

# **ABOUT METAPLECTIC WHITTAKER FUNCTIONS**

Whittaker functions are special functions that arise in *p*-adic number theory and representation theory. Metaplectic Whittaker functions are Whittaker functions on metaplectic covering groups, which are central extensions of a reductive group by the  $n^{th}$  roots of unity.

Unlike Whittaker functions on the base group  $GL_r(F)$ , metaplectic Whittaker functions are not unique up to multiplicity. To understand the behavior of the space, we need to know how the cover affects the dimension. Theorem (Brylinski-Deligne [2], Frechette [3])

Every *n*-fold metaplectic cover of  $GL_r(F)$  corresponds to a bilinear form  $B_{c,d}$  for some  $c, d \in \mathbb{Z}$  that acts on  $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{Z}^r \times \mathbb{Z}^r$  by

$$B_{c,d}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T \cdot \begin{pmatrix} c & d & \dots & d \\ d & c & & d \\ \vdots & & \ddots & \vdots \\ d & d & \dots & c \end{pmatrix} \cdot \boldsymbol{y}.$$

### Theorem (McNamara [4], Frechette [3])

The dimension of the space  $\mathfrak{W}$  of Whittaker functions on a metaplectic cover of  $GL_r(F)$  is  $\dim(\mathfrak{W}) =$ 

where  $\Lambda_{fin}$  is the set of solutions  $(x_1, \ldots, x_r)^T \in \mathbb{Z}_n^r$  to

$$\begin{pmatrix} c & d & \dots & d \\ d & c & & d \\ \vdots & & \ddots & \vdots \\ d & d & \dots & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{n}$$

These are the **cocharacter equations**.

Diagonal numbers

Let 
$$d_1 = \gcd(c - d, n)$$
 and  $d_2 = \gcd(c + (r - 1)d, n)$ .

**Example:** r = 3, n = 10

We record  $|\Lambda_{fin}|$  for each  $c, d \in \mathbb{Z}_{10}$  when r = 3 and n = 10:

$$d_1 = 10$$
 1 2 1 2 5 2 1 2 1

In this case (where *n* and *r* are coprime),  $|\Lambda_{fin}| = d_1^{r-1} d_2$ .

# MEASURING THE SPACE OF METAPLECTIC WHITTAKER FUNCTIONS

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# MAIN THEOREM (A)

### Theorem

$$|\Lambda_{fin}| = \mathscr{d}_1^{r-1} \operatorname{gcd}\left(\mathscr{d}_2, \frac{dn}{\mathscr{d}_1}\right)$$

### **Proof outline**

A vector  $x \in \mathbb{Z}_n^r$  solves the **inhomogenous cocharacter equations** if

$$B_{c,d}\boldsymbol{x} \equiv a\boldsymbol{1}_r \pmod{n}$$

for some  $a \in \mathbb{Z}_n$ . Here,  $\mathbf{1}_r = (1, 1, \dots, 1)^T$ . The solutions to this are precisely vectors of the form

$$x_1(1, 1, \dots, 1)^T + \frac{n}{\ell_1}(0, v_2, \dots, v_n)^T$$

for constants  $x_1, v_2, \ldots, v_n \in \mathbb{Z}_n$ . There are  $n \mathscr{A}_1^{r-1}$  of these, and the fraction with  $a \equiv 0$  is  $gcd(\mathcal{A}_2, \frac{dn}{\mathcal{A}_1})/n$ .

# MAIN THEOREM (B)

### Definition

The **coroot equations** are the system of *r* equations:

$$(c-d)(x_i - x_r) \equiv 0 \pmod{n}$$
 for all  $i \in \{1, ..., r-1\}$ ,  
 $(c+(r-1)d)(x_1 + \dots + x_r) \equiv 0 \pmod{n}$ .

These arise from the root structure of  $GL_r(F)$ , and any solution to the cocharacter equations also solves the coroot equations.

### Theorem

The number of solutions to the coroot equations is

$$S_{coroot}(c,d,r,n) = \mathscr{d}_1^{r-1} \mathscr{d}_2 \operatorname{gcd}\left(\frac{n}{\mathscr{d}_1},\frac{n}{\mathscr{d}_2},r\right)$$

Counting the proportion of these which also solve the cocharacter equations yields another expression for  $\Lambda_{fin}$ :

$$|\Lambda_{fin}| = \frac{\mathscr{d}_1^{r-1}\mathscr{d}_2}{n} \operatorname{gcd}\left(\frac{n}{\mathscr{d}_1}, \frac{n}{\mathscr{d}_2}, r\right) \operatorname{lcm}\left(\frac{n}{\operatorname{gcd}(r, n)}, \operatorname{gcd}\left(\mathscr{d}_2, \frac{dn}{\mathscr{d}_1}\right)\right).$$

### **COROLLARIES**

- We have  $\dim(\mathfrak{W}) = 1$  (that is, of minimum size) if and only if  $c \equiv d \equiv$  $0 \pmod{n}$ .
- We have  $\dim(\mathfrak{W}) = n^r$  (that is, of maximum size) if and only if c dand c + (r-1)d are coprime to n.
- If *n* and *r* are relatively prime,  $\dim(\mathfrak{W}) = n^r / (\mathfrak{d}_1^{r-1} \mathfrak{d}_2)$ .

### Definition

There is a homomorphism

This map translates the action of intertwining operators on  $\mathfrak{W}$  into that of the quantum group on the module.

However, this map depends on a particular choice of basis for  $\mathfrak{W}$ . When is it well-defined regardless of that choice? Theorem

The map  $\theta_z$  is well-defined for any choice of basis precisely when

# **FUTURE WORK**







## QUANTUM GROUPS

One application of our approach is to investigate the connection between m and quantum group modules found by Brubaker-Buciumas-Bump for the case c = d = 1 [1], and Frechette [3] in general.

 $U = U_q(\mathfrak{gl}(n/\mathfrak{d}_1))$  is a *quantum group*, an affine quasitriangular Hopf algebra built from the Lie algebra  $\mathfrak{gl}$ . For each  $z \in \mathbb{C}$ , this quantum group has an *evaluation module*  $V_+(z)$ , with basis parametrized by  $\mathbb{Z}/(n/\mathcal{A}_1)\mathbb{Z}$ .

Thoerem (Brubaker-Bucimas-Bump [1], Frechette [3])

$$\mathcal{D}_{\boldsymbol{z}}:\mathfrak{W}^{\boldsymbol{z}}\to V_+(z_1)\otimes\cdots\otimes V_+(z_r)$$

 $\boldsymbol{\nu} \mapsto (r-1,...,2,1,0) - \boldsymbol{\nu} \pmod{n/d_1},$ 

$$gcd\left(\mathcal{d}_2, \frac{nd}{\mathcal{d}_1}\right) = gcd(c, d, n).$$

• Find the dimension of the space of metaplectic Whitaker functions for other reductive groups (the coroot strategy may be helpful here).

• Investigate the kernel and image of  $\theta_z$  and classify when this map is an isomorphism (or injection or surjection).

• Use this to understand better what quantum group modules we should use for other reductive groups.

### REFERENCES

[1] Ben Brubaker, Valentin Buciumas, Daniel Bump, and Nathan Gray, A Yang-Baxter equation for metaplectic ice, Commun. Number Theory Phys. 13 (2019), no. 1, 101–148. MR 3951106

[2] Jean-Luc Brylinski and Pierre Deligne, Central extensions of reductive groups by  $K_2$ , Publ. Math. Inst. Hautes Études Sci. (2001), no. 94, 5–85. MR 1896177

[3] Claire Frechette, Yang-Baxter Equations for General Metaplectic Ice, arXiv:2009.13669, (2020).

[4] Peter J. McNamara, Metaplectic Whittaker functions and crystal bases, Duke Math. J. 156 (2011), no. 1, 1–31. MR 2746386

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