Bender-Knuth Moves on Linear Extensions of Posets

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BK Moves on Column Strict Tableaux

Definition

The **Bender-Knuth (BK) moves t_i** act on a column-strict tableau by swapping the contents of the letters i and $i+1$ in each row, fixing an i (resp. $i+1$) when there is an $i+1$ below (resp. i above) [2].

Illustration

$T =$

1	1	2	3
2	2	3	
3	4		

, $t_2(T) =$

1	1	2	2
2	3	3	
3	4		

.

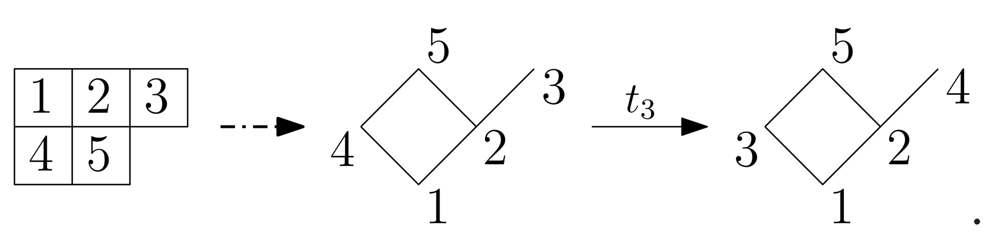
BK moves on Linear Extension of Posets

Definition

- **A linear extension of a poset P** is a linear order that is compatible with P. Denote by **LinExt(P)** set of all such linear extensions.
- The **BK moves t_i on LinExt(P)** swap two adjacent letters i and $i+1$ when they label incomparable elements of P and fix them otherwise. We denote the $\mathcal{BK}$ **group** on $\mathrm{LinExt}(P)$ as $\mathcal{BK}_P$ [3].

Illustration

For example. we can view an Standard Young Tableau as a linear extension of a Ferrers poset.



Main Questions

Definition: the **cactus relation** is $(t_i q_{jk})^2 = 1$ where $q_{jk} = q_{k-1}q_{k-j}q_{k-1}$ and $q_j = t_1(t_2t_1)\dots(t_jt_{j-1}\dots t_1)$ for all $i+1 < j < k$ [2].

Q1: For which posets P do the cactus relation hold inside $\mathcal{BK}_P$?

- Define an n-element poset P as **LE-cactus** if the cactus relation holds inside the group $\mathcal{BK}_P$.

Q2: For which posets P do we have $\mathcal{BK}_P = \mathfrak{S}_P$, the full symmetric group on $\mathrm{LinExt}(P)$?

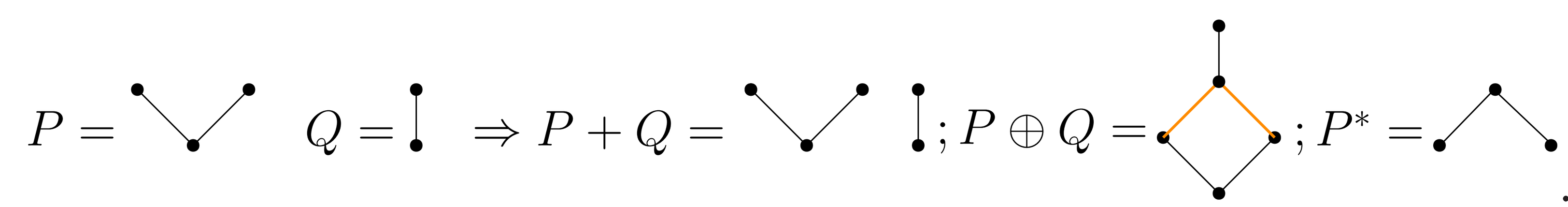
- Define **LE-symmetric posets** as the posets where $\mathcal{BK}_P$ equals the full symmetric group on $\mathrm{LinExt}(P)$

$\Rightarrow$ Which posets P are LE-cactus? What about LE-symmetric?

Results on LE-cactus posets

Poset operations

Given P, Q two finite posets, denote their ordinal sum as $P \oplus Q$, where each element of P is less than those of Q. We also denote their disjoint union $P+Q$ and dual P^* (opposite poset) as below:



Properties

Theorem. *P is LE-cactus iff every ordered ideal of P is LE-cactus.*

Theorem. *P and Q are LE-cactus iff $P+Q$ is LE-cactus.*

Theorem. *If P is LE-cactus, then $A_1 \oplus P$ and $A_2 \oplus P$ are LE-cactus, where A_m is an antichain of m elements. In particular, A_1 is a singleton.*

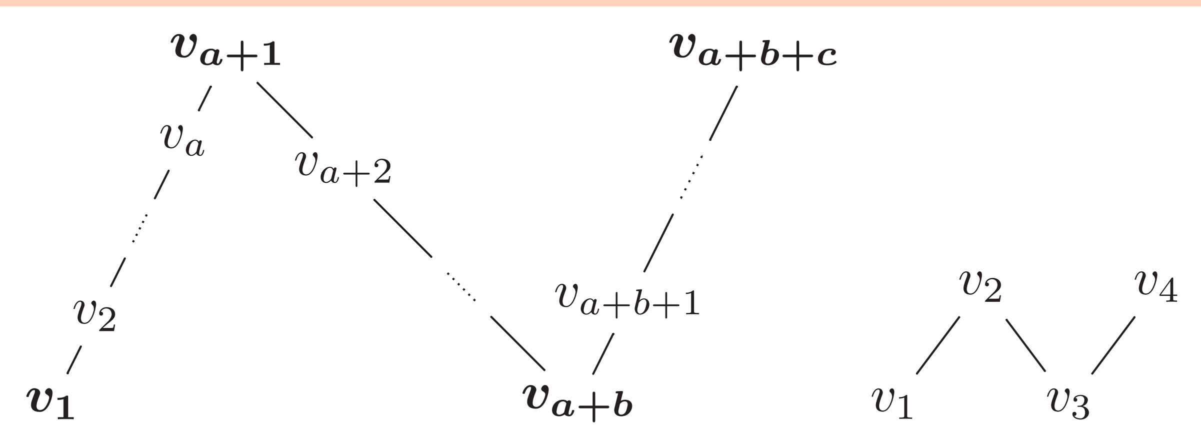
Theorem. *For any finite non-empty$P$, $A_m \oplus P$ is not LE-cactus if $m \geq 3$.*

LE-cactus Families

Theorem. *The Ferrers posets [2], shifted Ferrers posets, rooted trees, and the minuscule posets are all families of LE-cactus posets.*

Results on LE-symmetric posets

The $N_{a,b,c}$ Posets



[The poset $N_{a,b,c}$] [The poset $N_{1,2,1}$]

Properties

Theorem. *A disconnected poset P is LE-symmetric iff $P = C_n + A_1$, where C_n is the chain of n elements for some $n \geq 1$.*

Theorem. *If P is LE-symmetric, so is its dual P^*.*

Theorem. *There exists an isomorphism: $\mathcal{BK}_{P\oplus Q} \cong \mathcal{BK}_P \times \mathcal{BK}_Q$.*

LE-symmetric Families

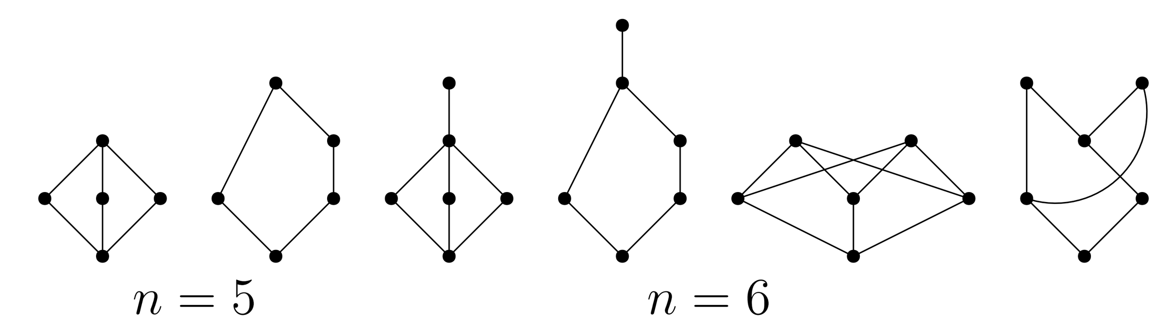
Theorem. *Posets $N_{1,2,c}$, $N_{1,b,1}$, and $N_{a,2,1}$ for all $a,b,c \geq 2$ are LE-symmetric.*

Theorem. *A series parallel poset is LE-symmetric iff is of the form $C_a \oplus (C_b + A_1) \oplus C_d$, where C_a, C_b and C_d are (possibly empty) chains.*

Conjectures and Future Directions

LE-cactus Posets

Based on *SageMath* computational data, there are more LE-cactus posets which are not covered by our previous results. We listed out some small examples here.



Motivated by Knuth's and Frame-Robinson-Thrall's hook-length formula that counts the linear extensions of rooted forest posets and Ferrers posets, Proctor defined a family of ***d*-complete posets** that include both.

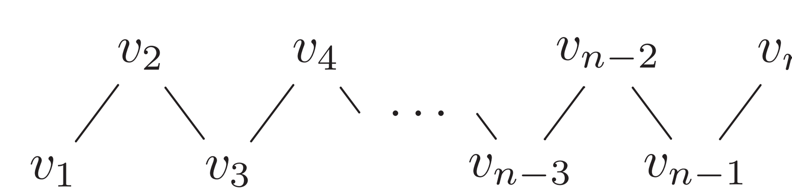
Conjecture. *All d-complete posets are LE-cactus.*

This family is computationally checked as LE-cactus for $n \leq 9$ by *SageMath*.

LE-symmetric Posets

Conjecture. *The posets $N_{a,b,c}$ are LE-symmetric for all $a, c \geq 1$ and $b \geq 2$.*

Conjecture. *The zigzag-poset Z_n below is LE-symmetric when n is even.*



Computationally, *SageMath* indicates poset $N_{a,b,c}$ is LE-symmetric for all $1 \leq a,c \leq 4$ and $2 \leq b \leq 5$; Z_n is LE-symmetric for all even $n \leq 10$

References

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[2] Michael Chmutov, Max Glick, and Pavlo Pylyavskyy. The Berenstein-Kirillov group and cactus groups. J. Comb. Algebra, 4(2):111–140, 2020.

[3] Richard P. Stanley. Promotion and evacuation. Electron. J. Combin., 16(2, Special volume in honor of Anders Björner):Research Paper 9, 24, 2009.

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