A q-analog of Random-to-Random Shuffling

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Random-to-Random Shuffling

Markov chains model the behavior of dynamical systems such as shuffling a deck of cards. To predict their long term behavior, we need to find their eigenvalues. Here we do so for a q-analog of random-to-random shuffling.

Definition: Random-to-Random Shuffling

- **Physical:** Given a deck of n uniformly distributed cards, **random-to-random shuffle** $\mathsf{R2R}_n$ moves a random card to a random position.
- **In the group algebra $\mathbb{C}[S_n]$:** In terms of adjacent transpositions,

$$\mathsf{R2R}_n = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} s_i \dots s_1 s_1 \dots s_j.$$

Example: $\mathsf{R2R}_3(123)$
$= \big((123)+(213)+(231)\big)+\big((213)+(123)+(132)\big)+\big((312)+(132)+(123)\big)$
$= 3\cdot(123)+2\cdot(132)+2\cdot(213)+1\cdot(231)+1\cdot(312).$

Dieker and Saliola [1] gave an explicit formula and recurrence for the eigenvalues of $\mathsf{R2R}_n$ indexing by horizontal strips λ/μ where $\lambda \vdash n$.[a] They also computed the S_n-representation structure of the eigenspaces.

[a] $\mu \neq (k)$, $\mu \neq (1^\ell)$ for ℓ odd

Our Questions

A q-analog of $\mathbb{C}[S_n]$ is the space $\mathbb{C}[G/B]$ of **complete flags**

$$0 \subsetneq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_{n-1} \subsetneq U_n = \mathbb{F}_q^n.$$

Brown-Diaconis [2] and Brauner-Commins-Reiner [3] defined a natural q-analog $\mathsf{R2R}_n^{(q)}$ arising from the theory of left regular bands. We ask

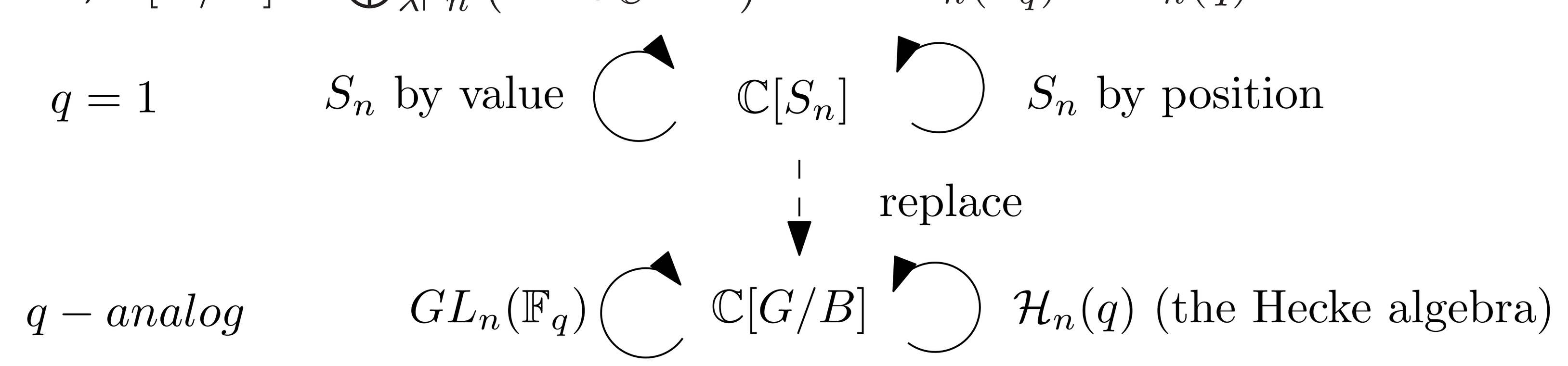
- What are the eigenvalues of $\mathsf{R2R}_n^{(q)}$?
- What is the $GL_n(\mathbb{F}_q)$ representation structure on $\mathsf{R2R}_n^{(q)}$ eigenspaces?

Our approach: The Hecke Algebra

The **Hecke algebra $\mathcal{H}_n(q)$** is the associative $\mathbb{C}$-algebra with generators $T_1, \dots, T_{n-1}$ satisfying

1. $(T_i - q)(T_i + 1) = 0$,
2. $T_i T_j = T_j T_i$ for $|i-j| \geq 2$,
3. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$.

Here, $\mathbb{C}[G/B] = \bigoplus_{\lambda \vdash n} \left(G^\lambda \otimes_{\mathbb{C}} H^\lambda\right)$ as a $GL_n(\mathbb{F}_q) \times \mathcal{H}_n(q)$ bimodule.



Main Results

$\mathsf{R2R}_n^{(q)}$ Expression in $End(\mathbb{C}[G/B])$

$$\mathsf{R2R}_n^{(q)} := \mathsf{T2R}_n^{(q)} \circ \mathsf{R2T}_n^{(q)},$$

where for $\mathbf{U} := 0 \subsetneq U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_n = \mathbb{F}_q^n$ with $U_i := \langle f_1, f_2, \cdots, f_i \rangle$,

- $\mathsf{R2T}_n^{(q)}(\mathbf{U}) := \sum_i \sum_{\substack{\text{line } \ell \subset U_i \\ \ell \not\subset U_{i-1}}} \ell \subsetneq \ell + U_1 \subsetneq \cdots \subsetneq \ell + U_{i-1} \subsetneq U_{i+1} \cdots \subsetneq U_n,$
- $\mathsf{T2R}_n^{(q)}(\mathbf{U}) := \sum_j \sum_{(a_1,\cdots,a_j)\in\mathbb{F}_q} \langle f_2+a_1f_1\rangle \subsetneq \langle f_2+a_1f_1,\cdots,f_{j+1}+a_jf_1\rangle \subsetneq U_{j+1} \subsetneq \cdots \subsetneq U_n$

$\mathsf{R2R}_n^{(q)}$ Expression in $\mathcal{H}_n(q)$

$$\mathsf{R2R}_n^{(q)} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} T_i \dots T_1 T_1 \dots T_j.$$

In the $\mathbb{C}$-basis for $\mathcal{H}_n(q)$ indexed by the reduced words of S_n,

$$\mathsf{R2R}_n^{(q)} = [n]_q + \sum_{p=1}^{n-1} q^{p-1}\Bigg((q+1)T_p + q\sum_{i=p+1}^{n-1} T_i \dots T_p + q\sum_{j=p+1}^{n-1} T_p \dots T_j$$
$$+ (q-1)\sum_{k=p+1}^{n-1}\sum_{\ell=p+1}^{n-1} T_k \dots T_{(p+1)} T_p T_{(p+1)} \dots T_\ell\Bigg).$$

Conjectural Explicit Formula and Representation Structure

Let λ be a partition of n and λ/μ be a horizontal strip. [a] We conjecture **all eigenvalues** of $\mathsf{R2R}_n^{(q)}$ are of the form:

$$\mathsf{eig}(\lambda/\mu)_q = \mathsf{diag}(\lambda/\mu)_q + \sum_{j=1}^{|\lambda/\mu|} q^{|\lambda/\mu|-j}[j+|\mu|]_q \quad \in \mathbb{Z}_{\geq 0}[q].$$

And for any eigenvalue ϵ, the ϵ-eigenspace has q-Frobenius characteristic

$$\sum_{\substack{(\lambda/\mu) \\ eig(\lambda/\mu)_q = \epsilon}} d^\mu s_\lambda.$$

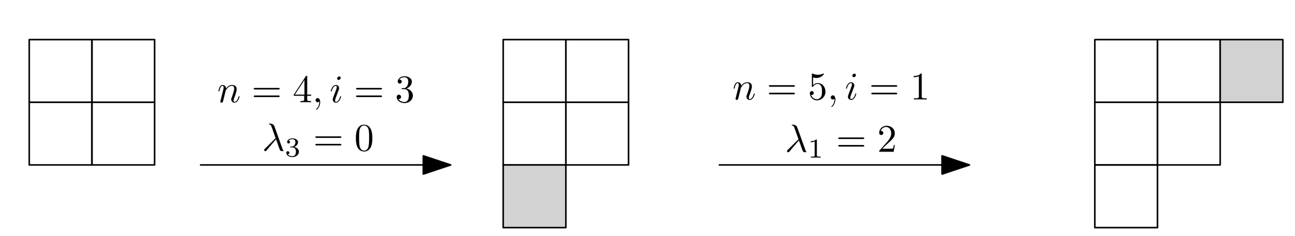
Note: $\mathsf{diag}(\lambda/\mu)_q$, d^μ are (q)-constants depending on the shape of λ, μ

Conjectural Recurrence Formula

For ϵ_λ an eigenvalue of $\mathsf{R2R}_n^{(q)}$ on $G^\lambda \otimes H^\lambda$, there's an eigenvalue $\epsilon_{\lambda+e_i}$ for $\mathsf{R2R}_{n+1}^{(q)}$ on $G^{\lambda+e_i} \otimes H^{\lambda+e_i}$ with

$$\epsilon_{\lambda+e_i} = q\epsilon_\lambda + [(n+1)+(\lambda_i+1)-i]_q.$$

Example: for $\lambda = (2,2)$,



$\epsilon_{(2,2)} = [0]_q$ $\quad$ $\epsilon_{(2,2,1)} = q[0]_q + [(4+1)+(0+1)-3]_q = [3]_q$ $\quad$ $\epsilon_{(3,2,1)} = q[3]_q + [(5+1)+(2+1)-1]_q = [8]_q + q[3]_q$

Computational Data

λ	Eigenvalues of $\mathsf{R2R}_{\lvert\lambda\rvert}^{(q)}$ on $G^\lambda \otimes H^\lambda$	Mult.
$(2,2)$	$[0]_q$, $[4]_q$	1
$(2,2,1)$	$[0]_q$	2
	$[3]_q$, $[5]_q$, $[5]_q + q[2]_q$	1
$(3,2,1)$	$[0]_q$	6
	$[4]_q$, $[6]_q$, $[8]_q$	2
	$[8]_q + q[3]_q$, $[8]_q + q[5]_q$, $[8]_q + q[5]_q + q^2[2]_q$, $[6]_q + q[3]_q$	1

**Note: Multiplicity here is normalized by $\dim(G^\lambda)$

Proof Strategy: The Algebra of Words

Let $M^{\langle n \rangle}$ be the $\mathbb{C}$-vector space generated by words of length n in the alphabet $\{1, 2, \dots, n\}$ allowing repeated letters.

Key idea: In [4], Mathas defines the right action by $\mathcal{H}_n(q)$ on $M^{\langle n \rangle}$ in terms of rules for how each T_i acts on words. We can use this action to find recurrences and eventually solve for the eigenvalues of $\mathsf{R2R}_n^{(q)}$.

Example: $(112)\cdot T_1 = q(112)$ and $(112)\cdot T_2 = (121)$.

For $q = 1$, Dieker and Saliola [1] also used $M^{\langle n \rangle}$ and defined two operators.

- $\mathsf{sh}_i(w) := \sum_{0 \leq j \leq n} w_1 \dots w_j \cdot i \cdot w_{j+1} \dots w_n.$
- $\Theta_{j,i}(w) := \sum_{\substack{1 \leq k \leq n \\ w_k = j}} w_1 \dots w_{k-1} \cdot i \cdot w_{k+1} \dots w_n.$

We define a q-analog of these operators and conjecture that for $i = 1$,

$$\mathsf{R2R}_{n+1}^{(q)}\mathsf{sh}_1^{(q)} = q\cdot\mathsf{sh}_1^{(q)}\mathsf{R2R}_n^{(q)} + [n+1]_q\mathsf{sh}_1^{(q)} + \sum_{j=1}^{n} q^{n+2-j}\cdot\mathsf{sh}_j^{(q)}\Theta_{j,1}^{(q)}.$$

We hope to find and prove the recursion for all i, then project them onto irreducible representations for our conjectural recurrence formula.

References

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Acknowledgements

This research is based on a project carried out at the 2022 University of Minnesota Twin Cities Combinatorics and Algebra REU. We are extremely thankful for UMN Math Department and the support from RTG grant NSF/DMS-1745638.