\binom{n-2}{d+1} < \binom{n-2}{d}$, so this is not possible.

$\square$

Proof. Proof of proposition 4.3

1. r is odd. It suffices to consider the multiplication by ℓ map from $A_{\frac{r-1}{2}} \to A_{\frac{r+1}{2}}$. As mentioned above, the determinant of this matrix is $\alpha(c_1 \dots c_r)^{\frac{1}{r}\binom{r}{\frac{r-1}{2}}}$ for some $\alpha \in \mathbb{Z}$. To show $\alpha \neq 0$, it suffices to make the substitution $c_i = 1$ for all i. The resulting matrix is the incidence matrix for the two middle ranks of the the Boolean poset on an odd number of elements, which is well known to be invertible [1].

2. r is even. It suffices to consider the multiplication by ℓ map from $A_{\frac{r}{2}-1} \to A_{\frac{r}{2}}$. We will reduce this case to the odd case to show that there exists a maximal minor of $M(r, \frac{r}{2} - 1)$ of form $\alpha c_1^{e_1} c_2^{e_2} \dots c_{r-1}^{e_{r-1}}$, with $\alpha \neq 0$. Then we are done, since by symmetry, there exists a maximal minor with nonzero determinant and $e_i = 0$ for any i, and by lemma 4.13 we cannot have $e_i = e_j = 0$ for $i \neq j$.
   First, assume $c_1, \dots, c_{r-1} \neq 0$. Thus $M(r-1, r/2-1)$ is invertible and $M(r-1, r/2-2)$ contains a nonzero maximal minor. Say this minor is corresponds to rows $\mathcal{J} \subset \binom{[r-1]}{r/2-1}$. Let $\mathcal{J}' = \{J \cup \{r\} | J \in \mathcal{J}\}$. Then the desired minor in $M(r, r/2-1)$ corresponds to rows $\binom{[r-1]}{r/2} \cup \mathcal{J}' \subset \binom{[r]}{r/2}$. Let H be the subgraph of $G(r, r/2-1)$ obtained by restricting to vertices $\mathcal{J}'$ and $\binom{[r-1]}{r-2}$. Note that H decomposes into two disjoint bipartite graphs H_A and H_B on vertices containing and not containing r respectively, so a matching in H can be thought of as a pair of matchings, one in H_A and one in H_B. Say that A_+ and A_- are the number of even and odd matchings respectively of H_A, and that B_+ and B_- are the number of even and odd matchings respectively of H_B. Without loss of generality, say that $A_+ > A_-$ and $B_+ > B_-$ (otherwise we could just swap the ordering of two vertices/rows). But then the number of even and odd matchings of H is $A_+B_+ + A_-B_- > A_+B_- + A_-B_+$ respectively, and in particular these two quantities are unequal, so we conclude that $\alpha \neq 0$.
   Finally, since all $I \in \binom{[r]}{\frac{r}{2}-1}$ such that $r \in I$ must be matched to $J \in \binom{[r]}{\frac{r}{2}}$ such that $r \in J$, we know that $e_r = 0$.

$\square$

4.4 Investigating when a minor is 0

Here, we investigate conditions for when a minor of $M(r,d)$ is zero. In light of lemma 4.12, we can specialize each c_i to be 1. Call this new matrix $M^*(r,d)$. Then, we are working with a matrix with rows and columns indexed by $\binom{[r]}{d+1}$ and $\binom{[r]}{d}$, respectively, where

$$M^*(r,d)_{J,I} = \begin{cases} 1 & I \subset J \\ 0 & \text{otherwise} \end{cases}$$

Now, let $\mathcal{M}$ be the linear matroid over $\mathbb{R}$ on the rows of this matrix. Let $R_J \in \mathbb{R}^{\binom{r}{d}}$ be the Jth row of M^* and let $V = \binom{[r]}{d}$ be the vertex set of $\mathcal{M}$.

In this section, assume we are working in a field of characteristic 0.

4.4.1 The relations simplex

Here, we discuss how to associate a simplex Φ to a set $\mathcal{J} \subset 2^V$ and state some basic results about how they relate to dependencies between the R_Js.

Definition. The relations simplex Φ of $\mathcal{J}$ is the simplex on $[r]$ generated by $J \in \mathcal{J}$.

Corollary 4.14. *Φ is pure and the facets of Φ are $\mathcal{J}$.*

The primary motivation for the preceding definition is that assigning weights to the facets of Φ is in correspondence with dependencies between the R_Js

Definition. Let a weighting of Φ be a function wt from facets of Φ $(= \mathcal{J})$ to K. We extend wt to all of Φ via

$$\mathrm{wt}(I) = \sum_{J \in \mathcal{J} : J \supset I} \mathrm{wt}(J)$$

Definition. If $\mathrm{wt}(I) = 0$ for all $|I| = d$, call wt a dependent weighting

Proposition 4.15. *Given $\mathcal{J} \subset 2^V$, there is a linear dependency between between the R_Js if and only if there exists a non-zero dependent weighting of Φ*

Proof. First, we prove

Lemma 4.16. *For $\beta_J \in K$,*

$$\sum_{J \in \mathcal{J}} \beta_J R_j = 0 \iff \mathrm{wt}(J) = \beta_J, \text{ defines a dependent weighting}$$

Proof. We have $\sum_{J \in \mathcal{J}} \beta_J R_J = 0$ if and only if $\sum_{J \supset I} \beta_J = 0$ for all $|I| = d$ if and only if $\mathrm{wt}(J) = \beta_J$ defines a dependent weighting. ☐

The claim follows from the lemma. ☐

Corollary 4.17. *If $\mathcal{J}$ is a circuit in $\mathcal{M}$, then Φ is connected.*

Proof. If Φ were not connected, one of the connected components of Φ must have a non-zero dependent weighting. This component describes a dependent subset of $\mathcal{J}$, thus $\mathcal{J}$ is not a circuit. ☐

4.4.2 The case $d = 1$

First, we investigate $\mathcal{M}$ when $d = 1$ but for general $r > 2$. We characterize the independent sets of $\mathcal{M}$, and, as a corollary, say exactly when $M(r,d)$ is non-zero.

In the $d = 1$ case, Φ is a graph. Then, let $E(\Phi)$ be the set of edges in Φ and $V(\Phi)$ the set of vertices.

Proposition 4.18. *$\mathcal{J}$ is an independent set if and only if for each connected component $\Psi \subset \Phi$, $|E(\Psi)| \leq |V(\Psi)|$ and Φ contains no even cycles.*

Proof. We prove the contrapositive, that $\mathcal{J}$ is a dependent set if and only if one of the following holds:

1. There is a connected component $\Psi \subset \Phi$ with $|E(\Psi)| > |V(\Psi)|$
2. Φ contains an even cycle

First, let $\mathcal{J}$ be a set with a component $\Phi \subset \Phi$ with $|E(\Psi)| > |V(\Psi)|$. Then, for $J \subset E(\Psi)$, the row vectors R_J have only $|V(\Psi)|$ columns in which they have non-zero values. Then, the span of the R_J is at most of dimension $|V(\Psi)|$ but we have $|E(\Psi)| > |V(\Psi)|$ vectors, meaning there is a dependency between them.

Now, say Φ contains an even cycle. Consider the weighting wt which assigns weight 0 to edges not in the cycle, and alternates between weights 1 and -1 for edges in the cycle. We see this is a dependent weighting.

Now, let $\mathcal{J}$ be a dependent set. If condition (1) holds, we are done. Moving forward, assume condition (1) fails. Let $\mathcal{C} \subset \mathcal{J}$ be a circuit, which exists since $\mathcal{J}$ is dependent. Let Ψ be the relations simplex to associated $\mathcal{C}$. From corollary 4.17, we know Ψ is connected.

Let L be a matrix with row vectors R_Js for $J \in \mathcal{C}$. Let q be the number of 1s in L, which we know is equal to $2|E(\Psi)|$. If such a matrix has a column with only 1 one in it, then that row must have coefficient 0 in a linear dependency. This would contradict $\mathcal{C}$ being a circuit, so every column with at at least one 1 must have at least two. Then, since there are $|V(\Psi)|$ columns which are not all zeroes, $q \geq 2|V(\Psi)|$, so in particular $|E(\Psi)| \geq |V(\Psi)|$. Since we are not in case 1, $|E(\Psi)| = |V(\Psi)|$.

Then, there are exactly 2 ones in each column of L. Then, this says each vertex of Ψ has degree 2. Since Ψ is connected, Ψ is a cycle.

Now we must show Ψ is an even cycle. Say the linear dependency between the R_Js, $J \in E(\Psi)$ is

$$\sum_{J \in \mathcal{J}} \beta_J R_J = 0.$$

Then, if R_J, $R_{J'}$ have a one in the same column, we have $\beta_J = -\beta_J$. Since Φ is connected, we can take all β_Js to be ± 1. Now, since the weight at each vertex in $V(\Phi)$ is 0, the sum of the weights of all vertices is 0, so

$$2 \sum_{J \in \mathcal{J}} \beta_J = 0.$$

Thus, since $\beta_J = \pm 1$, the number of β_Js is even, so $|\mathcal{J}| = |V(\mathcal{J})|$ is even. Thus, Φ contains an even cycle. ☐

Corollary 4.19. *A minor with rows $\mathcal{J} \subset 2^V$ is non-zero if and only if each component of Φ is odd unicyclic.*

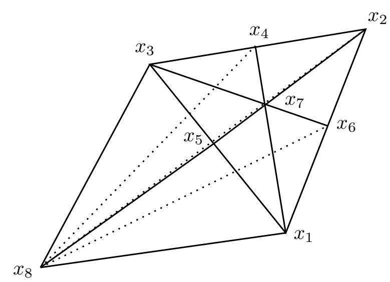
Proof. A minor with rows $\mathcal{J} \subset 2^V$ has the constraint that $|\mathcal{J}| = r$. Then, since $V(\mathcal{J}) = [r]$, this says $|E(\Phi)| = |\mathcal{J}| = r = |V(\mathcal{J})|$. We know a minor is non-zero if and only if $\mathcal{J}$ is an independent set. If every component of Φ is odd unicyclic, we see this satisfies the conditions of proposition 4.18, so the minor is non-zero. If the minor is non-zero, then $|E(\Phi)| = |V(\Phi)|$ in combination with condition (1) of proposition 4.18 says that each connected component of Φ is unicyclic. Further, as to not violate condition (2) each connected component is odd unicyclic. ☐

5 Two non-matroidal counterexamples

We know that the weak Lefschetz property will not hold for a general simplicial polytope and geometric lsop $\underline{\theta}$, as seen in the following counterexamples.

5.1 Tetrahedron Boundary Counterexample

Let $x_1, \ldots, x_8$ be vertices in $\mathbb{R}^3$, given by the coordinates $(1,0,0), (0,1,0)$, $(0,0,1), (0,1,1), (1,0,1), (1,1,0)$, $(1,1,1), (-1,-1,-1)$, respectively. Consider the boundary of the tetrahedron Δ_0 given by the following facets: $\{348\}, \{358\}, \{158\}$, $\{168\}, \{268\}, \{248\}, \{247\}, \{267\}, \{347\}, \{357\}, \{157\}, \{167\}$. This gives rise to a geometric interpretation of Δ_0 as



We let the linear system of parameters $(\underline{\theta})$ be given by the coordinates of our vertices, such that θ_i is a vector corresponding to the i^{th} coordinate of each v_j, in order from v_1 to v_8. In order for an element $\ell = a_1x_1 + \cdots + a_nx_n$ to be Weak Lefschetz, in particular the middle map from $A_1 \to A_2$ given by multiplication by ℓ must be invertible, as both A_1 and A_2 have dimension five. After establishing bases for the graded components A_1 and A_2, a simple computation gives the determinant of our middle map as $(-1)(a_7+a_8)(a_4^2+a_5^2+a_6^2+a_7a_8)$. In order for our desired map to be invertible, we need this determinant to be nonzero.

To see that the invertibility of the desired map is not determined by our matrix $\hat{M}(\underline{l}, \ell)$, and hence the Weak Lefshetz property is not in general matroidal for a simplicial complex, it is sufficient to find two evaluations of the coefficients of ℓ such that both evaluations produce the same set of minors of $\hat{M}(\underline{l}, \ell)$ being zero or nonzero, while one evaluation makes the ℓ-map determinant zero and the other makes this determinant nonzero. Indeed, it is not difficult to check that if we set $(a_1, \ldots, a_8) = (0,0,0,2,0,0,1,-4)$, we get a zero determinant for our ℓ-map, whereas when we set $(a_1, \ldots, a_8) = (0,0,0,6,0,0,3,-5)$, we get that our desired map is invertible. However, both matrices corresponding to the stated linear system of parameters and the respective coefficients of ℓ produce the same sets of zero and nonzero minors.

5.2 A Remark on Cross-polyotpes

In fact, depending on $\underline{\theta}$, the weak Lefschetz property for ℓ can be arbitrarily "far" from being matroidal. In particular, tale our simplicial complex Δ to be the boundary of the cross polytope in n dimensions, which has $2n$ vertices. We can pick a valid lsop given by choosing $2n$ generic points in the ambient n-dimensional space. Running code in `Sage` to calculate the determinant of the middle map referenced in Corollary 3.3 reveals that the weak Lefschetz property for $\ell = \sum_{i=1}^{2n} a_ix_i$ is controlled by a polynomial that seems to scale at least linearly with the dimension n. For example, letting $n = 3$ yields a middle map whose determinant is an irreducible polynomial of the form

$$c_1a_4^2a_5 + c_2a_4a_5^2 + c_3a_4^2a_6 + c_4a_4a_5a_6 + c_5a_5^2a_6 + c_6a_4a_6^2 + c_7a_5a_6^2,$$

where all c_i are constants (which will depend on $\underline{\theta}$). The $n = 4$ case requires a_8 to be nonzero, as well as the following relies on an irreducible degree 4 polynomial:

$$\begin{aligned} &c_1a_6^4 + c_2a_6^3a_8 + c_3a_5^2a_7a_8 + c_4a_5a_6a_7a_8 + c_5a_6^2a_7a_8 + c_6a_5a_7^2a_8 + c_7a_6a_7^2a_8 + c_8a_7^3a_8 \\ &\quad + c_9a_5^2a_8^2 + c_{10}a_5a_6a_8^2 + c_{11}a_6^2a_8^2 + c_{12}a_5a_7a_8^2 + c_{13}a_7a_8^2 + c_{14}a_5a_8^3 + c_{15}a_6a_8^3 + c_{16}a_7a_8^3. \end{aligned}$$

Following this pattern, the $n = 5$ case is even more complicated, depending on the vanishing of an irreducible degree 10 polynomial. The particular polynomials themselves are not as relevant for this discussion, and details will be spared.

These calculations imply that the choice of $\underline{\theta}$ has important ramifications in simplifying the condition for ℓ to be weak Lefschetz. Indeed, we saw in Proposition 4.3 that if we pick $\underline{\theta}$ to be the geometric lsop, we do obtain a matroidal condition for the weak Lefschetz property to hold. The core difference between the generic and geometric $\underline{\theta}$ are the linear dependencies. In the latter case, there are n pairs of vectors that are multiples of one another, and as we saw in the case of $n = 3$, breaking all such dependencies led to a weak Lefschetz condition that depended on an irreducible polynomial of degree 3. Would breaking fewer dependencies lead to polynomials of lower degree? The answer is in the affirmative. Recall that with no dependencies broken, the determinant controlling ℓ being weak Lefschetz is precisely the product $a_4 \cdot a_5 \cdot a_6$. Breaking one dependency by moving the vertex associated to a_6, the determinant of the middle map takes the form

$$a_5 \cdot a_4 \cdot (c_1 a_4 + c_2 a_6).$$

Curiously, we see that the a_6 term in the original determinant has been replaced with a more complicated linear term. If we shift the vertex associated to a_5, we obtain

$$a_4 \cdot (c_1 a_4 a_5 + c_2 a_4 a_6 + c_3 a_5 a_6),$$

which follows a similar logic as before. It appears that the more generic our $\underline{\theta}$, the more unwieldy our determinant. We do not understand this phenomenon very well at all, so needless to say, there is more to uncover in this direction.

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References

[1] William M Kantor, *On incidence matrices of finite projective and affine spaces*, Mathematische Zeitschrift **124** (1972), no. 4, 315–318.

[2] Juan Migliore and Uwe Nagel, *Survey article: a tour of the weak and strong Lefschetz properties*, J. Commut. Algebra **5** (2013), no. 3, 329–358. MR 3161738

[3] R. Stanley, *Weyl groups, the hard lefschetz theorem, and the sperner property*, SIAM J. Algebraic Discret. Methods **1** (1980), 168–184.

[4] Richard P. Stanley, *Combinatorics and commutative algebra*, second ed., Progress in Mathematics, vol. 41, Birkhäuser Boston, Inc., Boston, MA, 1996. MR 1453579

[5] Günter M. Ziegler, *Lectures on polytopes*, Springer-Verlag, New York, 1995.