Measuring the space of Metaplectic Whittaker functions

Ilani Axelrod-Freed[1] and Veronica Lang[2]
Mentor: Claire Frechette[3]
Massachusetts Institute of Technology[1], Smith College[2], Boston College[3]

About Metaplectic Whittaker functions

Whittaker functions are special functions that arise in p-adic number theory and representation theory. Metaplectic Whittaker functions are Whittaker functions on metaplectic covering groups, which are central extensions of a reductive group by the n^{th} roots of unity.

Unlike Whittaker functions on the base group $GL_r(F)$, metaplectic Whittaker functions are not unique up to multiplicity. To understand the behavior of the space, we need to know how the cover affects the dimension.

Theorem (Brylinski-Deligne [2], Frechette [3])

Every n-fold metaplectic cover of $GL_r(F)$ corresponds to a bilinear form $B_{c,d}$ for some $c, d \in \mathbb{Z}$ that acts on $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{Z}^r \times \mathbb{Z}^r$ by

$$B_{c,d}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T \begin{pmatrix} c & d & \dots & d \\ d & c & & d \\ \vdots & & \ddots & \vdots \\ d & d & \dots & c \end{pmatrix} \boldsymbol{y}.$$

Theorem (McNamara [4], Frechette [3])

The dimension of the space W of Whittaker functions on a metaplectic cover of $GL_r(F)$ is

$$\dim(\mathrm{W}) = \frac{n^r}{|\Lambda_{fin}|}$$

where Λ_{fin} is the set of solutions $(x_1, \dots, x_r)^T \in \mathbb{Z}_n^r$ to

$$\begin{pmatrix} c & d & \dots & d \\ d & c & & d \\ \vdots & & \ddots & \vdots \\ d & d & \dots & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{n}$$

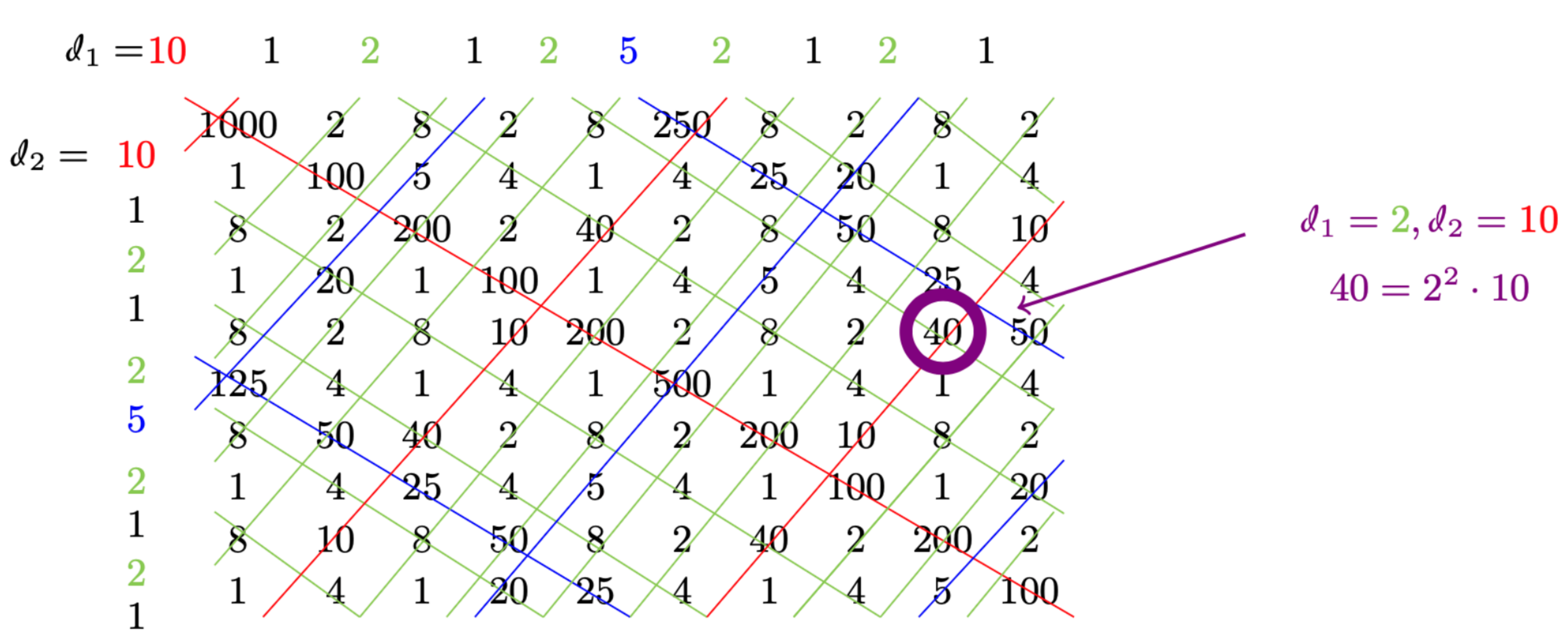
These are the **cocharacter equations**.

Diagonal numbers

Let $\partial_1 = \gcd(c - d, n)$ and $\partial_2 = \gcd(c + (r-1)d, n)$.

Example: $r = 3, n = 10$

We record $|\Lambda_{fin}|$ for each $c, d \in \mathbb{Z}_{10}$ when $r = 3$ and $n = 10$:



In this case (where n and r are coprime), $|\Lambda_{fin}| = \partial_1^{r-1}\partial_2$.

Main Theorem (A)

Theorem

$$|\Lambda_{fin}| = \partial_1^{r-1} \gcd\left(\partial_2, \frac{dn}{\partial_1}\right)$$

Proof outline

A vector $\boldsymbol{x} \in \mathbb{Z}_n^r$ solves the **inhomogenous cocharacter equations** if

$$B_{c,d}\boldsymbol{x} \equiv a\mathbf{1}_r \pmod{n}$$

for some $a \in \mathbb{Z}_n$. Here, $\mathbf{1}_r = (1, 1, \dots, 1)^T$.
The solutions to this are precisely vectors of the form

$$x_1(1, 1, \dots, 1)^T + \frac{n}{\partial_1}(0, v_2, \dots, v_n)^T$$

for constants $x_1, v_2, \dots, v_n \in \mathbb{Z}_n$. There are $n\partial_1^{r-1}$ of these, and the fraction with $a \equiv 0$ is $\gcd(\partial_2, \frac{dn}{\partial_1})/n$.

Main Theorem (B)

Definition

The **coroot equations** are the system of r equations:

$$(c-d)(x_i - x_r) \equiv 0 \pmod{n} \quad \text{for all } i \in \{1, \dots, r-1\},$$
$$(c + (r-1)d)(x_1 + \dots + x_r) \equiv 0 \pmod{n}.$$

These arise from the root structure of $GL_r(F)$, and any solution to the cocharacter equations also solves the coroot equations.

Theorem

The number of solutions to the coroot equations is

$$S_{coroot}(c, d, r, n) = \partial_1^{r-1}\partial_2 \gcd\left(\frac{n}{\partial_1}, \frac{n}{\partial_2}, r\right).$$

Counting the proportion of these which also solve the cocharacter equations yields another expression for Λ_{fin}:

$$|\Lambda_{fin}| = \frac{\partial_1^{r-1}\partial_2}{n} \gcd\left(\frac{n}{\partial_1}, \frac{n}{\partial_2}, r\right) \operatorname{lcm}\left(\frac{n}{\gcd(r, n)}, \gcd\left(\partial_2, \frac{dn}{\partial_1}\right)\right).$$

Corollaries

- We have $\dim(\mathrm{W}) = 1$ (that is, of minimum size) if and only if $c - d \equiv 0 \pmod{n}$.
- We have $\dim(\mathrm{W}) = n^r$ (that is, of maximum size) if and only if $c - d$ and $c + (r-1)d$ are coprime to n.
- If n and r are relatively prime, $\dim(\mathrm{W}) = n^r / (\partial_1^{r-1}\partial_2)$.

Quantum Groups

One application of our approach is to investigate the connection between W and quantum group modules found by Brubaker-Buciumas-Bump for the case $c = d = 1$ [1], and Frechette [3] in general.

Definition

$U = U_q(\widehat{\mathfrak{gl}}(n/\partial_1))$ is a *quantum group*, an affine quasitriangular Hopf algebra built from the Lie algebra $\mathfrak{gl}$. For each $z \in \mathbb{C}$, this quantum group has an *evaluation module* $V_+(z)$, with basis parametrized by $\mathbb{Z}/(n/\partial_1)\mathbb{Z}$.

Theorem (Brubaker-Bucimas-Bump [1], Frechette [3])

There is a homomorphism

$$\theta_{\boldsymbol{z}} : \mathrm{W}^{\boldsymbol{z}} \to V_+(z_1) \otimes \dots \otimes V_+(z_r)$$
$$\boldsymbol{\nu} \mapsto (r-1, \dots, 2, 1, 0) - \boldsymbol{\nu} \pmod{n/\partial_1},$$

This map translates the action of intertwining operators on W into that of the quantum group on the module.

However, this map depends on a particular choice of basis for W. When is it well-defined regardless of that choice?

Theorem

The map θ_z is well-defined for any choice of basis precisely when

$$\gcd\left(\partial_2, \frac{nd}{\partial_1}\right) = \gcd(c, d, n).$$

Future Work

- Find the dimension of the space of metaplectic Whitaker functions for other reductive groups (the coroot strategy may be helpful here).
- Investigate the kernel and image of $\theta_{\boldsymbol{z}}$ and classify when this map is an isomorphism (or injection or surjection).
- Use this to understand better what quantum group modules we should use for other reductive groups.

References

[1] Ben Brubaker, Valentin Buciumas, Daniel Bump, and Nathan Gray, *A Yang-Baxter equation for metaplectic ice*, Commun. Number Theory Phys. 13 (2019), no. 1, 101–148. MR 3951106
[2] Jean-Luc Brylinski and Pierre Deligne, *Central extensions of reductive groups by K_2*, Publ. Math. Inst. Hautes Études Sci. (2001), no. 94, 5–85. MR 1896177
[3] Claire Frechette, *Yang-Baxter Equations for General Metaplectic Ice*, arXiv:2009.13669, (2020).
[4] Peter J. McNamara, *Metaplectic Whittaker functions and crystal bases*, Duke Math. J. **156** (2011), no. 1, 1–31. MR 2746386

Acknowledgements

This research is based on a project carried out at the 2022 University of Minnesota Twin Cities Combinatorics and Algebra REU. We are extremely thankful for UMN Math Department and the support from RTG grant NSF/DMS-1745638.