Minimal Matchings for dP3 Cluster Variables

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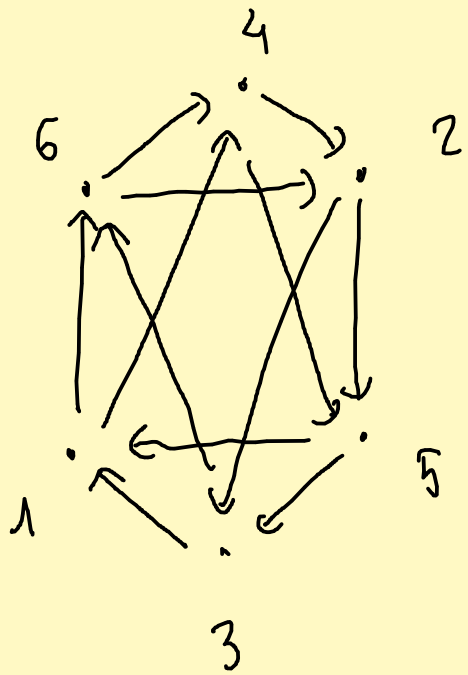
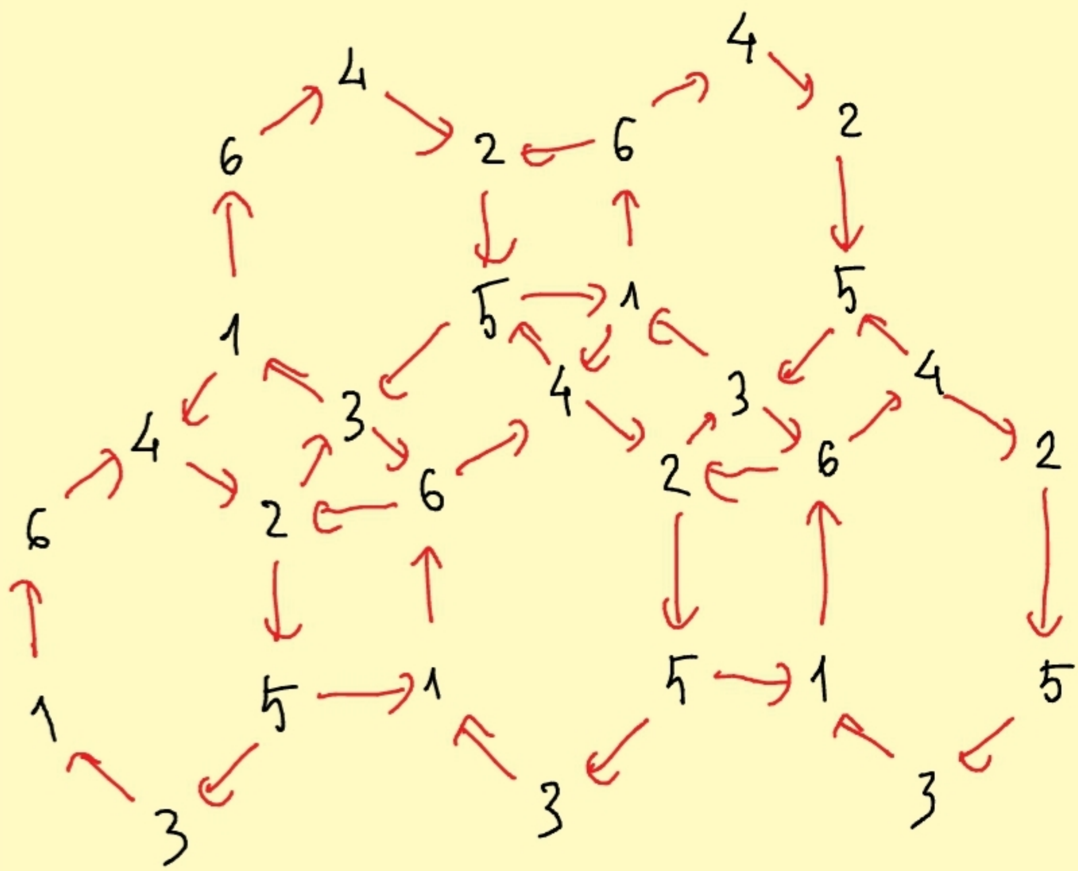
0. Overview:

- Aztec Castles and dP3 cluster variables
- Framed dP3 quiver
- Minimal matching of Aztec Castles
- Proof sketch

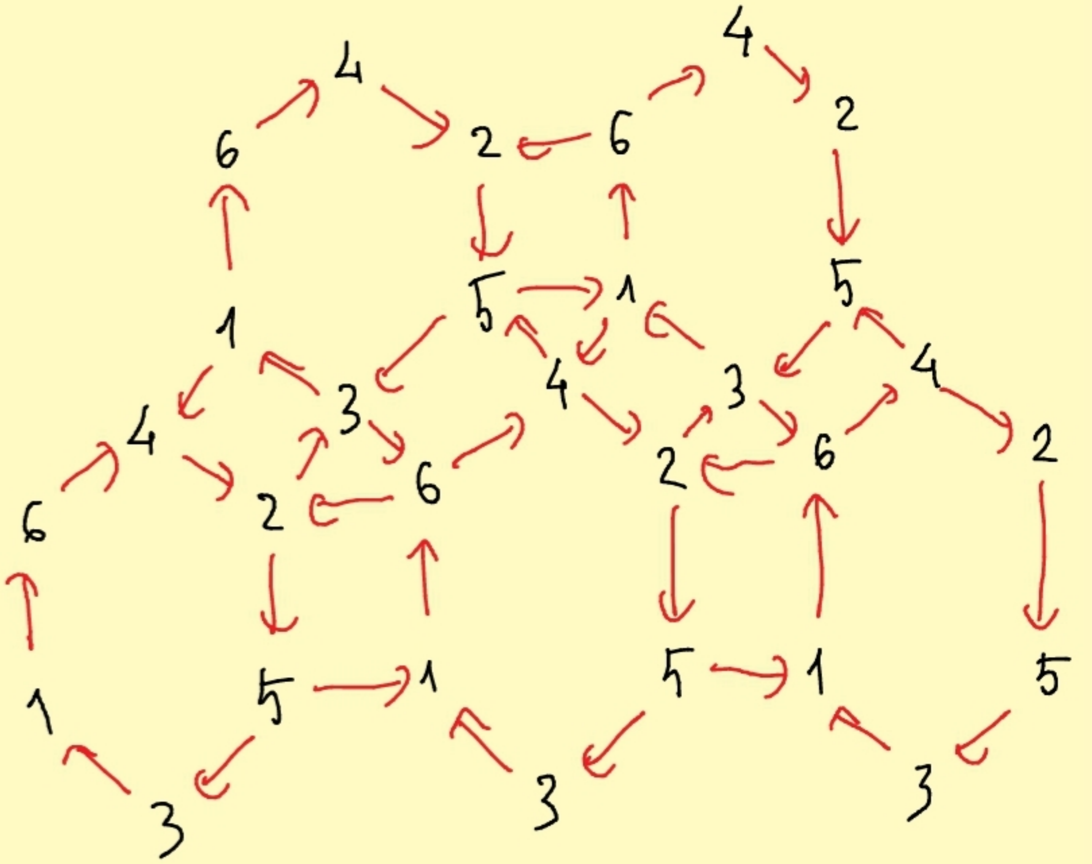
1. Aztec Castles and dP3 cluster variables:



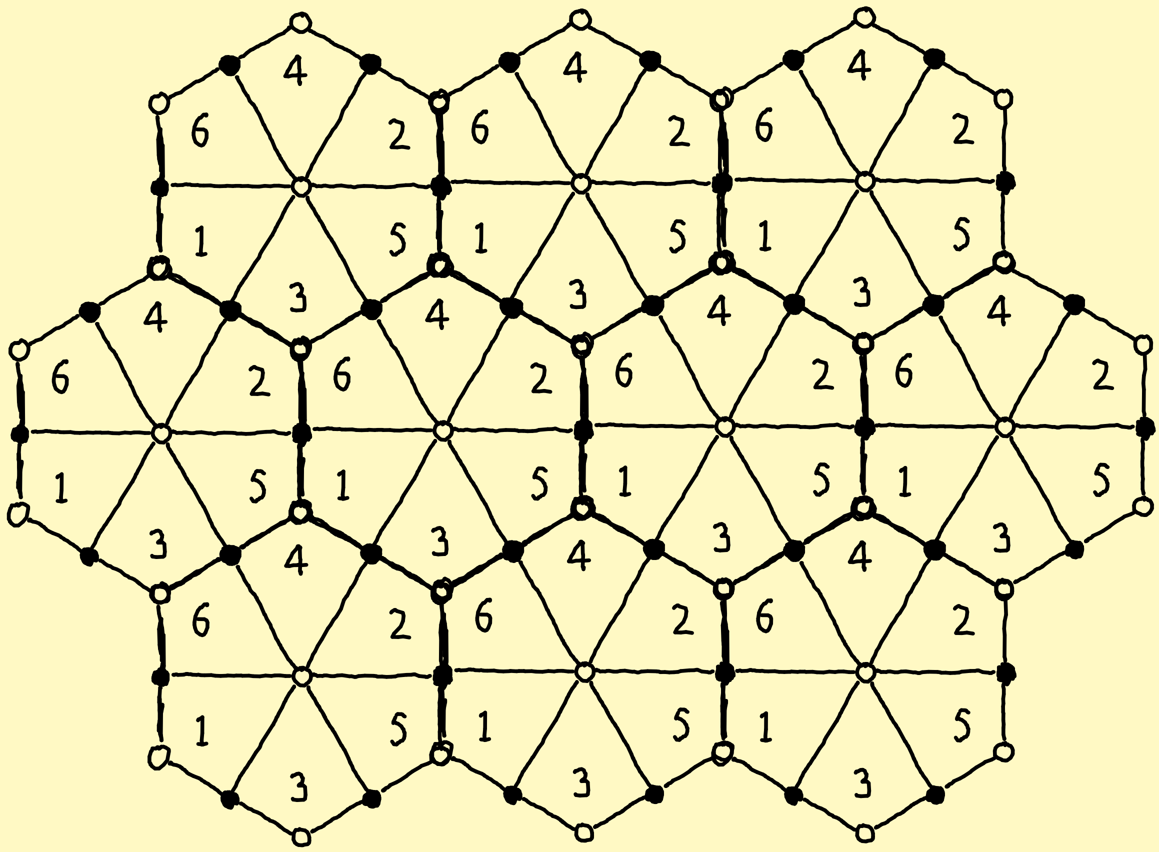
dP3 quiver



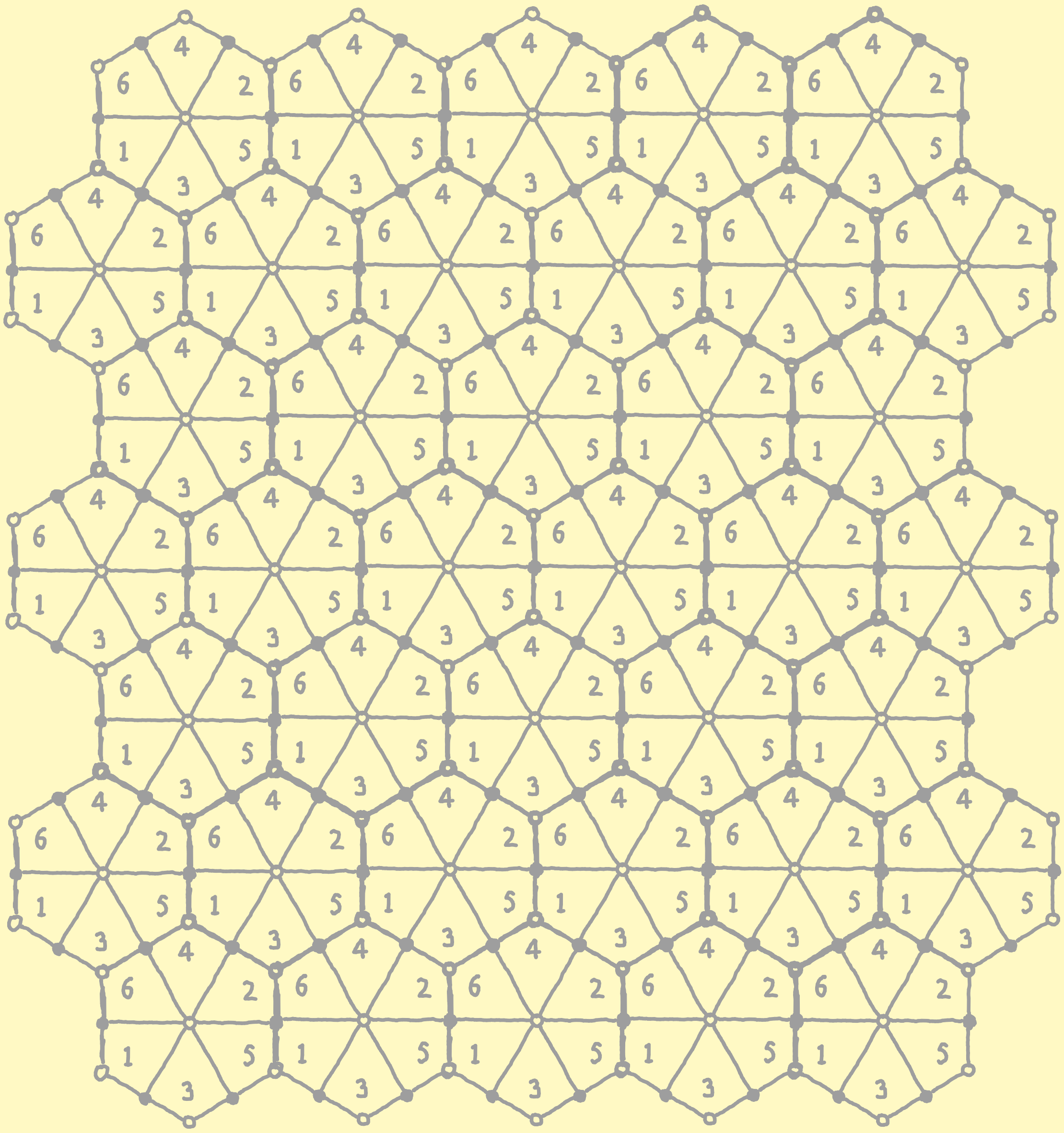
Unfolded dP3 quiver



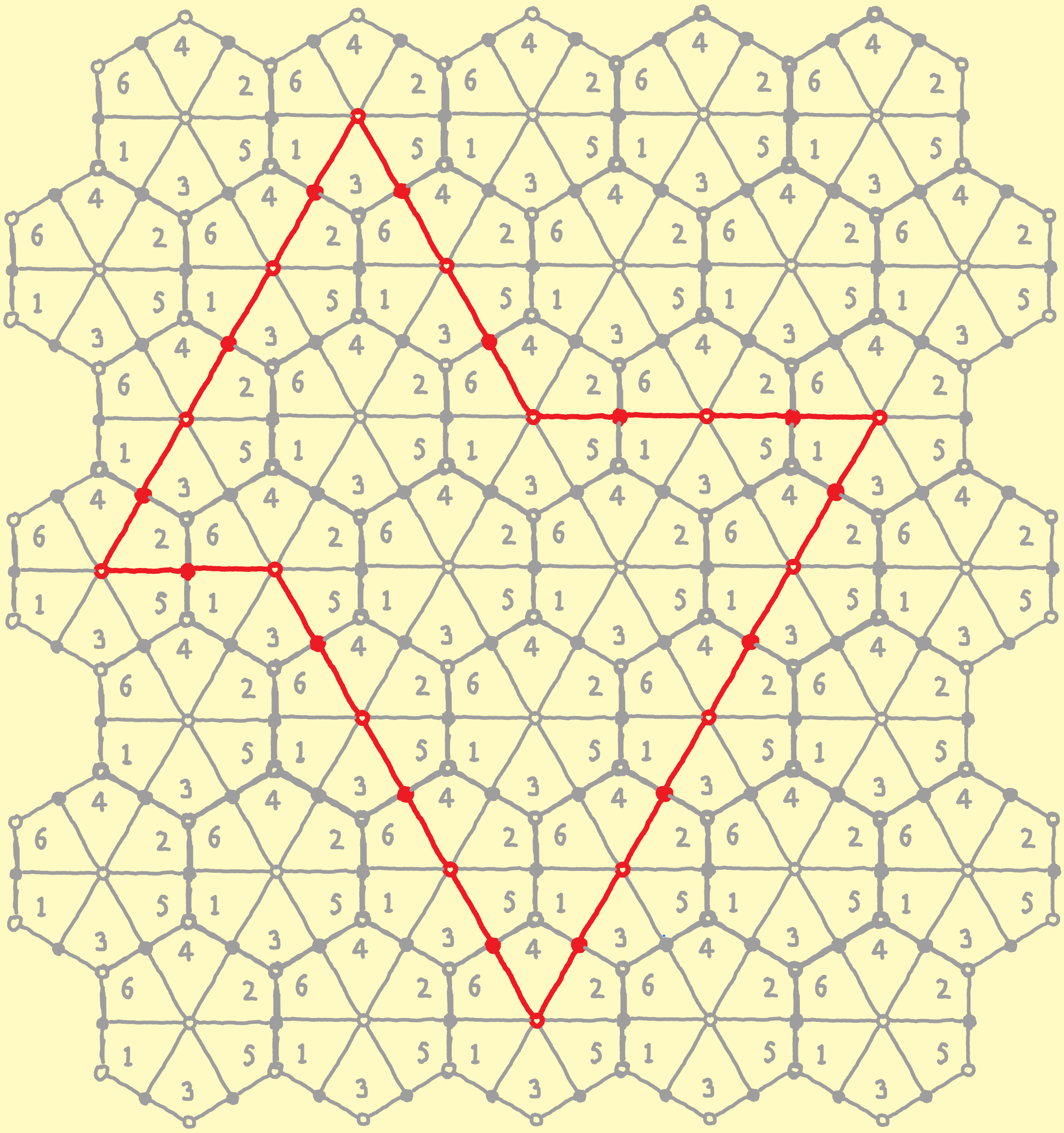
Unfolded dP3 quiver



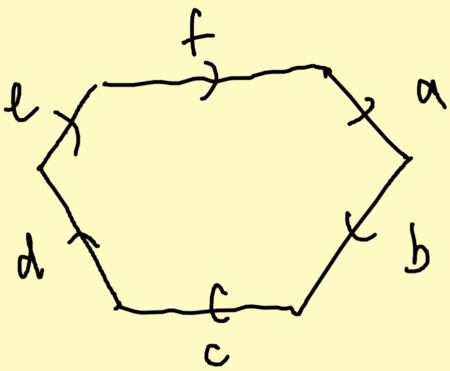
Brane tiling (dP3 lattice)



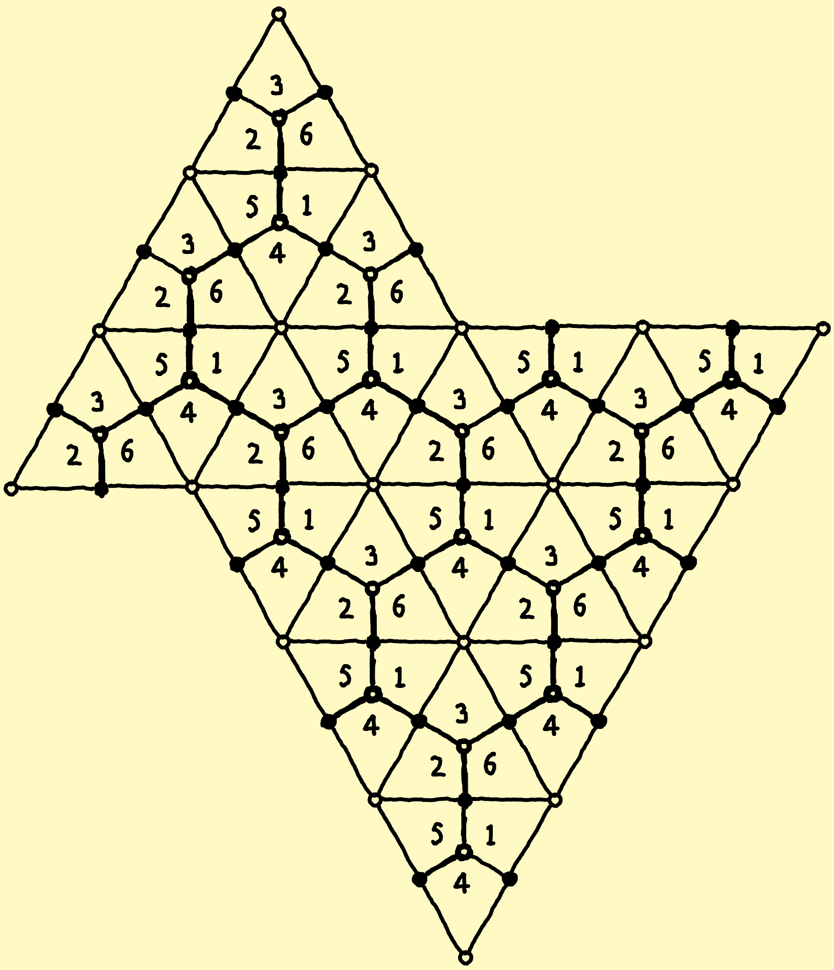
Start with a tuple (a, b, c, d, e, f)



Go along the lattice in clockwise direction



Ex : (3, -4 , 2, 2, -3, 1)

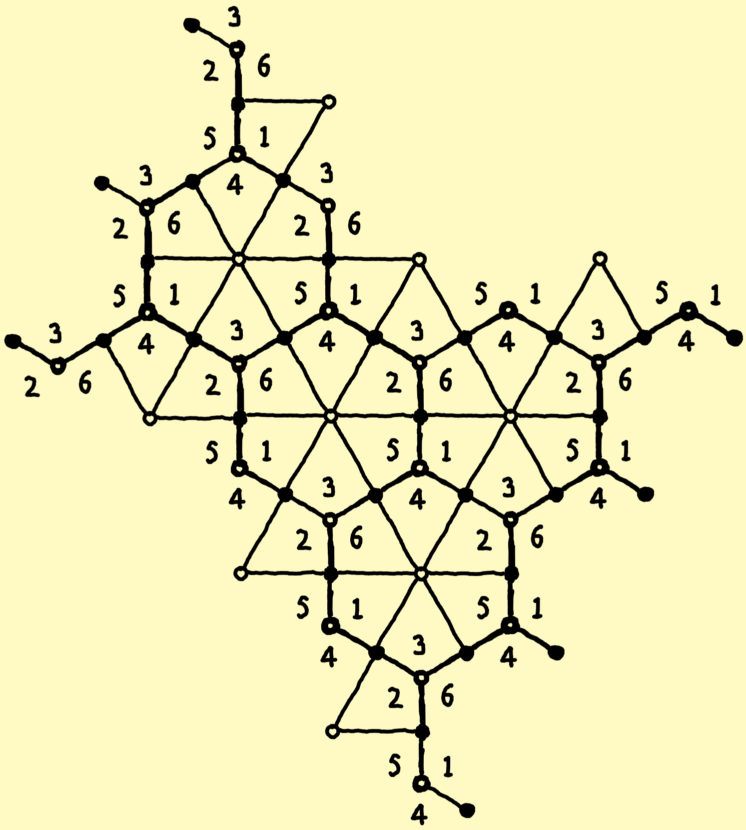


Side w/ positive length: delete all black vertices

Side w/ negative length: delete all white vertices

Side w/ 0 length:

keep if between 2 positive lengths

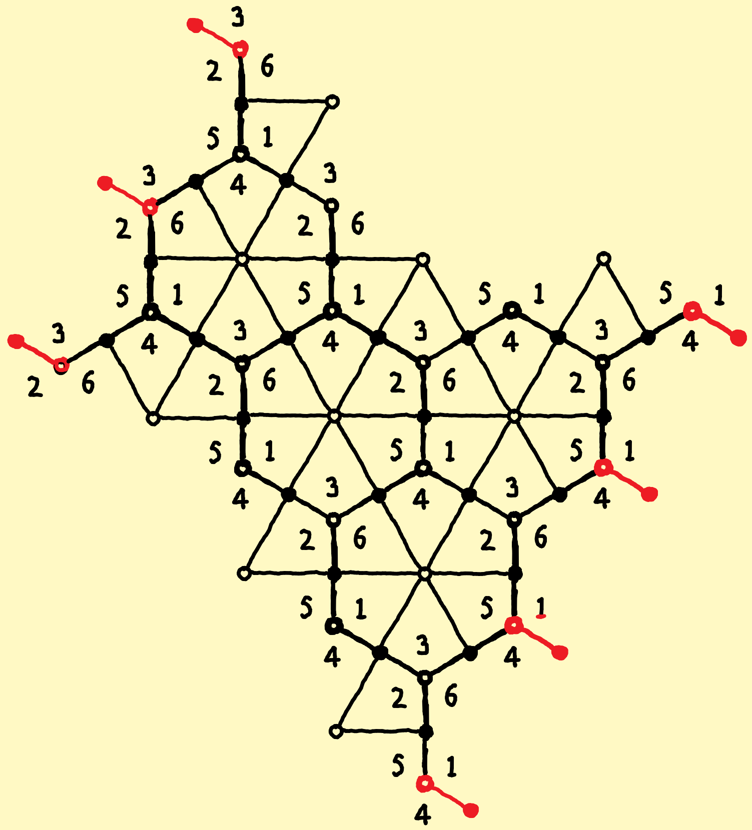


Side w/ positive length: delete all black vertices

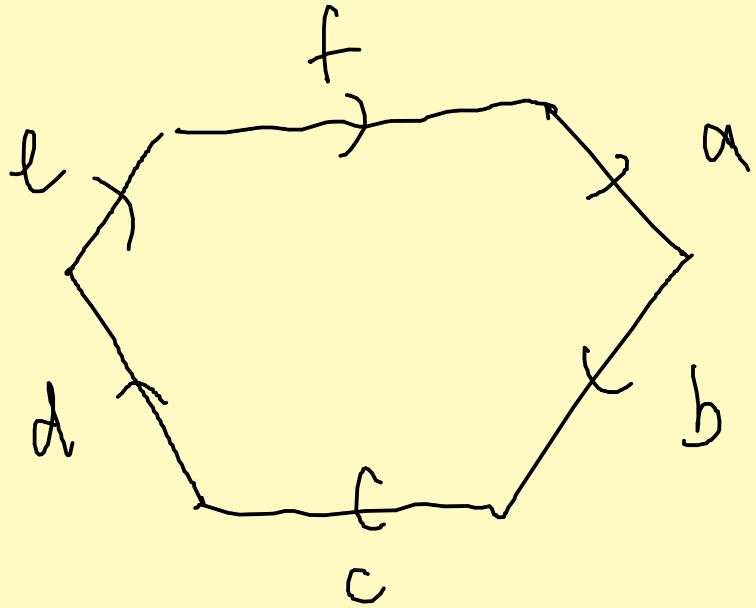
Side w/ negative length: delete all white vertices

Side w/ 0 length:

keep if between 2 positive lengths



Dangling edges : delete or keep up to you



1) $a+b=d+e$

2) $b+c=e+f$

Black = white

3) $a+b+c+d+e+f=1$

$\Rightarrow$ Depend on 3 variables only

$$C_{i,j,k} = (j+k,\ -i-j-k,\ i+k,$$
$$j+1-k,\ -i-j-1+k,\ i+1-k)$$

Tl;dr:

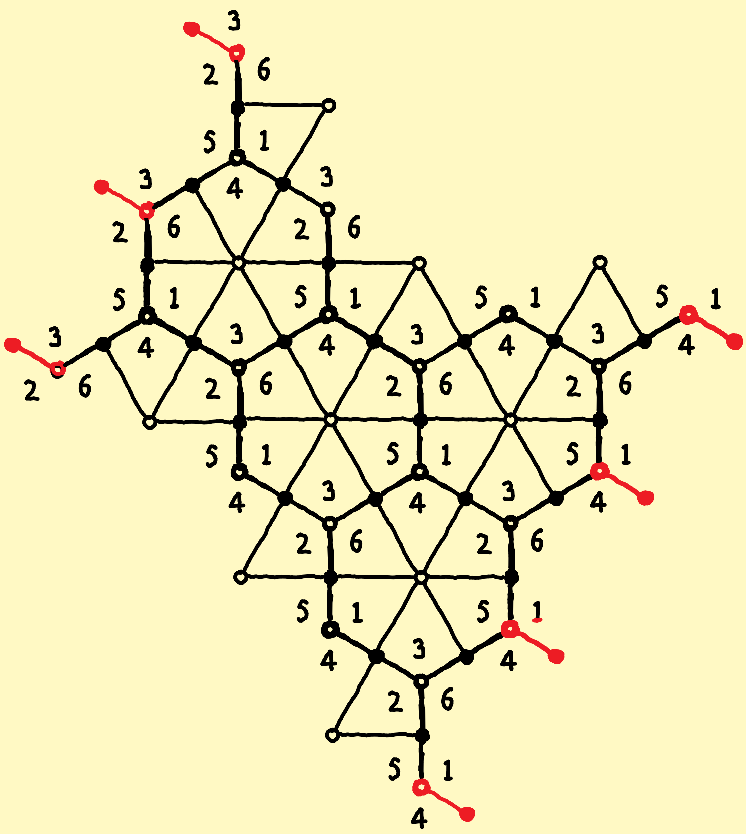
$$(i, j, k)$$

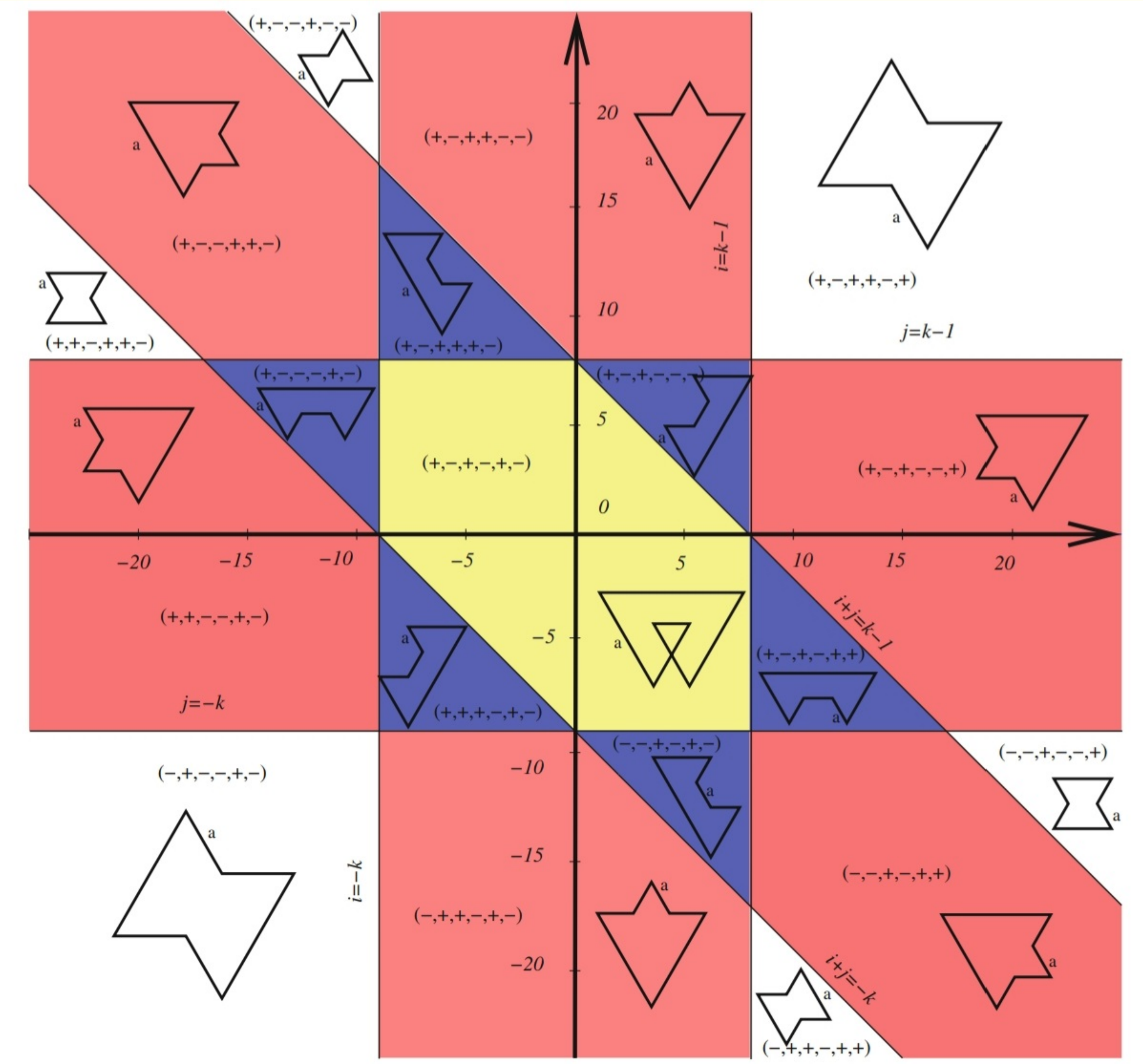
$$\downarrow$$

$$(j+k,\ -i-j-k,\ i+k,\ j+1-k,\ -i-j-1+k,\ i+1-k)$$

$$\downarrow$$

$$C_{i,j,k}$$

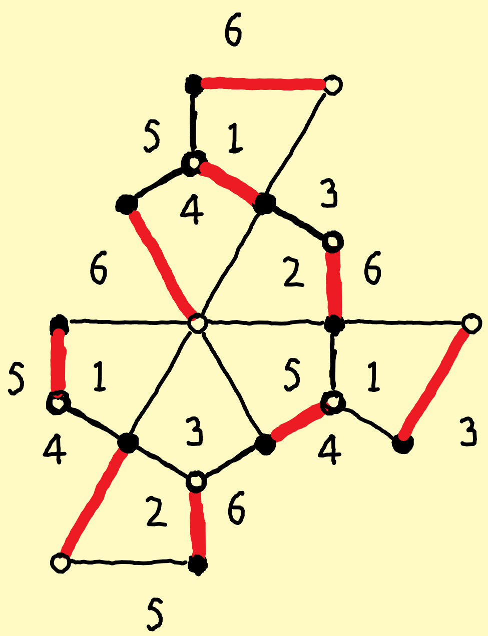




Possible Castle shapes

Def: A perfect matching is a set of edges M s/t each vertex is incident to exactly one edge in M.

Example:

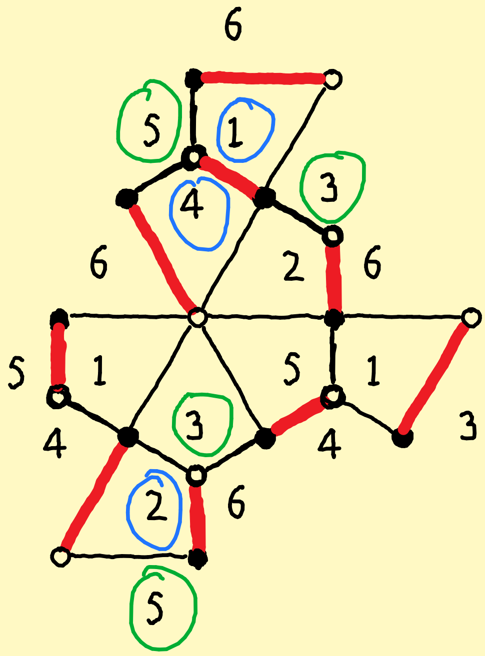


Weight of a perfect matching:

$$\mathrm{wt}(m) = \frac{\prod \text{covering faces}}{\prod \text{faces incident to edges}}$$

$$= \prod_{\text{face } i} x_i^{\,1-e(i)}, \quad e(i) = \#\text{ edges incident to } i$$

Example:



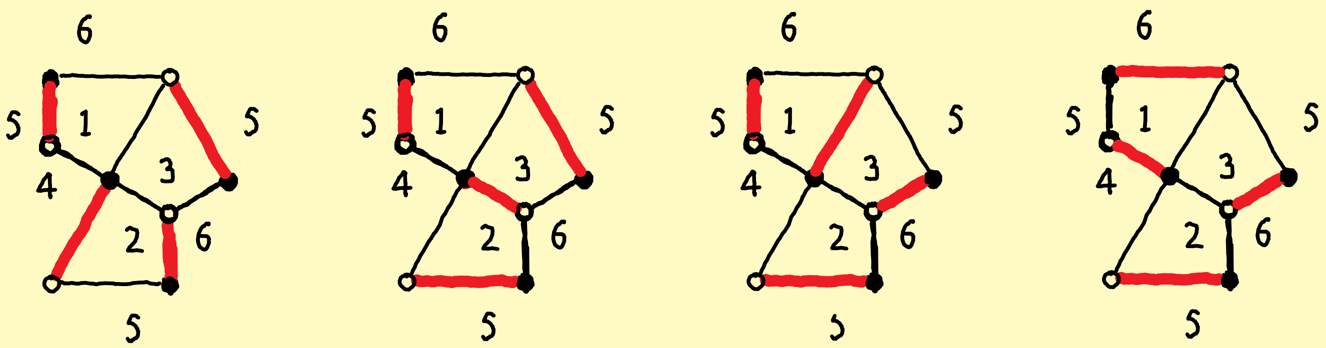
$$\mathrm{wt}(m) = \frac{x_1^3\, x_2^2\, x_3^3\, x_4^3\, x_5^4\, x_6^4}{x_1^4\, x_2^3\, x_3\, x_4^4\, x_5^2\, x_6^4}$$

$$= \frac{x_3^2\, x_5^2}{x_1\, x_2\, x_4}$$

Weight of a Castle:

$$\mathrm{wt}(C) = \sum_{\text{PM } m} \mathrm{wt}(m)$$

Example:



$$\frac{x_5 x_6}{x_2} + \frac{x_4 x_6^2}{x_2 x_3} + \frac{x_4 x_5 x_6}{x_1 x_3} + \frac{x_5^2}{x_1}$$

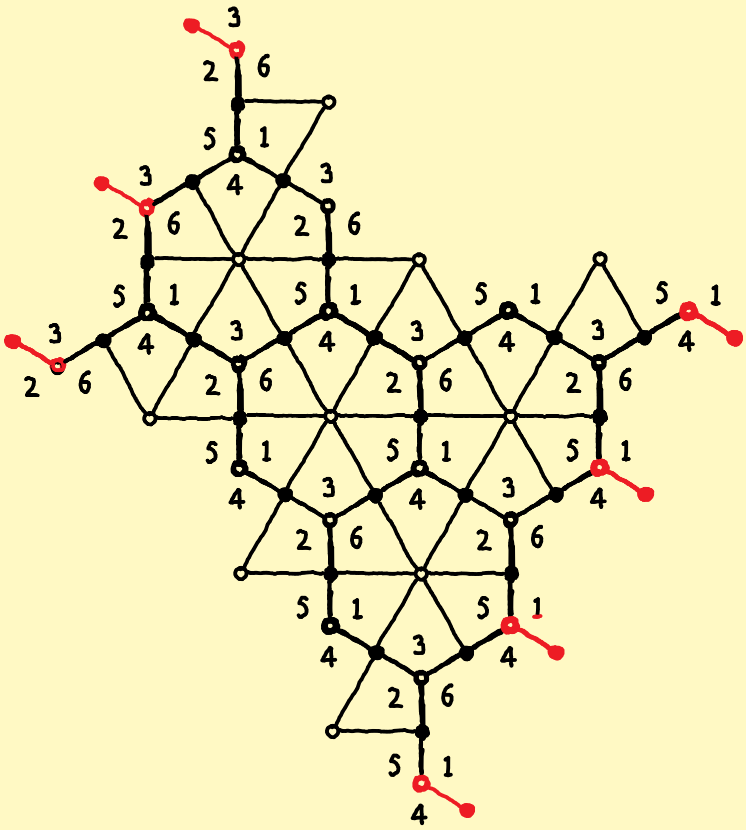
$$= \frac{x_1 x_3 x_5 x_6 + x_1 x_4 x_6^2 + x_2 x_4 x_5 x_6 + x_2 x_3 x_5^2}{x_1 x_2 x_3}$$

TL;dr:

$$(i,j,k)$$

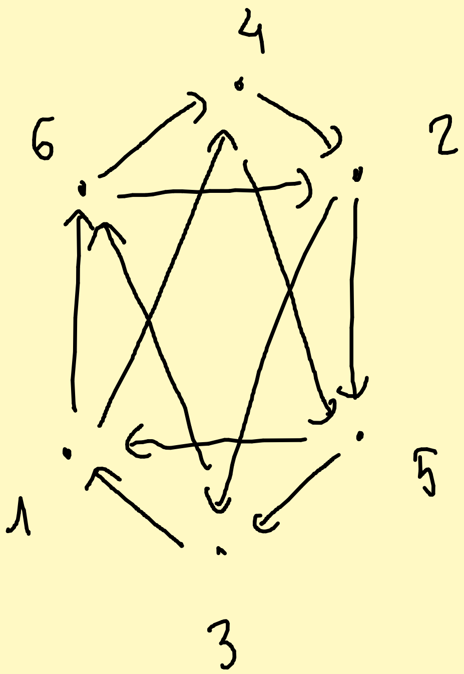
$$\Big\downarrow$$

$$C_{i,j,k}$$



$$\Big\downarrow$$

$$\mathrm{wt}(C_{i,j,k}) = \frac{P(x_1,\dots,x_6)}{x_1^{d_1}\cdots x_6^{d_6}}$$



We will study the following actions:

$$\tau_1 = \mu_1 \circ \mu_2 \circ (12)$$

$$\tau_2 = \mu_3 \circ \mu_4 \circ (34)$$

$$\tau_3 = \mu_5 \circ \mu_6 \circ (56)$$

$$\tau_4 = \mu_1 \circ \mu_4 \circ \mu_1 \circ \mu_5 \circ \mu_1 \circ (145)$$

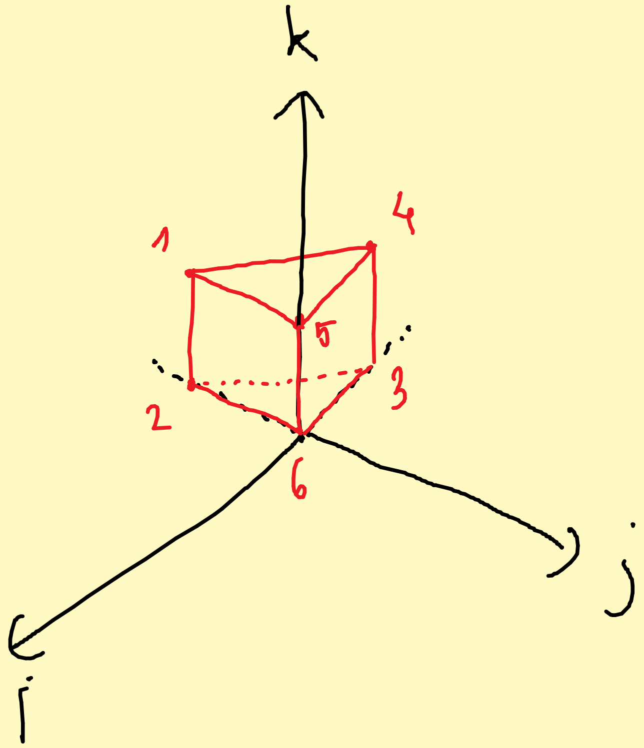
$$\tau_5 = \mu_2 \circ \mu_3 \circ \mu_2 \circ \mu_6 \circ \mu_2 \circ (236)$$

Defn. Cluster mutation:

μ_i changes x_i to x_i' satisfying

$$x_i x_i' = \prod_{i \to j} x_j + \prod_{j \to i} x_j$$

Now start with a prism:

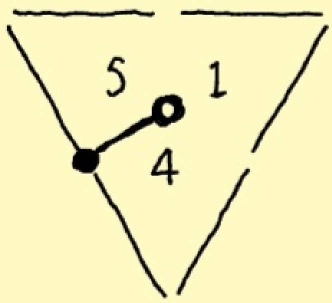


$1 = (0, -1, 1)$ $\quad 3 = (-1, 0, 0)$ $\quad 5 = (0, 0, 1)$

$2 = (0, -1, 0)$ $\quad 4 = (-1, 0, 1)$ $\quad 6 = (0, 0, 0)$

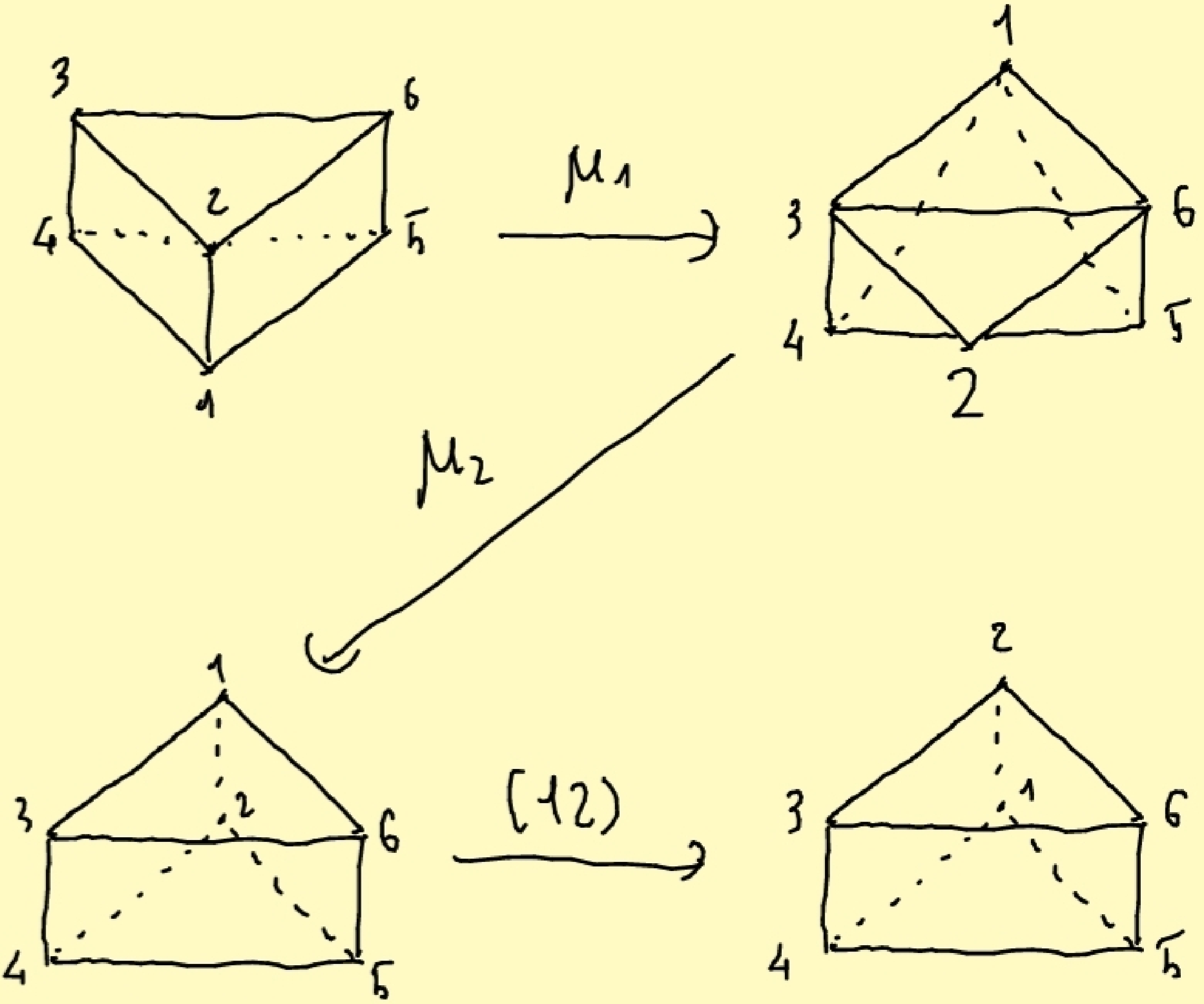
Why?

$$C_0^{-1,1} = (0, 0, 1, -1, 1, 0)$$

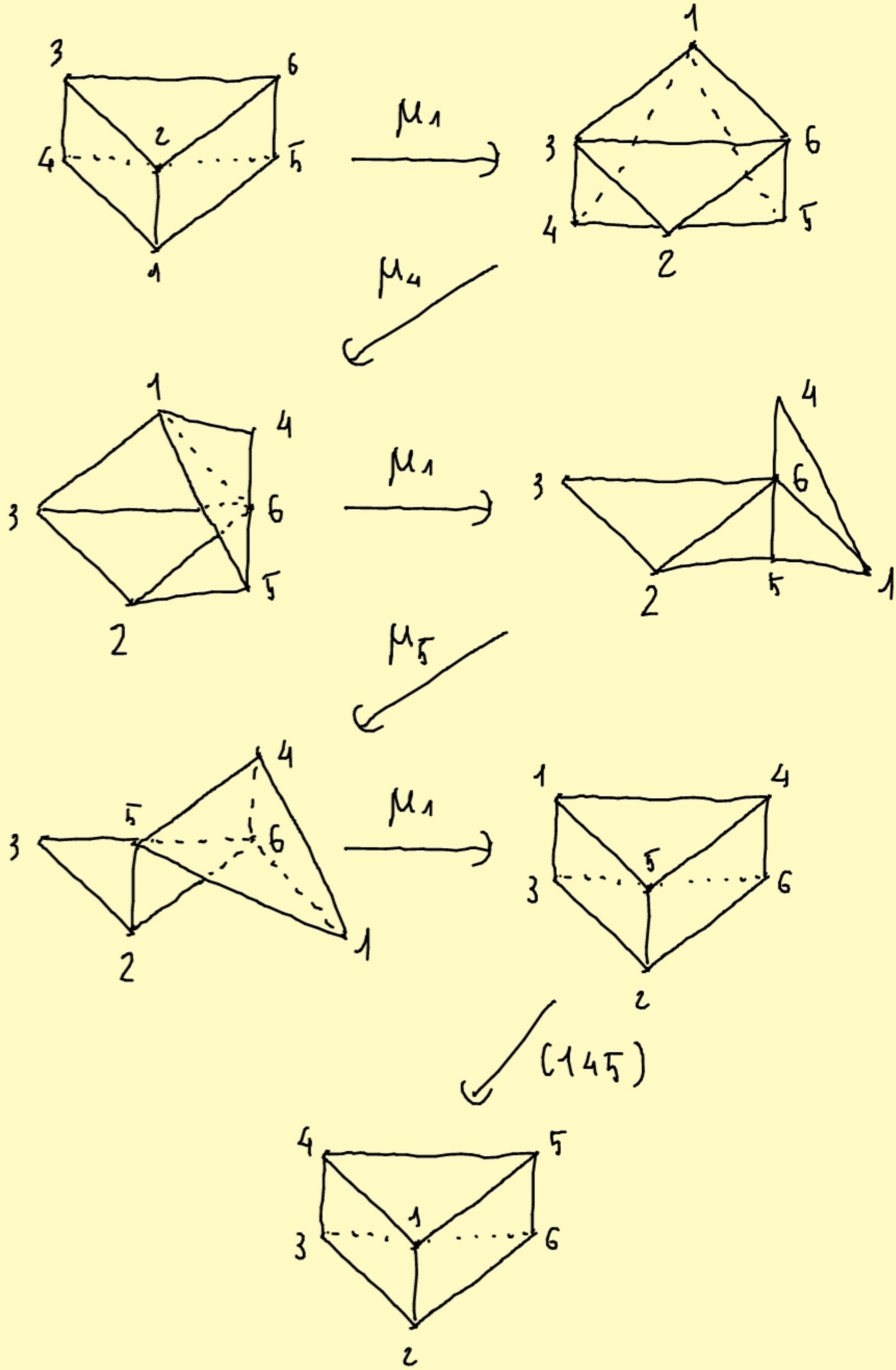


$$\frac{x_1 x_5 x_4}{x_5 x_4} = x_1$$

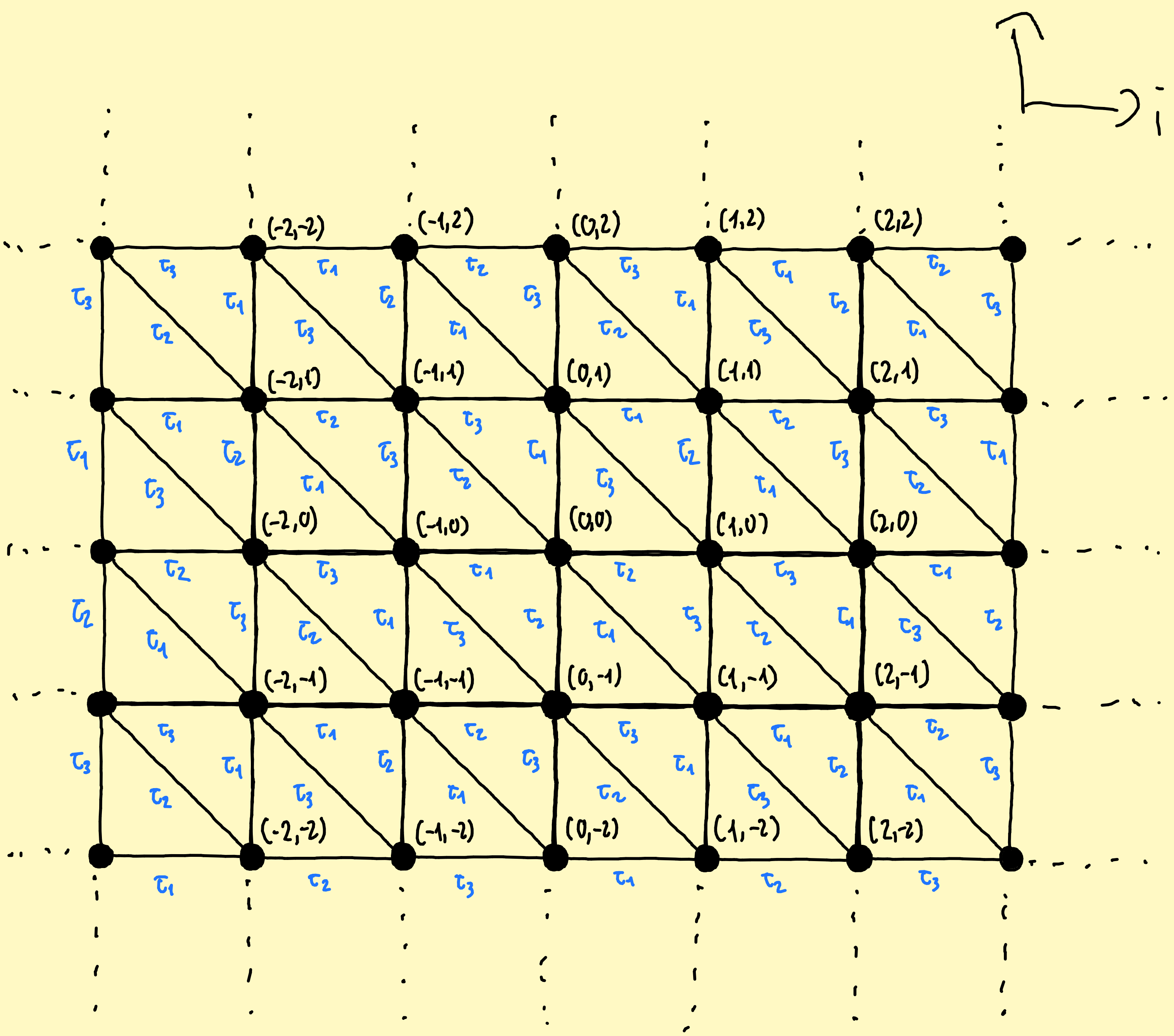
τ_1 :



τ_4



How to move the prism around?



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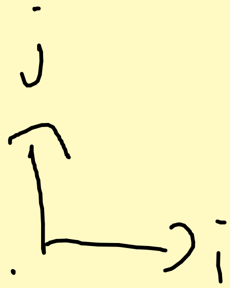

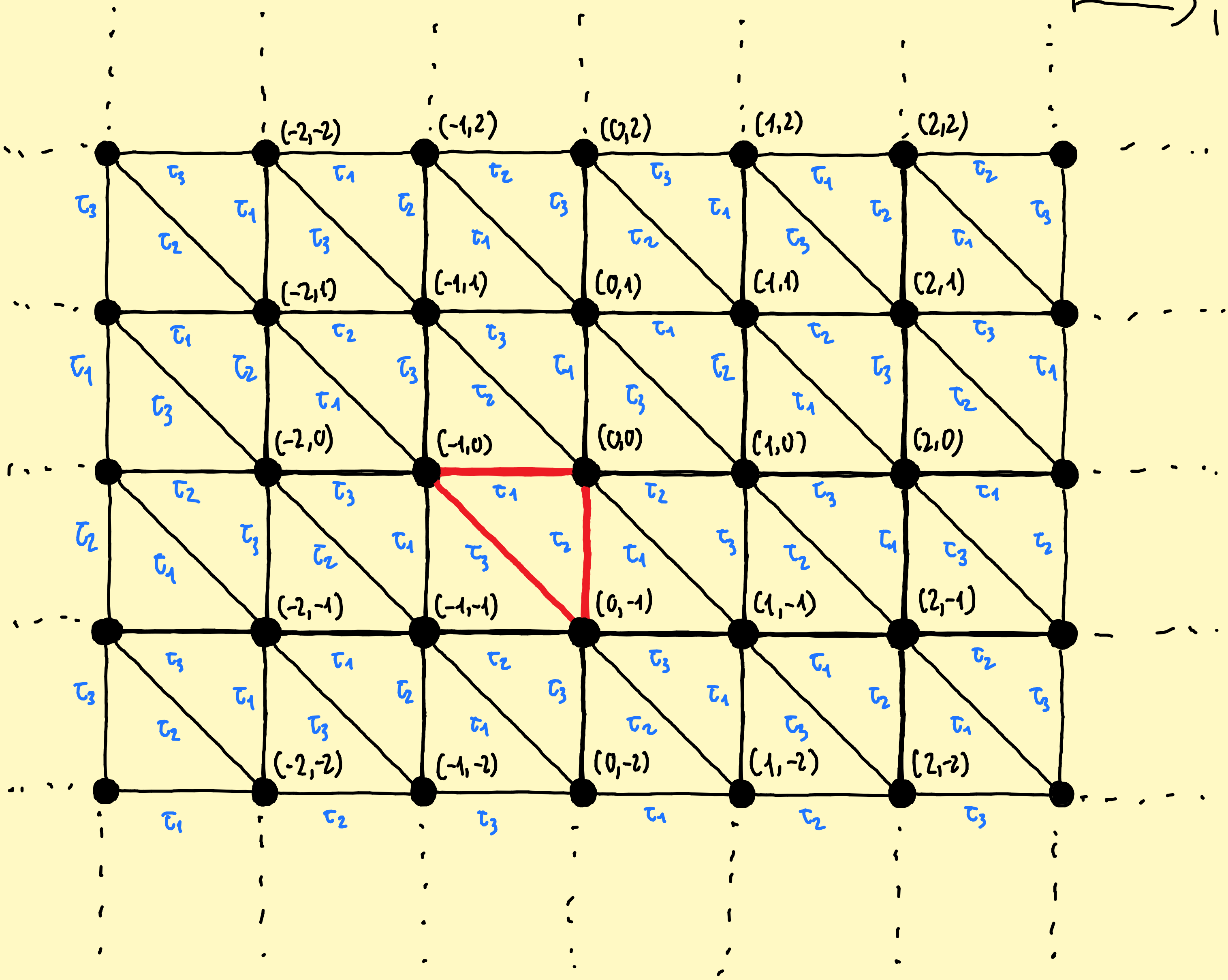

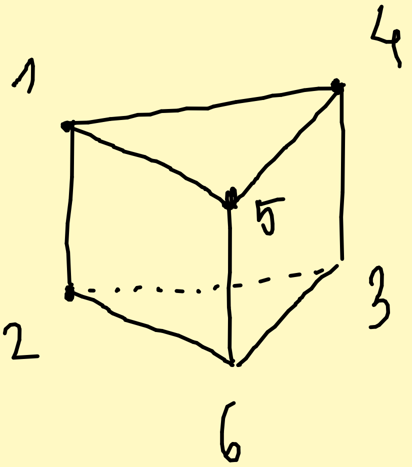

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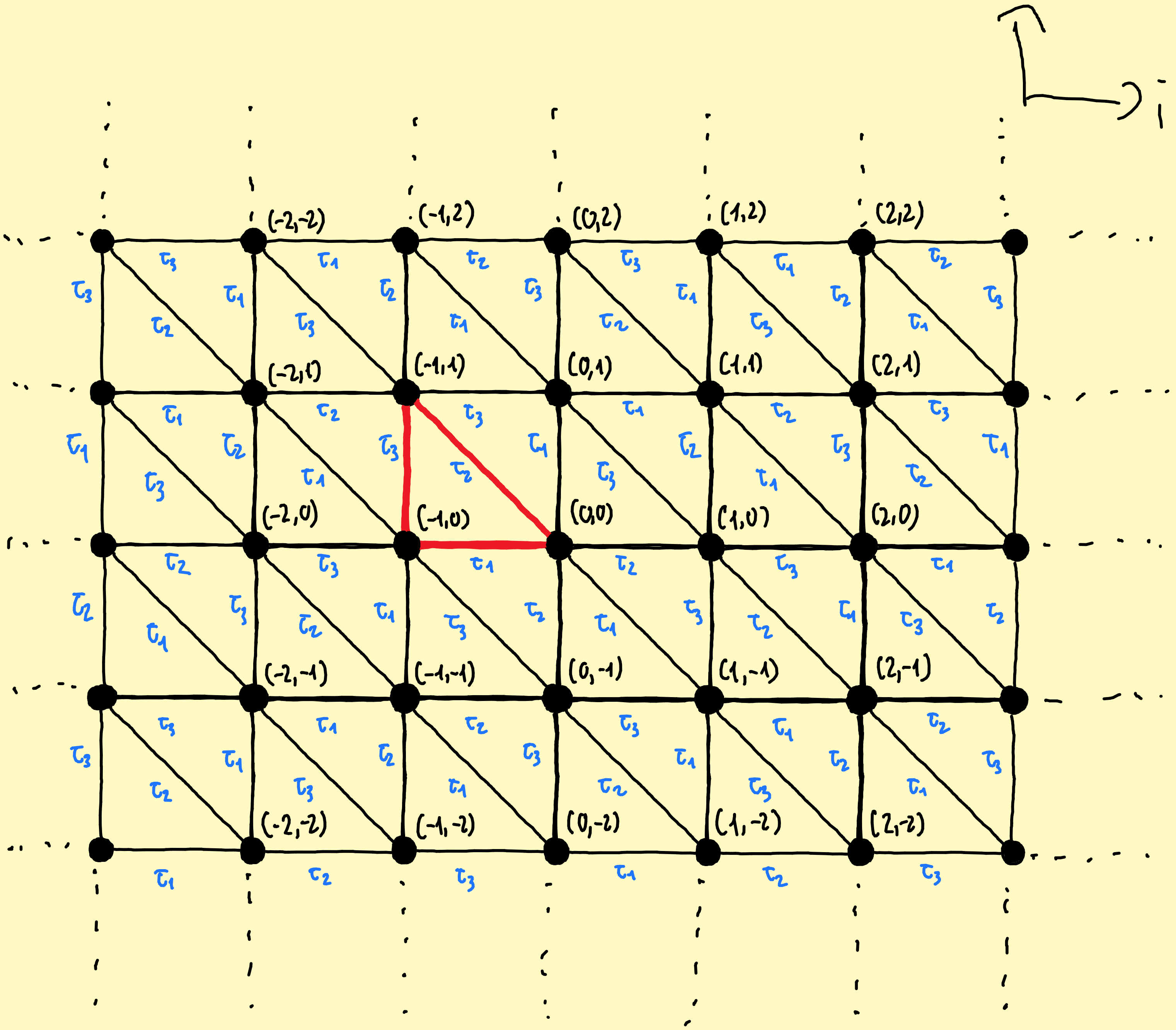

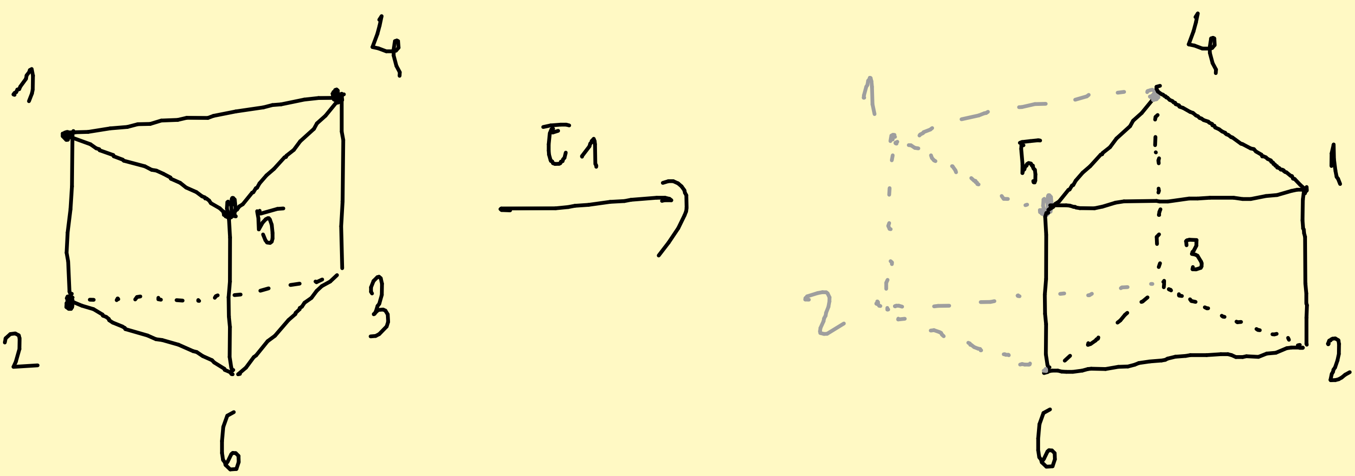

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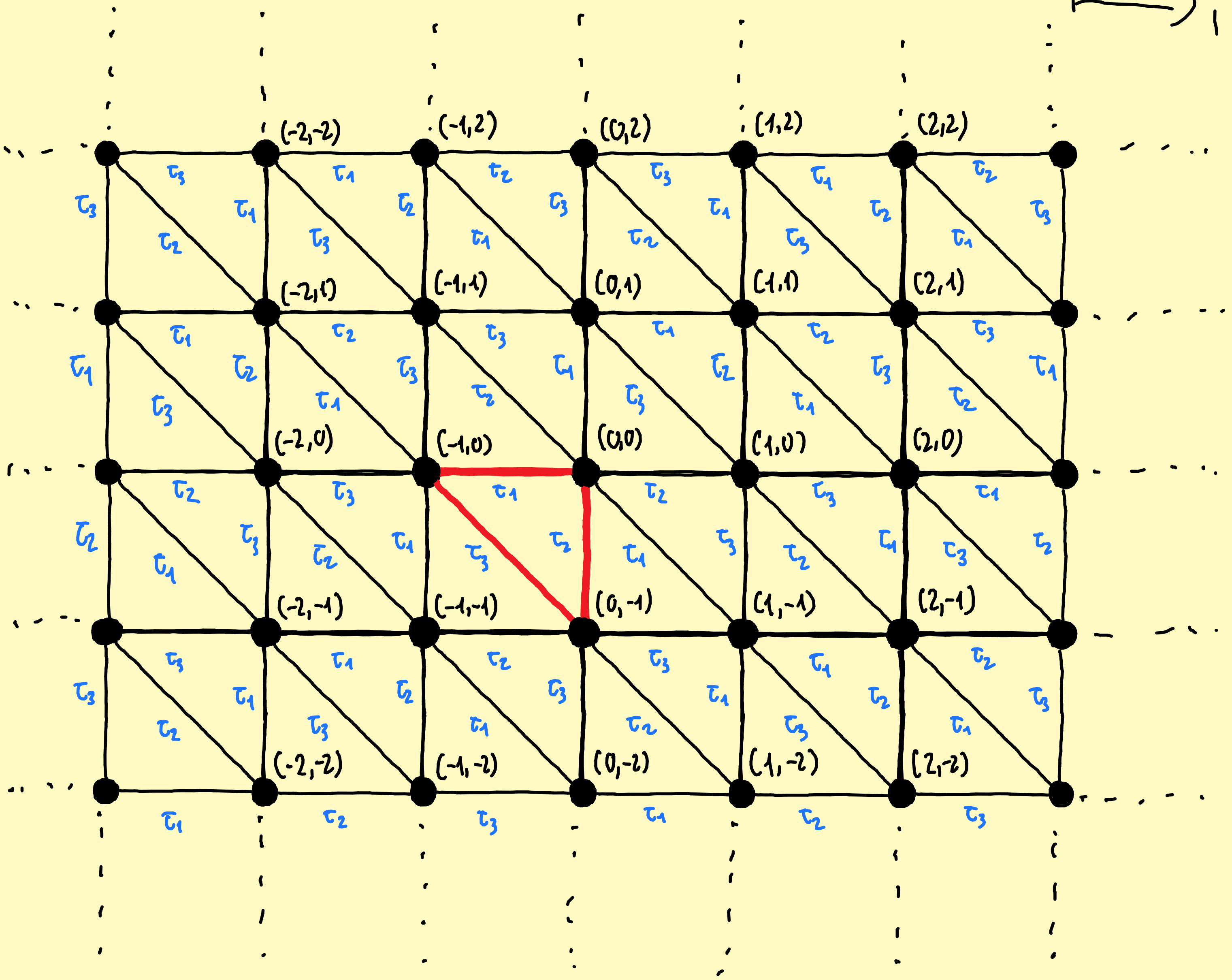

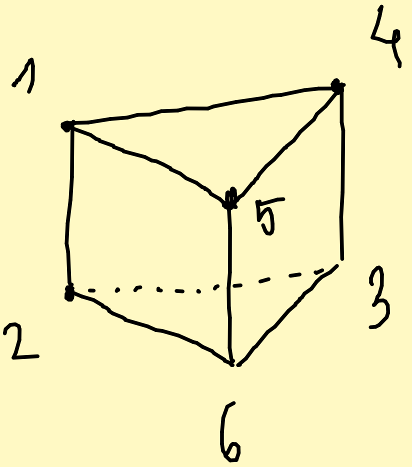

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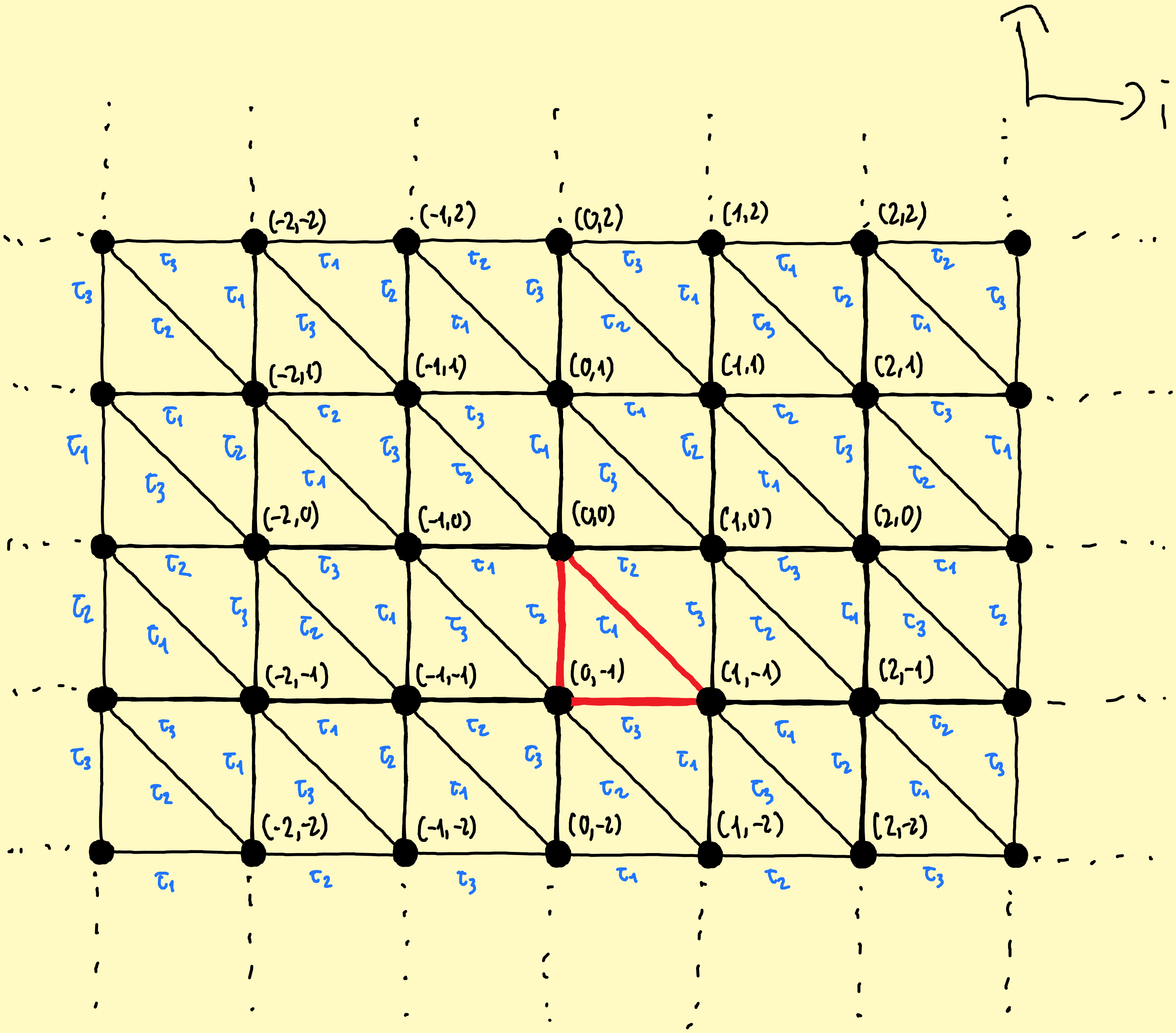

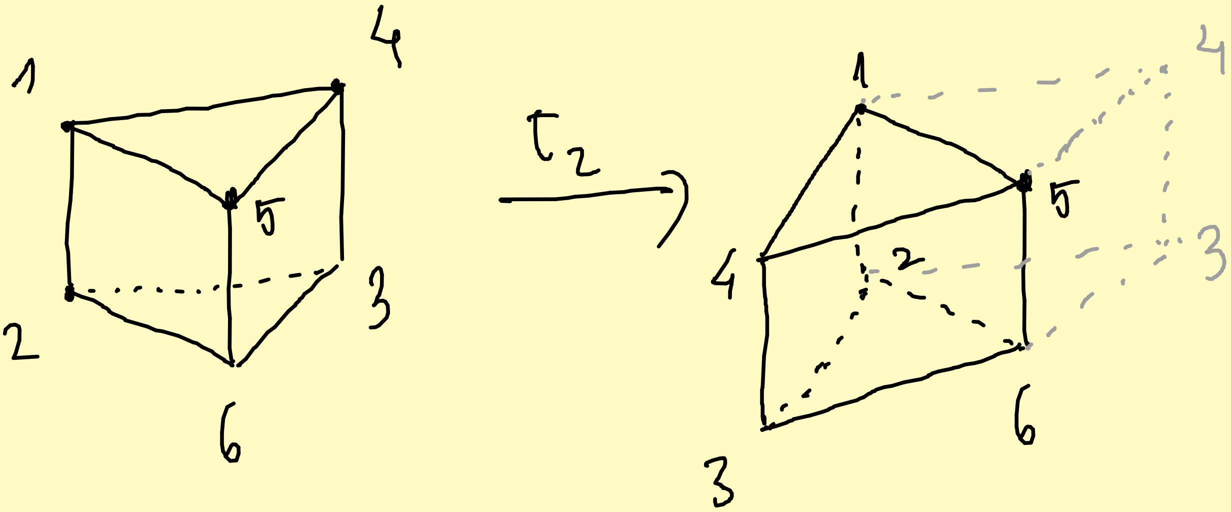

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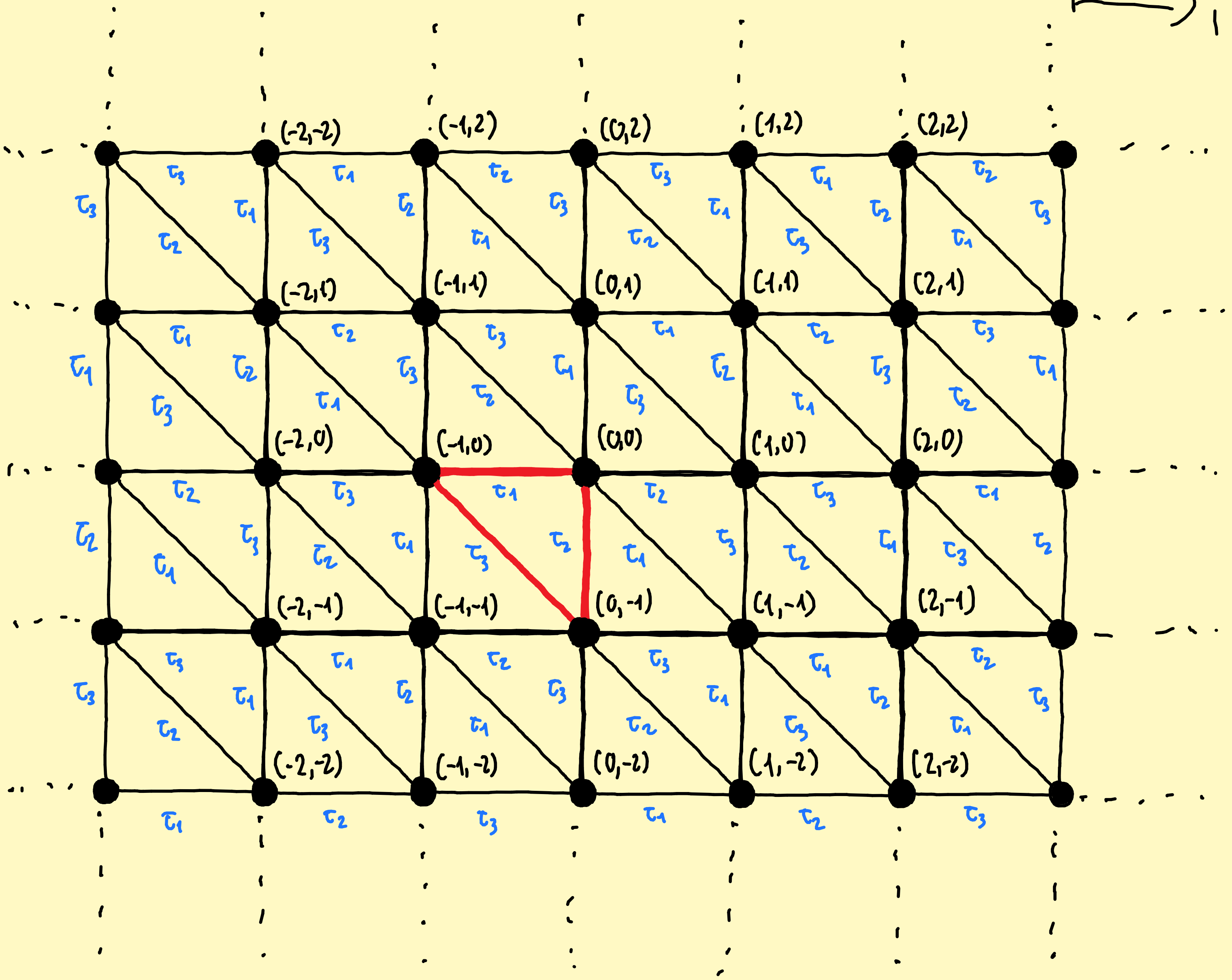
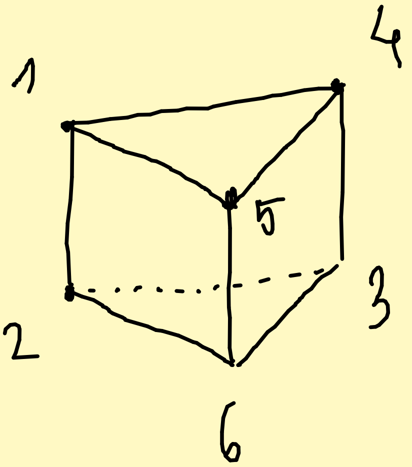
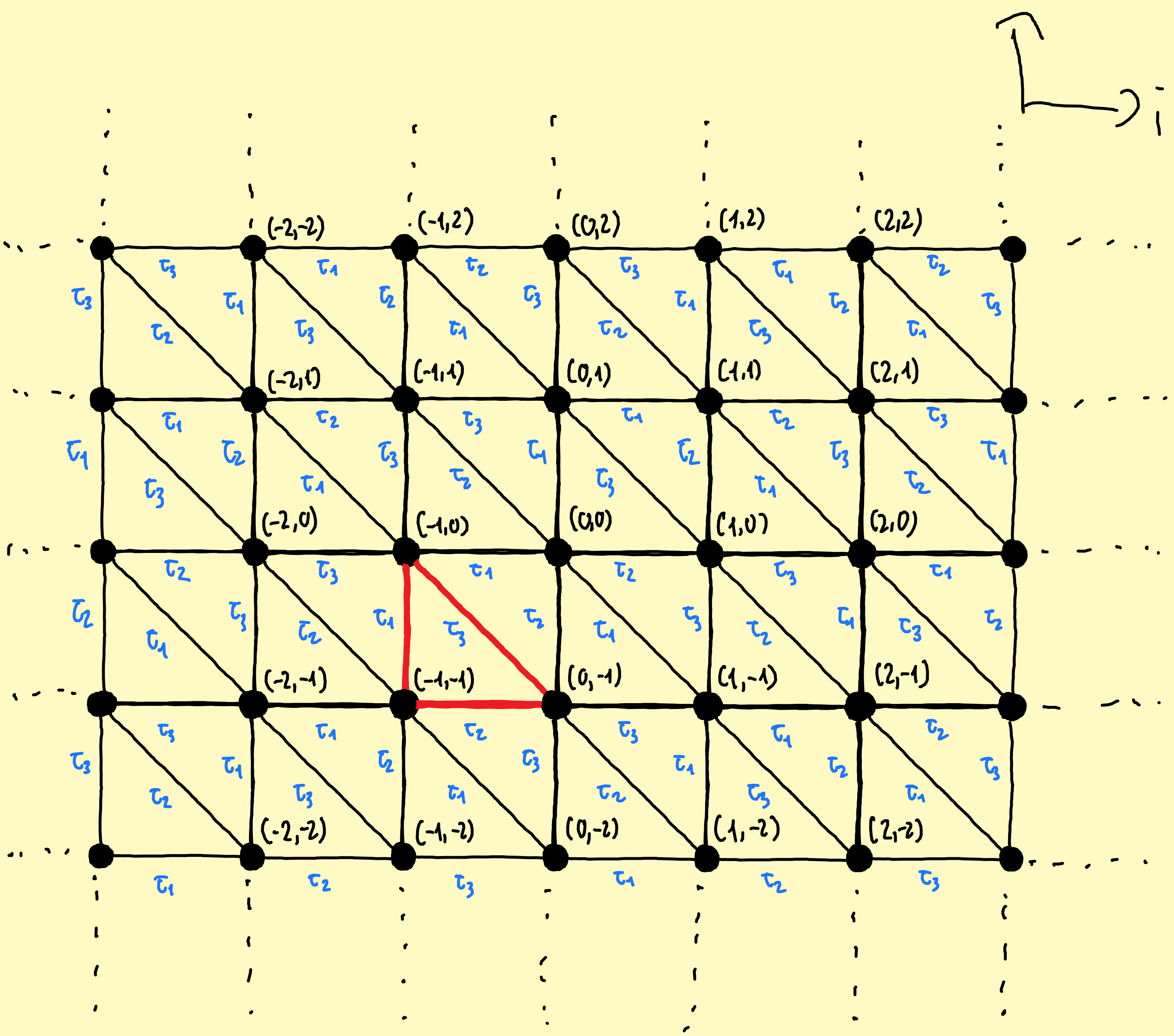
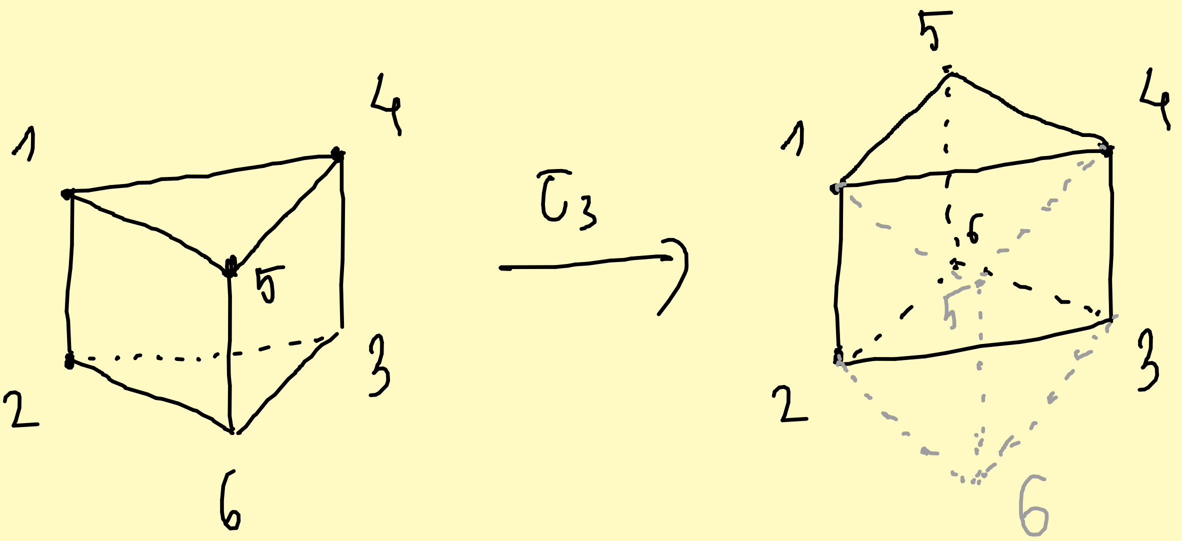


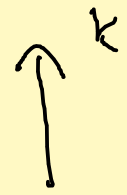


How to move the prism around?

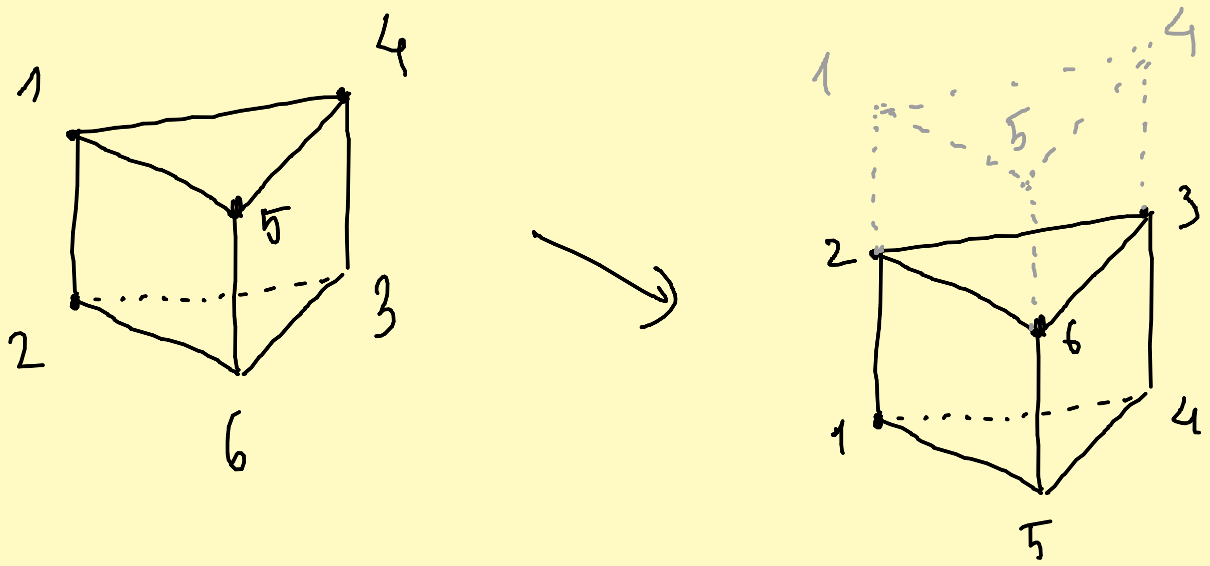




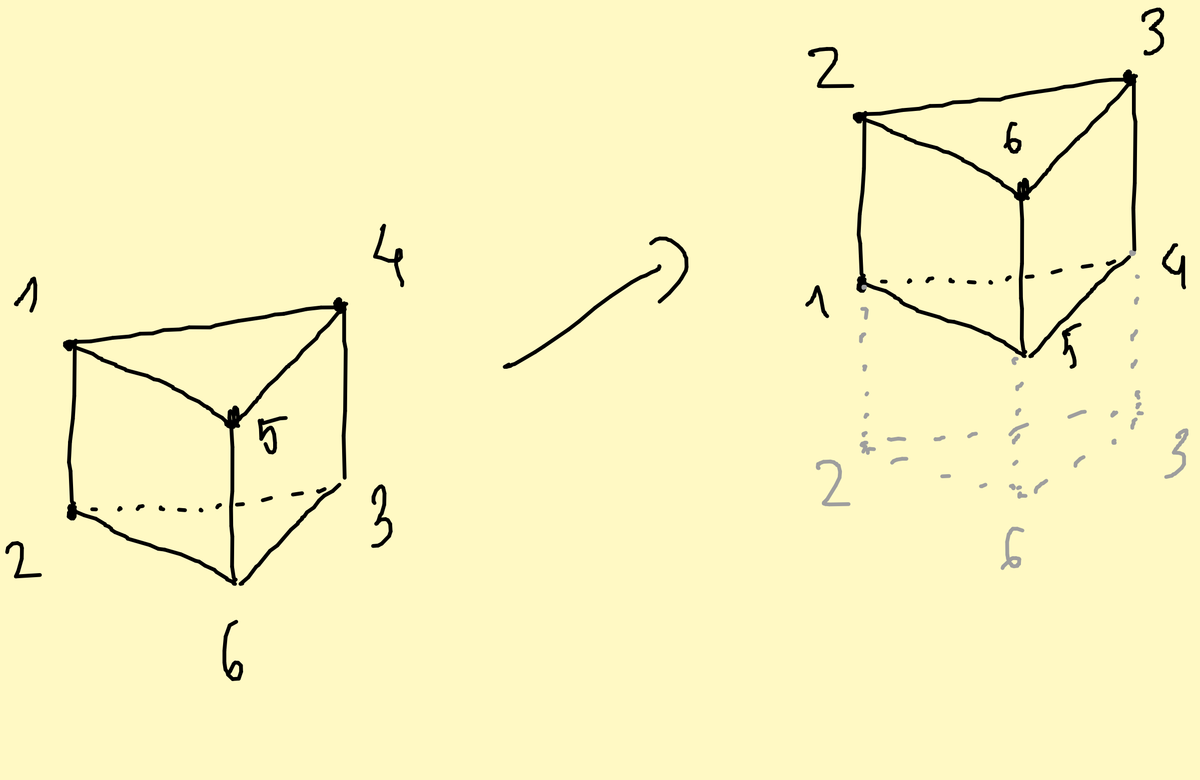
How about τ_4, τ_5?



τ_4:



τ_5:



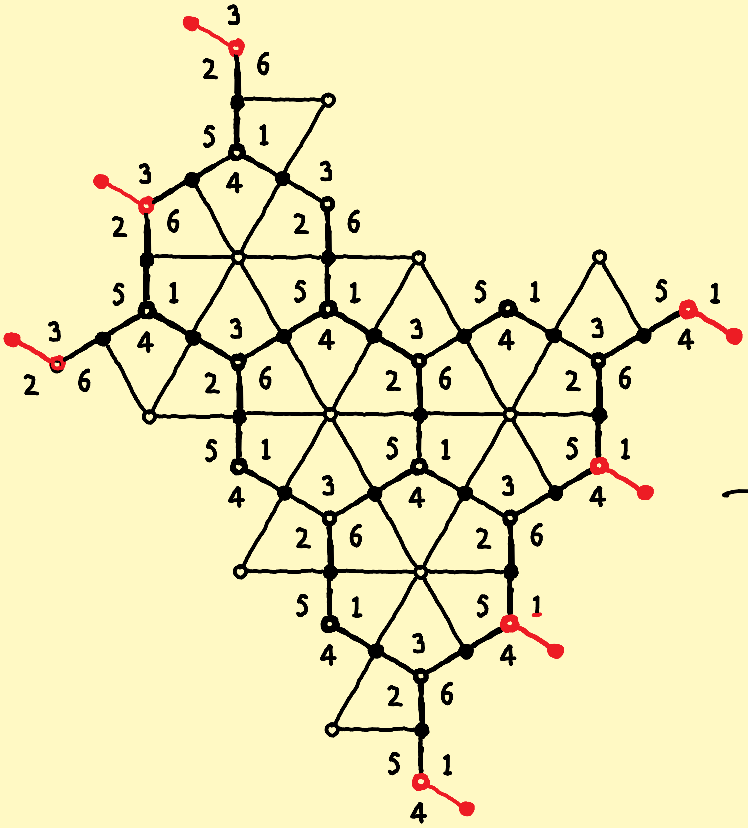
Tl;dr:

(i,j,k)

$C_{i,j,k}$

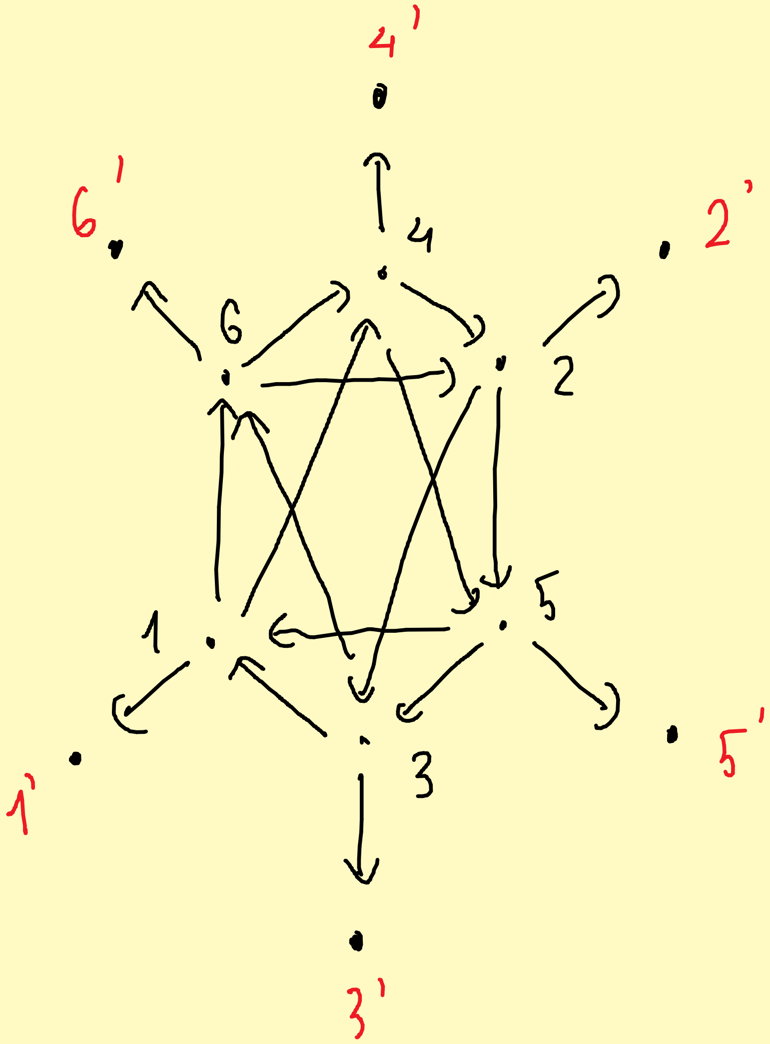
$$Z_{i,j,k} = \frac{P'(x_1,\dots,x_6)}{x_1^{d_1'}\cdots x_6^{d_6'}}$$

Lai Musiker $\Vert$



$$\longrightarrow \mathrm{wt}\left(C_{i,j,k}\right) = \frac{P(x_1,\dots,x_6)}{x_1^{d_1}\cdots x_6^{d_6}}$$

2. Framed dP3 quiver:



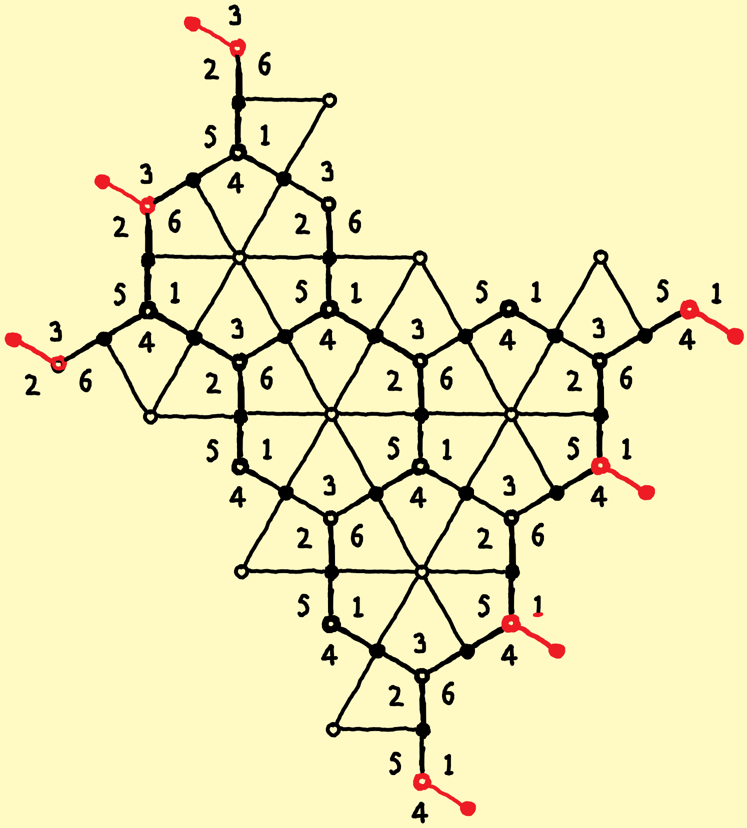
New cluster variables $y_1, \dots, y_6$

$$z_{i,j,k} = \frac{P(x_1, \dots, x_6, y_1, \dots, y_6)}{x_1^{d_1} \dots x_6^{d_6}}$$

$\Pi_i d_r$:

(i,j,k)

$C_{i,j,k}$ $\qquad$ $Z_{i,j,k}$



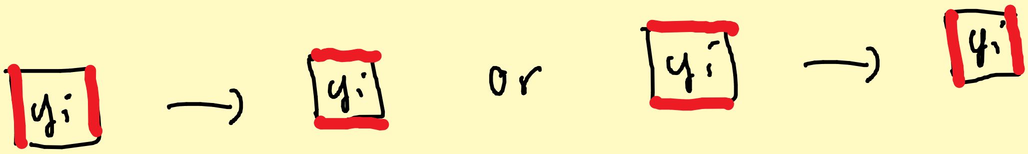
$$= \frac{\varphi(x_1,\dots,x_6,y_1,\dots,y_6)}{x_1^{d_1}\cdots x_6^{d_6}}$$

?

Back to day 1, we have the Poset Rule

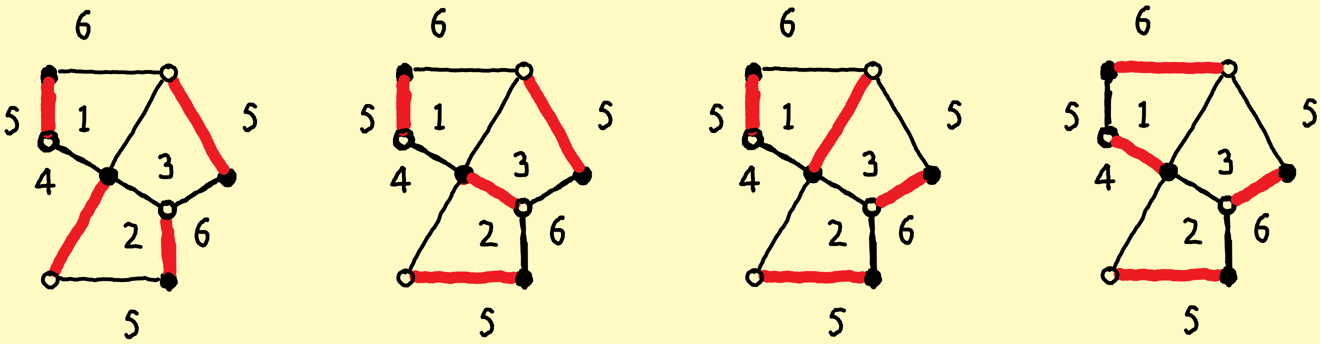
Poset rule:

- Pick a minimal matching
- $M_1 \to M_2$ if M_1, M_2 differ by a flip



- The number of y_i flips is the exponent of y_i.

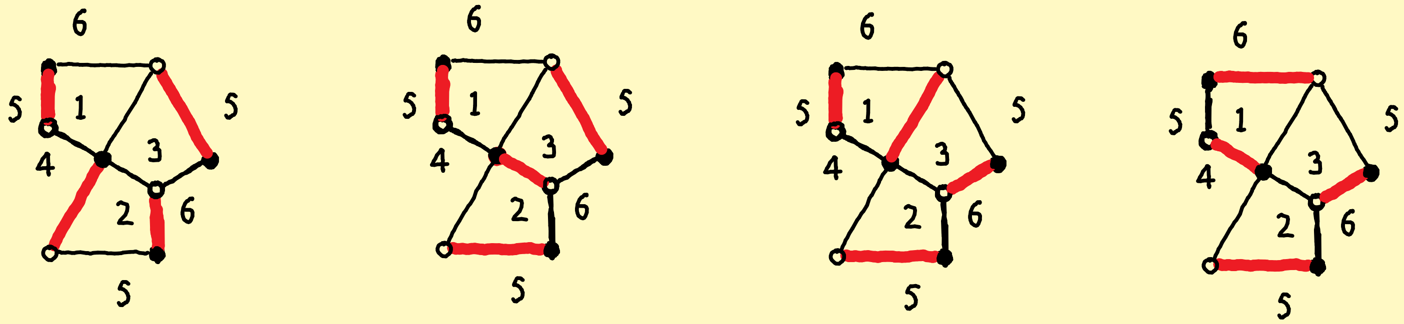
Example: $\tau_1 \tau_2$



$$\frac{x_5 x_6}{x_2} + \frac{x_4 x_6^2}{x_2 x_3} + \frac{x_4 x_5 x_6}{x_1 x_3} + \frac{x_5^2}{x_1}$$

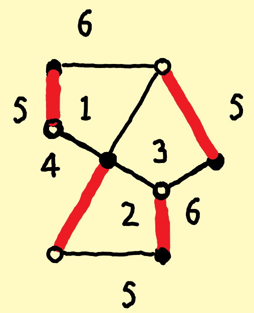
$$= \frac{x_1 x_3 x_5 x_6 + x_1 x_4 x_6^2 + x_2 x_4 x_5 x_6 + x_2 x_3 x_5^2}{x_1 x_2 x_3}$$

Framed :



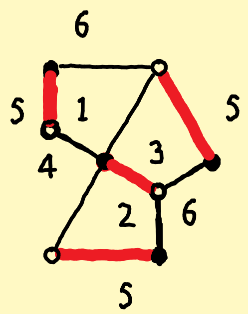
$$\frac{x_5 x_6}{x_2}\, y_1 y_2 y_3 + \frac{x_4 x_6^2}{x_2 x_3}\, y_1 y_3 + \frac{x_4 x_5 x_6}{x_1 x_3}\, y_1 + \frac{x_5^2}{x_1}\, 1$$

Framed :



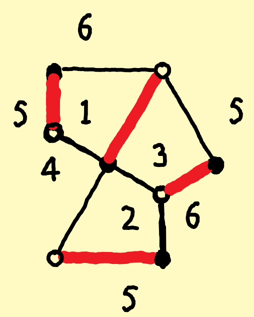
$\frac{x_5x_6}{x_2} \longrightarrow y_1y_2y_3$

$\Big|\ y_2$



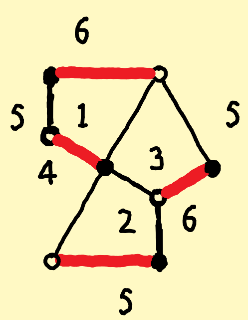
$\frac{x_4x_6^2}{x_2x_3}$ y_1y_3

$\Big|\ y_3$



$\frac{x_4x_5x_6}{x_1x_3}$ y_1

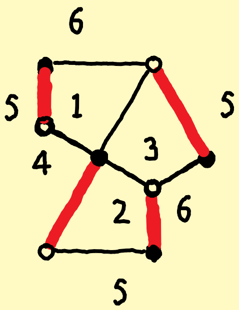
$\Big|\ y_1$



$\frac{x_5^2}{x_1}$ 1

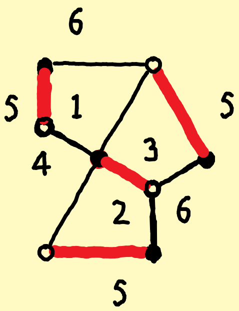
Framed :

Wrong minimal matching

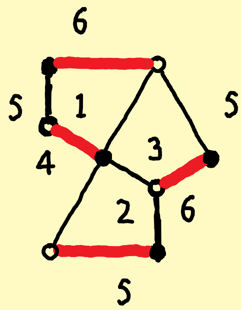


$$\frac{x_5 x_6}{x_2} \quad {\color{red} y_3 y_2}$$

y_2

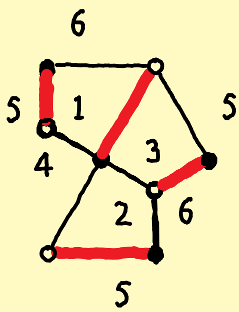


$$\frac{x_4 x_6^2}{x_2 x_3} \quad {\color{red} y_3}$$



$$\frac{x_5^2}{x_1} \quad {\color{red} y_1}$$

y_3 $\qquad$ y_1



$$\frac{x_4 x_5 x_6}{x_1 x_3} \quad {\color{red} 1}$$

Question: What is the correct minimal matching?

3. Minimal matching of Aztec Castles:

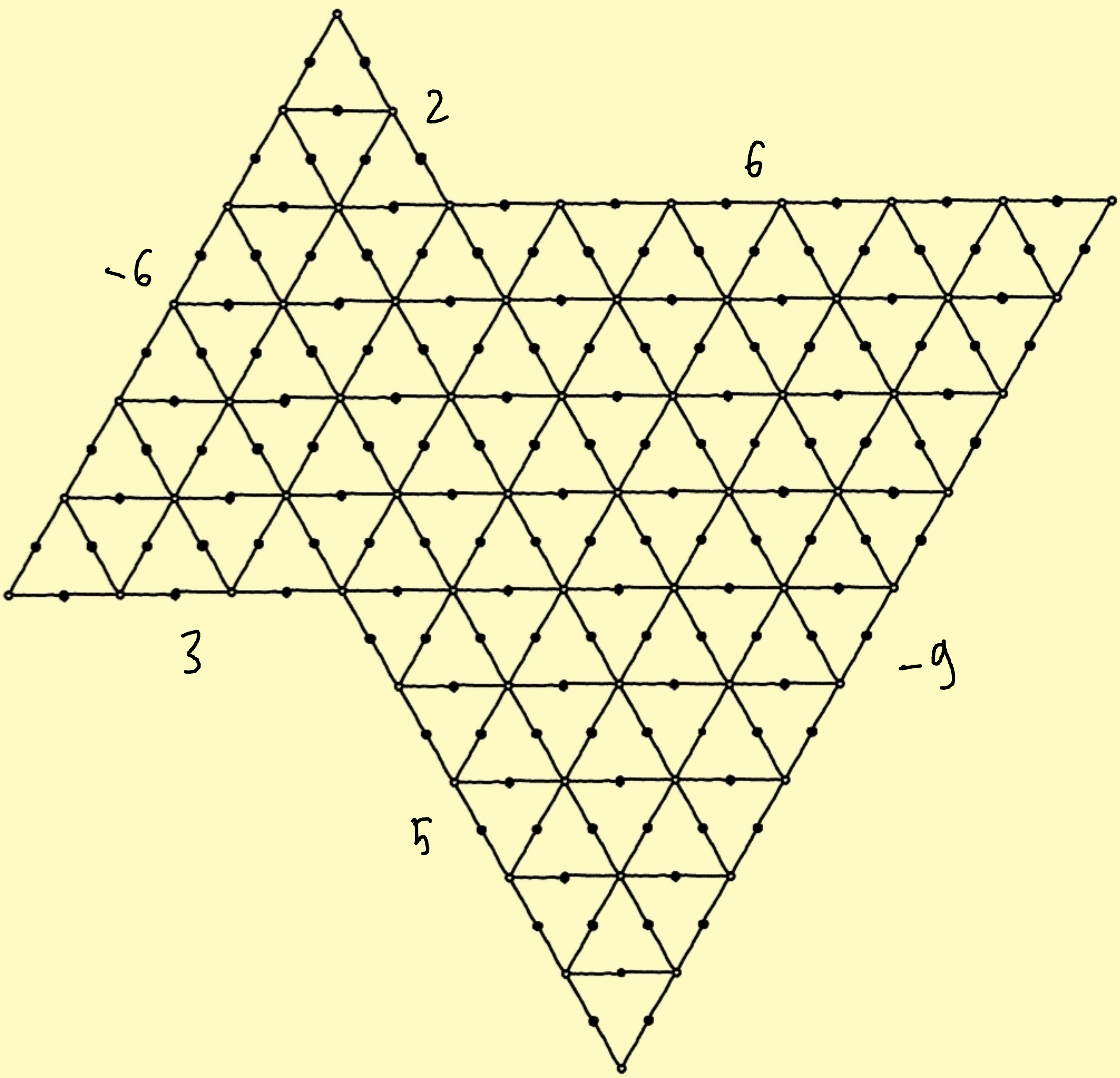
Construction:

Step 1: Divide into regions

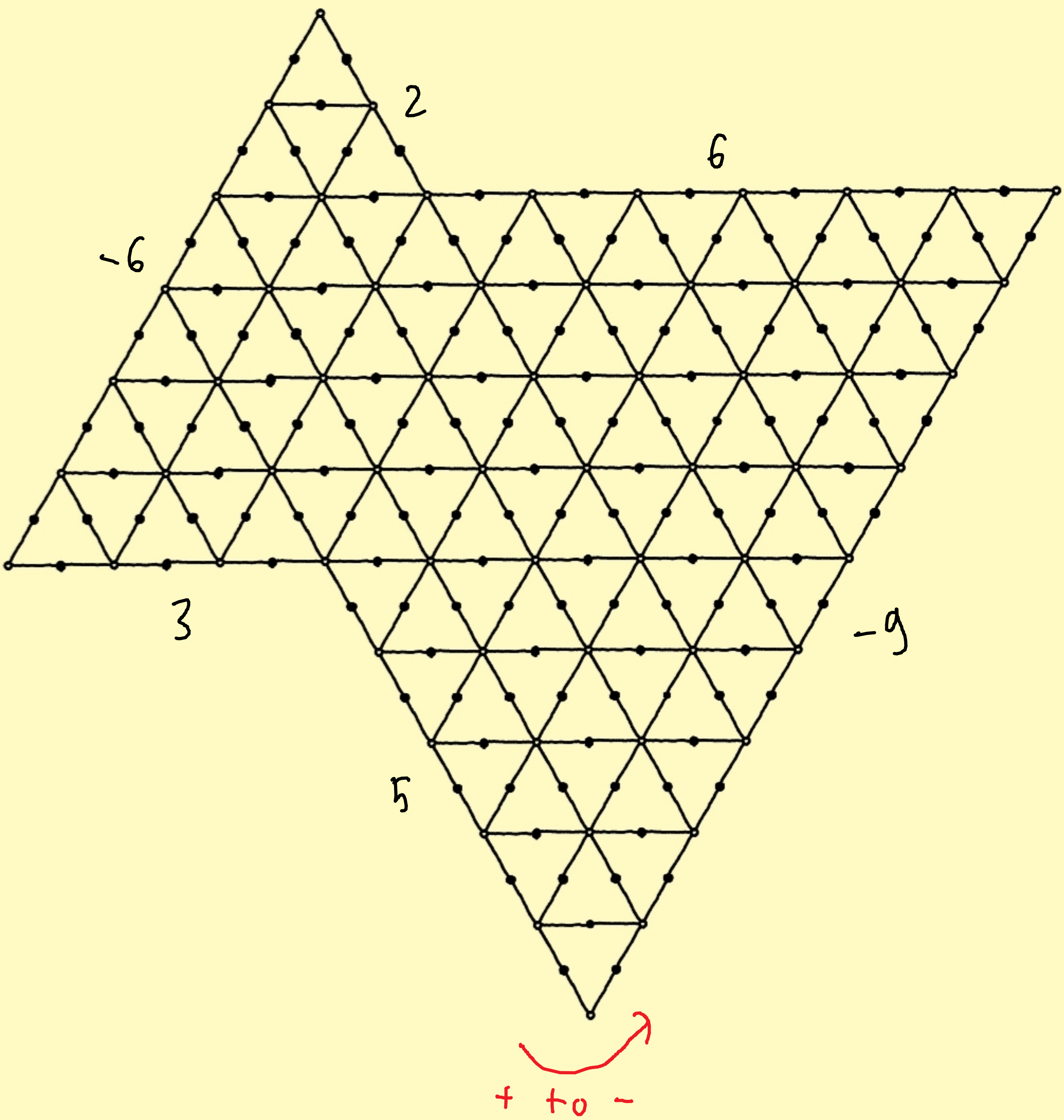
Go along the contour in $a \to f$ direction

- Positive to positive: straight line along the second side
- Negative to negative: straight line along the first side
- Negative to positive: diagonal staircase with first step on the positive side
- Positive to negative: do nothing

- Positive to positive : straight line along the second side
- Negative to negative : straight line along the first side
- Negative to positive : diagonal staircase with first step on the positive side
- Positive to negative : do nothing



- Positive to positive : straight line along the second side
- Negative to negative : straight line along the first side
- Negative to positive : diagonal staircase with first step on the positive side
- Positive to negative : do nothing

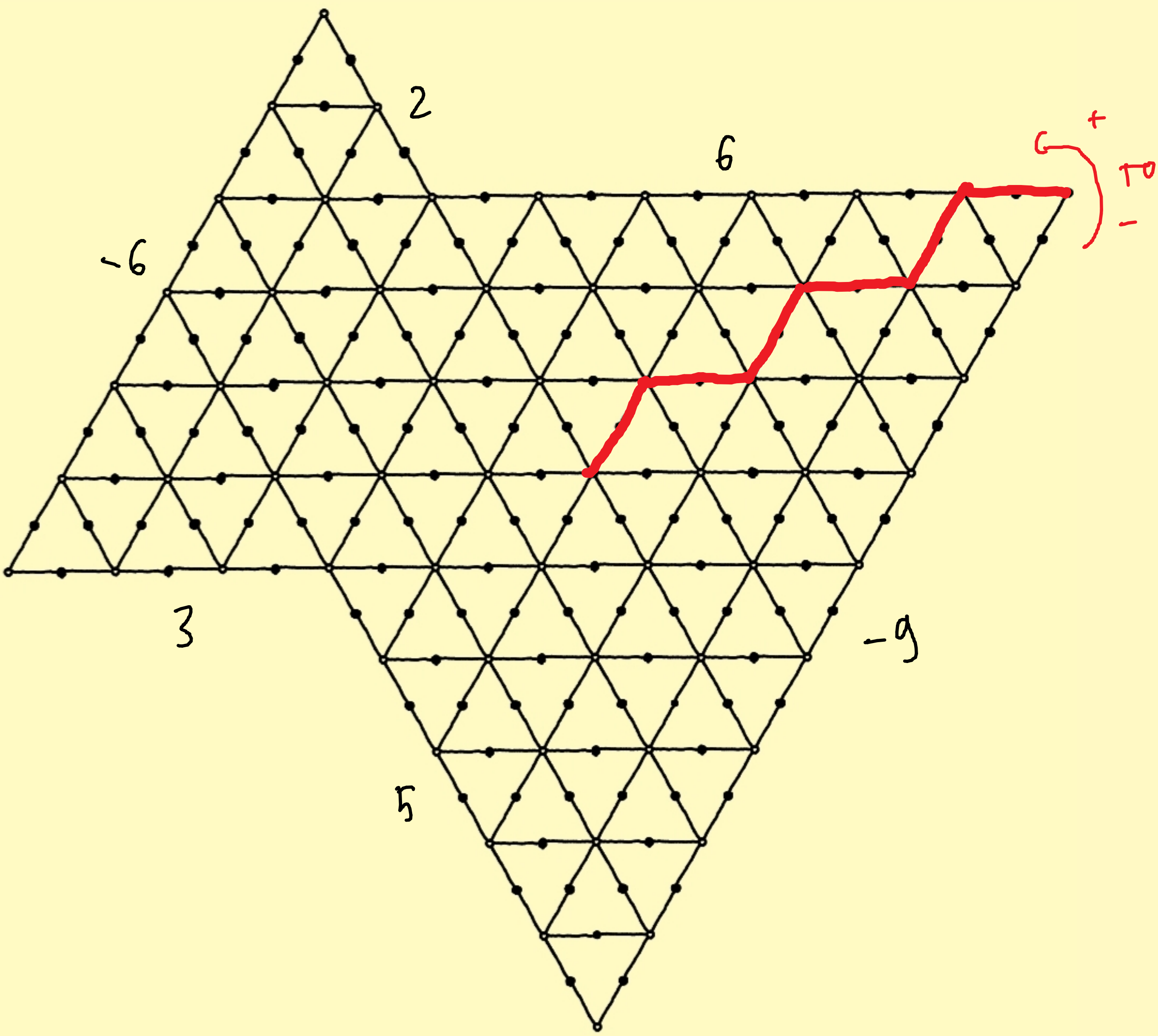


- Positive to positive : straight line along the second side

- Negative to negative : straight line along the first side

- Negative to positive : diagonal staircase with first step on the positive side

- Positive to negative : do nothing

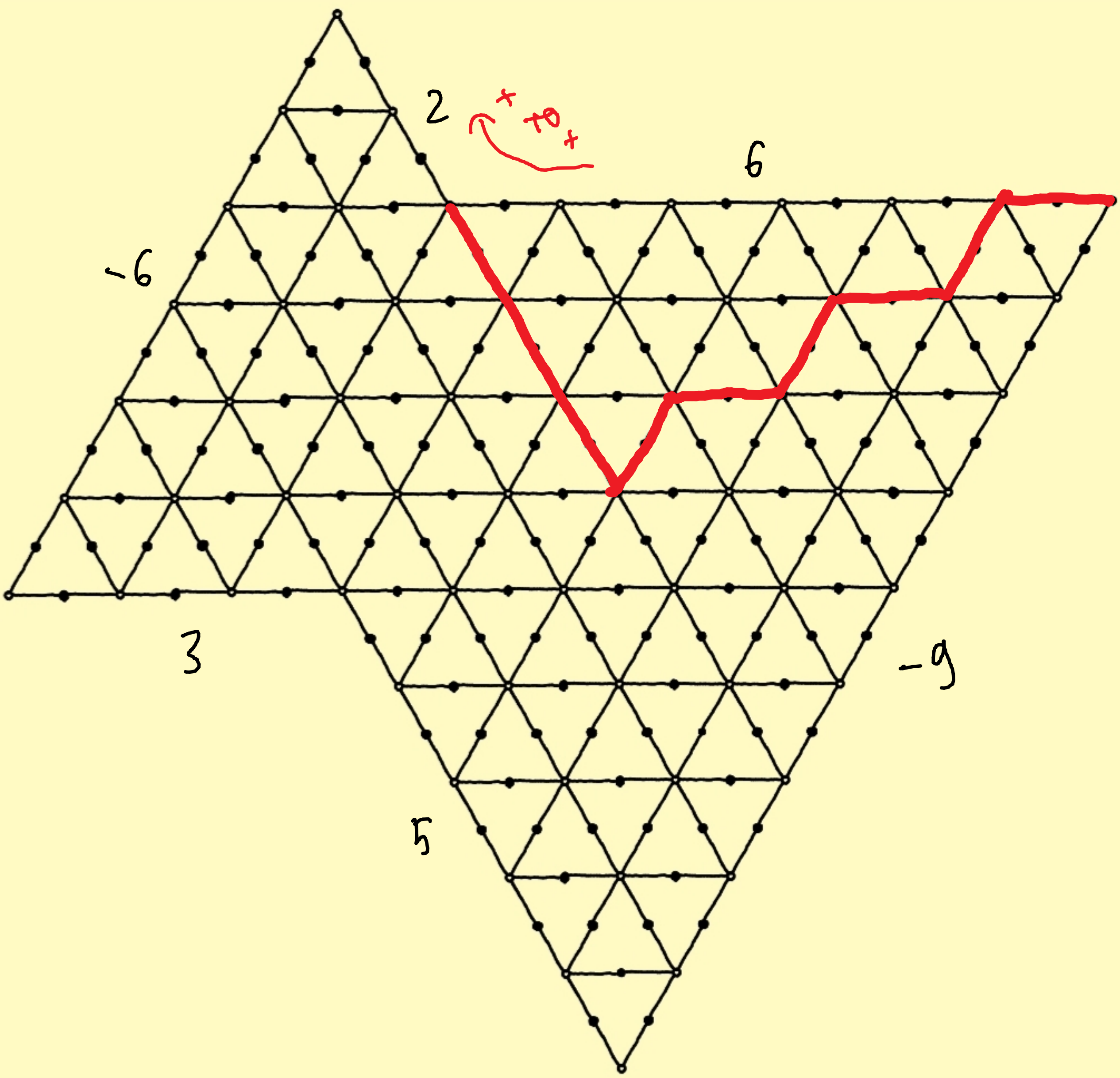


- Positive to positive : straight line along the second side

- Negative to negative : straight line along the first side

- Negative to positive : diagonal staircase with first step on the positive side

- Positive to negative : do nothing



- Positive to positive : straight line along the second side

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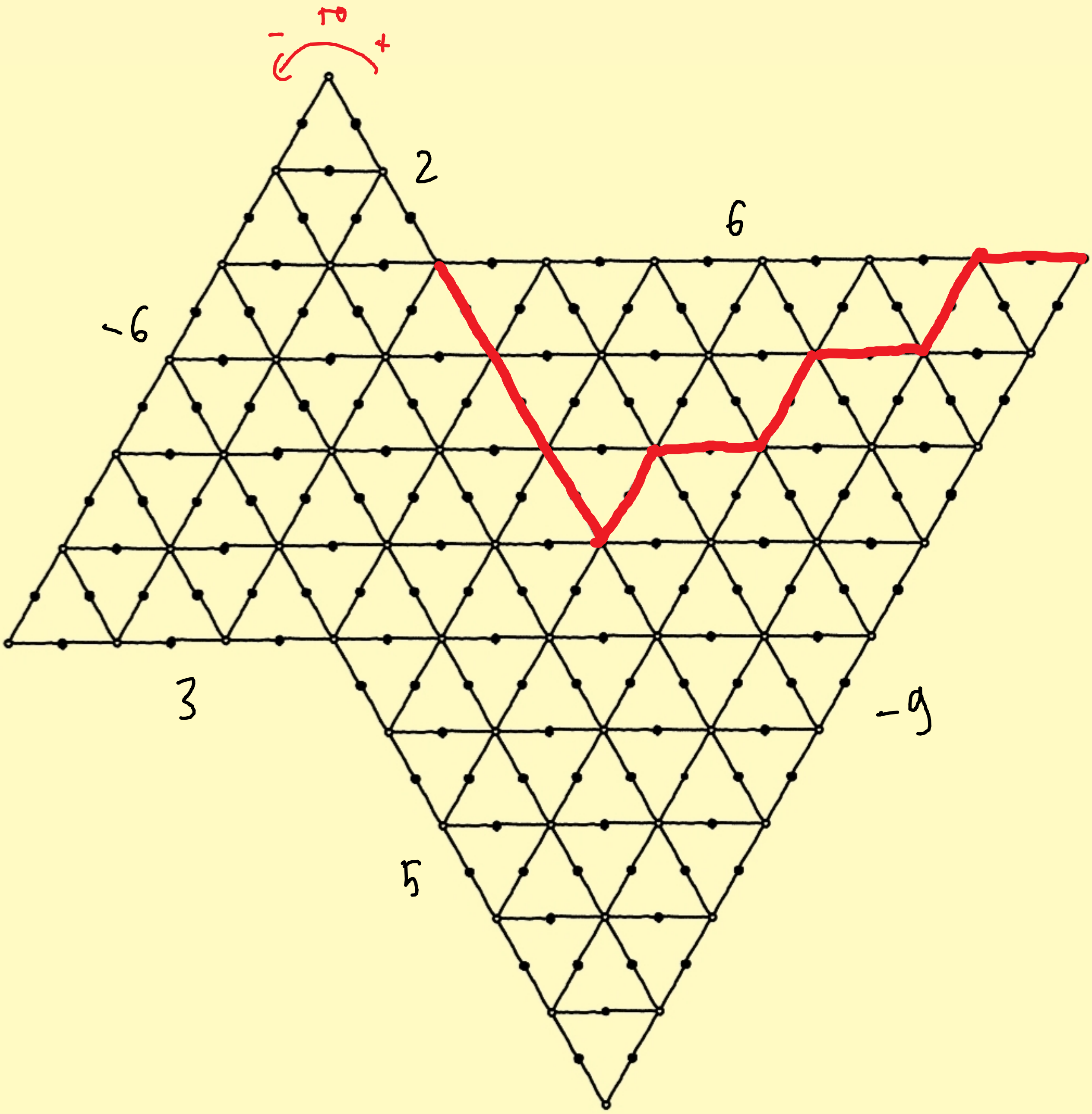

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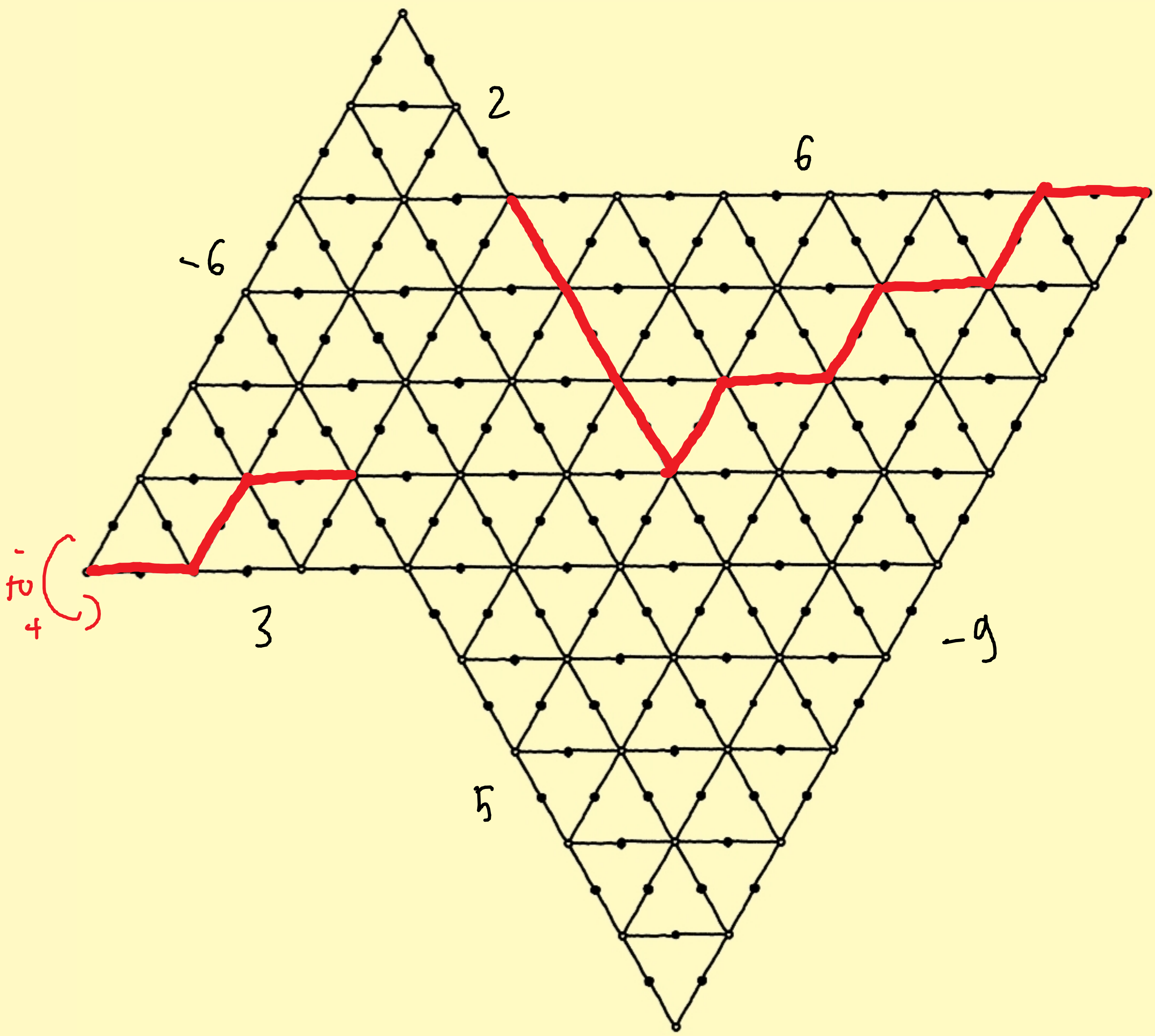

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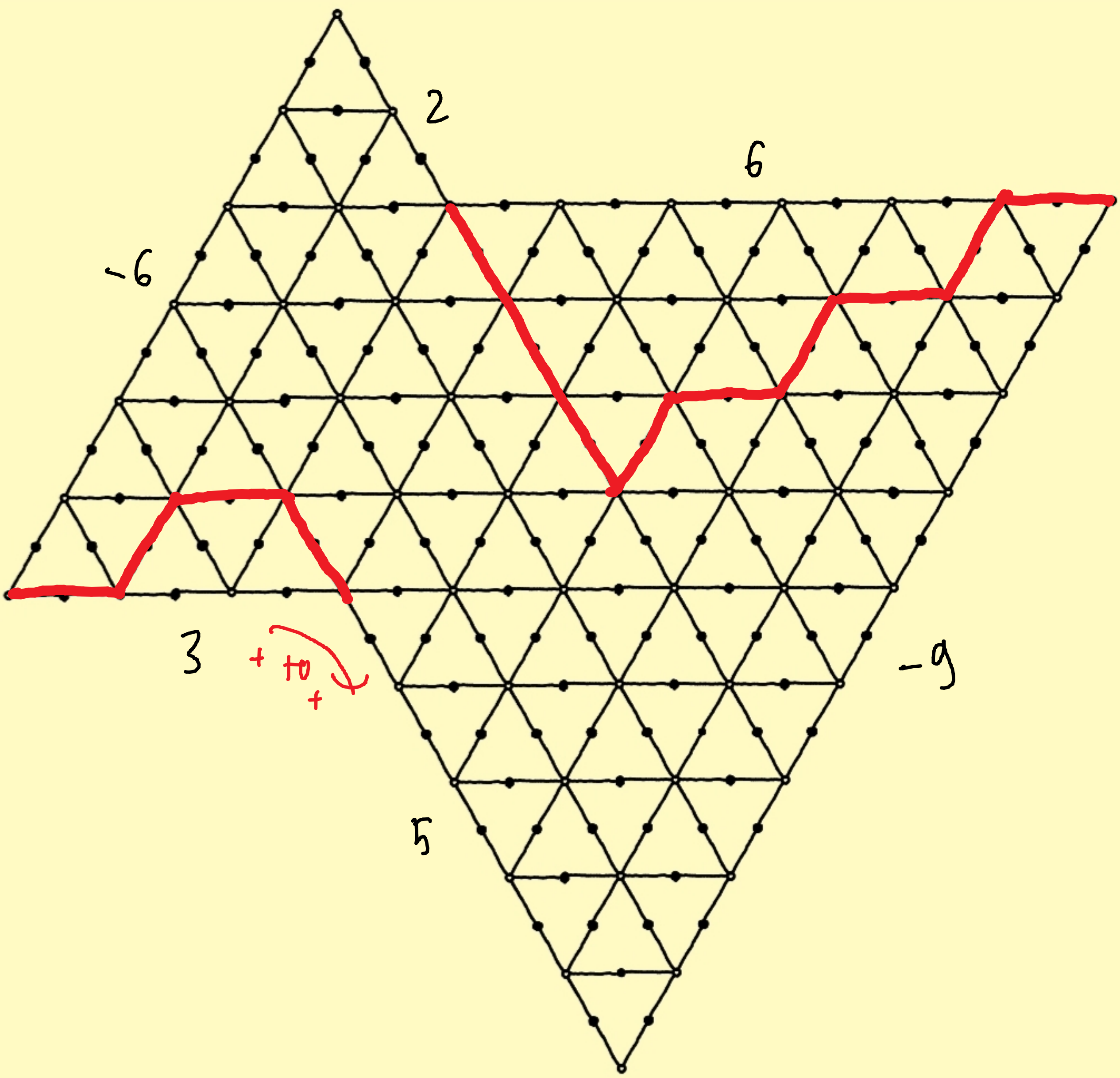
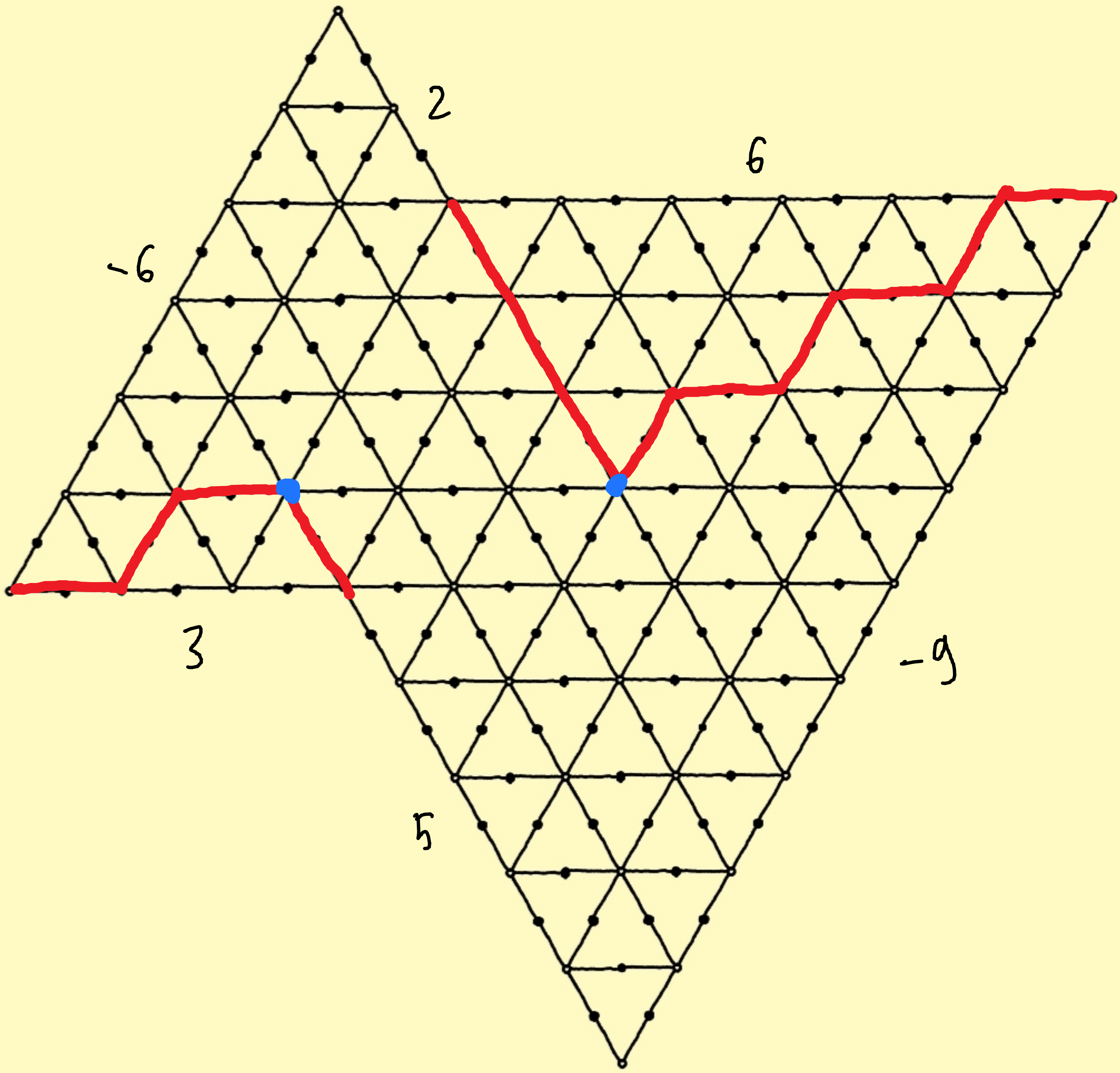
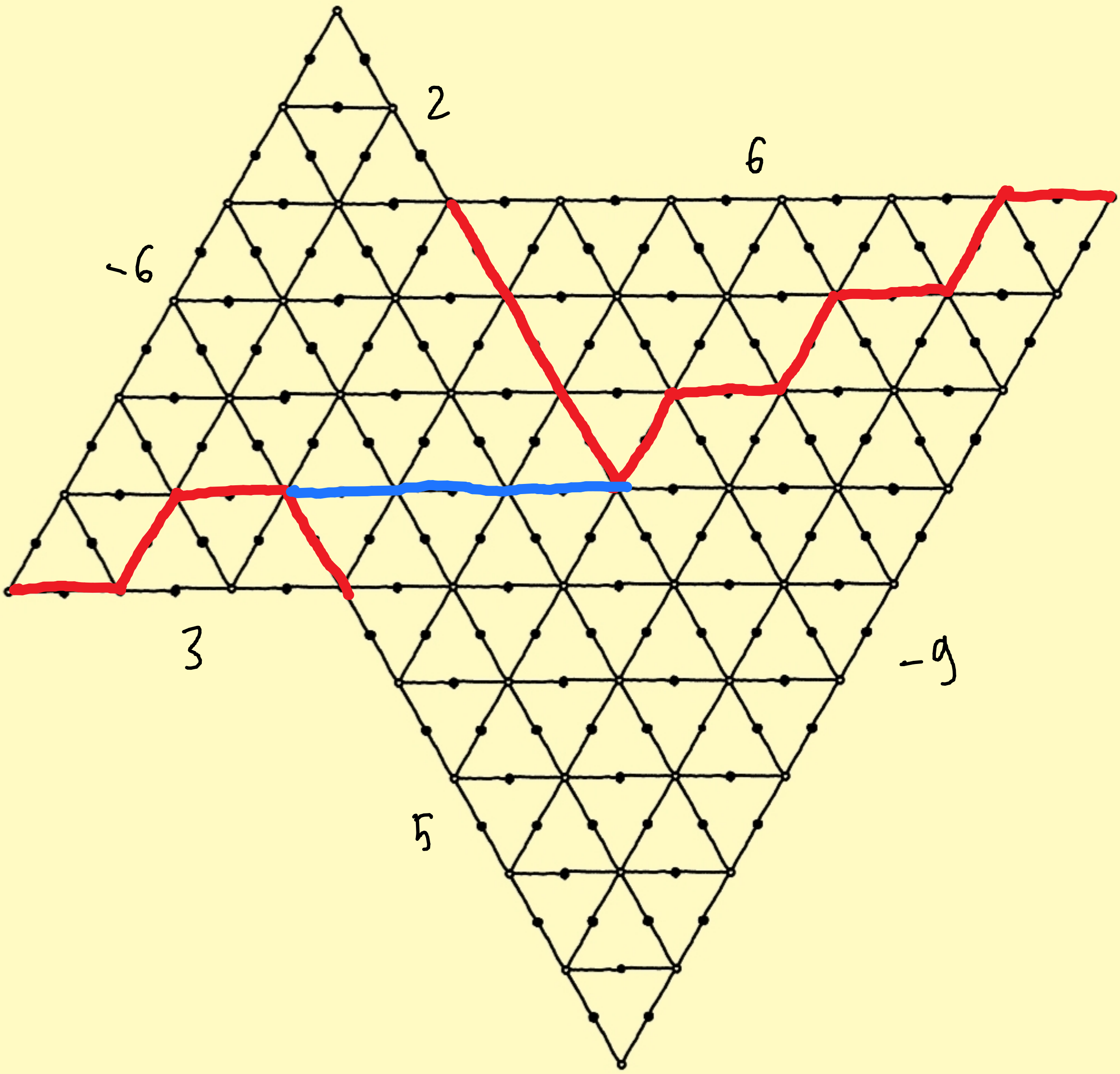


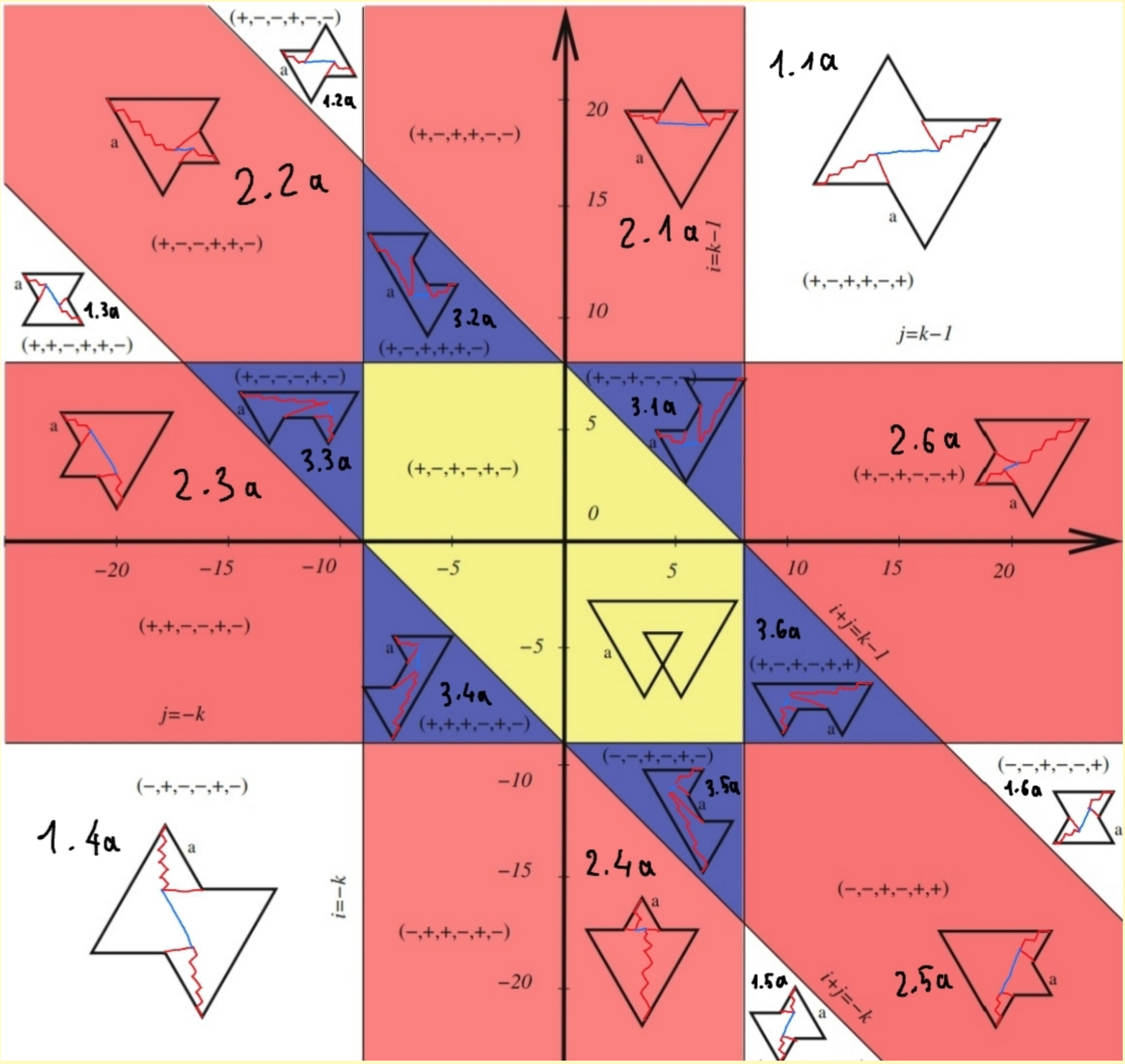


2
6
-6
3
-9
5



Connect the two intersection

→ "zero line"

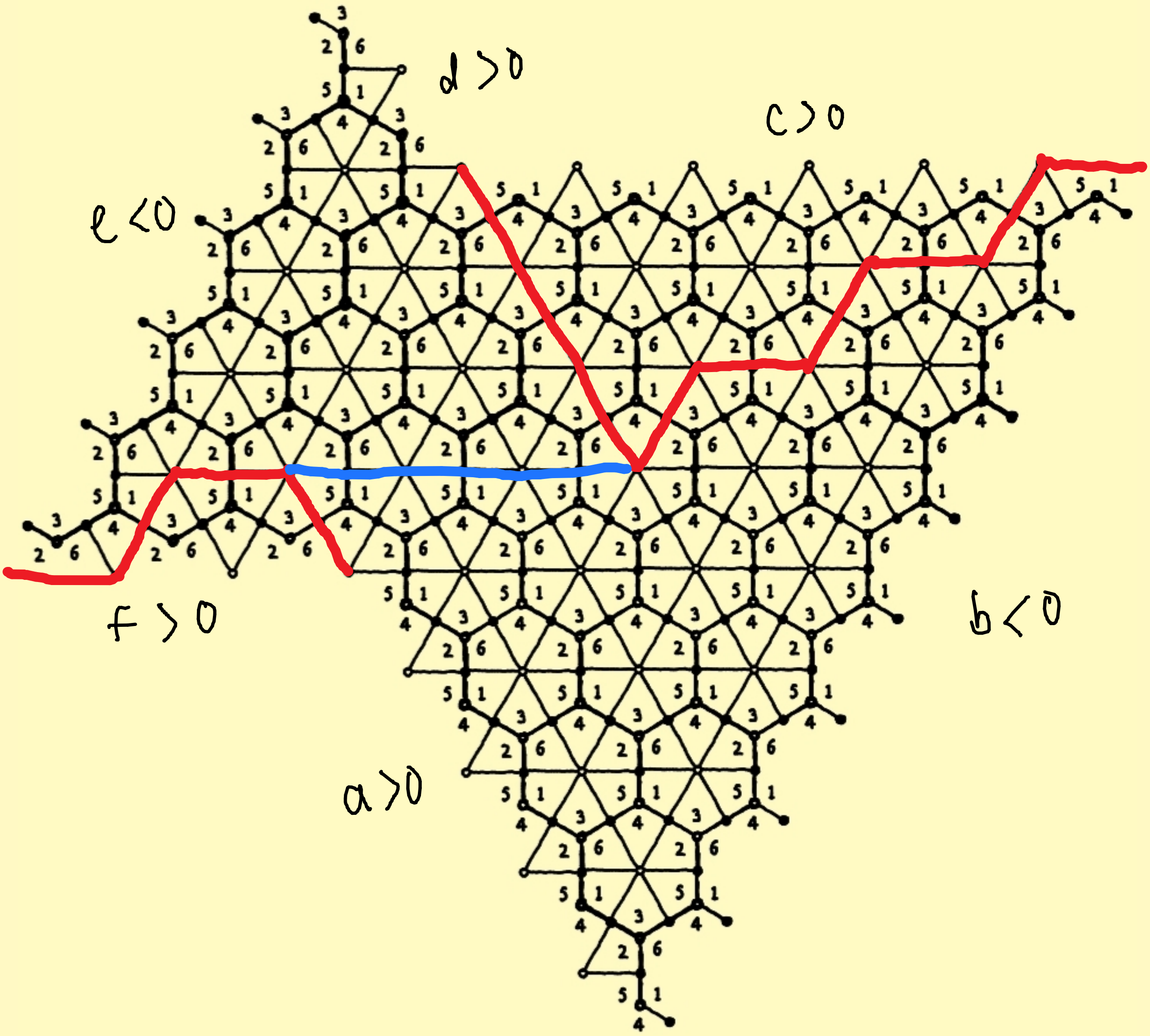


Four regions for all shapes

Step 2: Cover each region according to the sign of their sides as in table 1:

Side	positive	negative
a	$1-4, 2-5, 3-6$	$1-5, 2-4, 3-6$
b	$1-4, 2-6, 3-5$	$1-4, 2-5, 3-6$
c	$1-3, 2-6, 4-5$	$1-4, 2-6, 3-5$
d	$1-6, 2-3, 4-5$	$1-3, 2-6, 4-5$
e	$1-5, 2-3, 4-6$	$1-6, 2-3, 4-5$
f	$1-5, 2-4, 3-6$	$1-5, 2-3, 4-6$

Table 1: Universal covering for each case



Step 2: Cover each region according to the sign of their sides as in table 1:

Side	positive	negative
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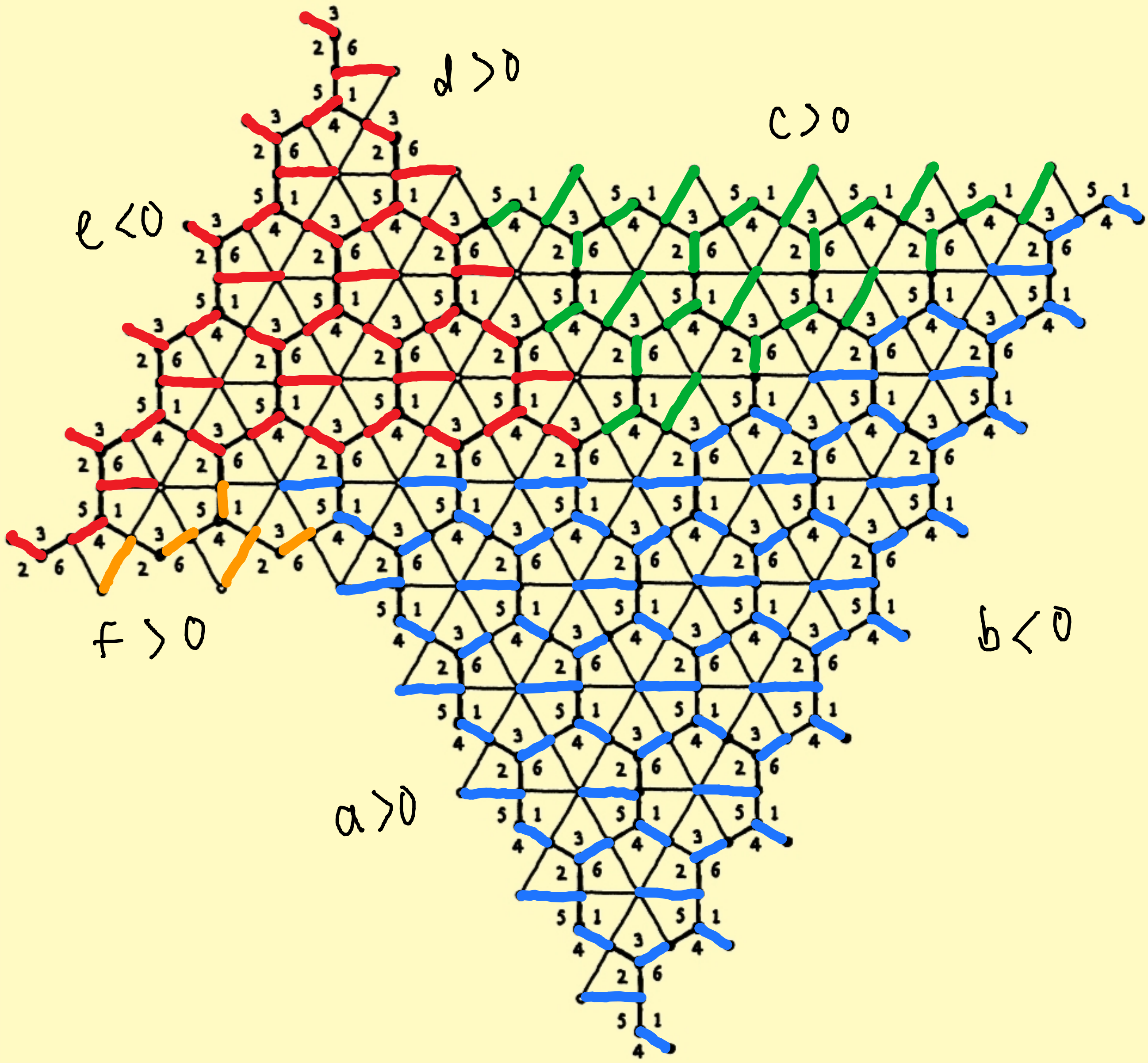
b	$1-4, 2-6, 3-5$	$1-4, 2-5, 3-6$
c	$1-3, 2-6, 4-5$	$1-4, 2-6, 3-5$
d	$1-6, 2-3, 4-5$	$1-3, 2-6, 4-5$
e	$1-5, 2-3, 4-6$	$1-6, 2-3, 4-5$
f	$1-5, 2-4, 3-6$	$1-5, 2-3, 4-6$

Table 1: Universal covering for each case



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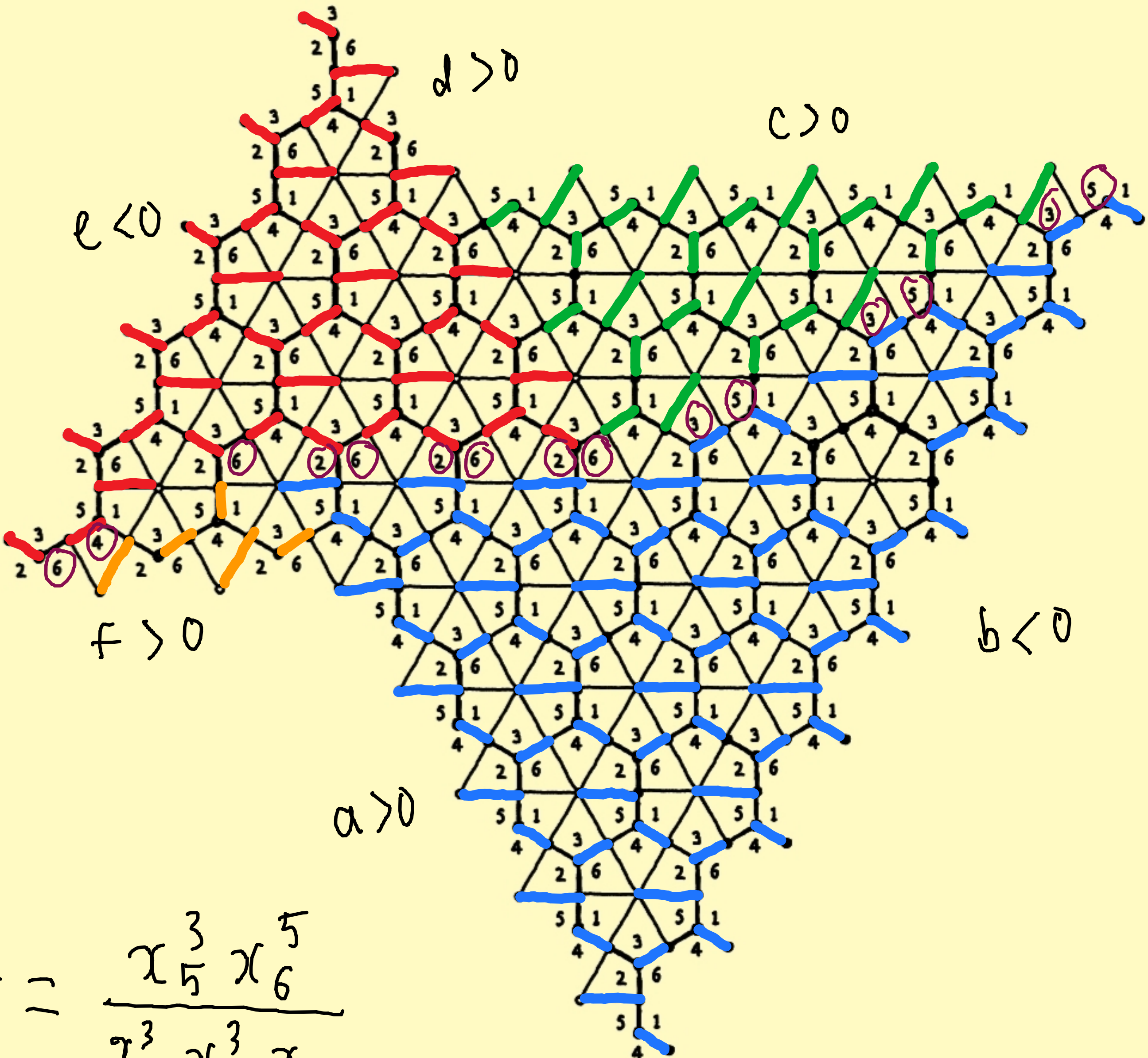
c	$1-3, 2-6, 4-5$	$1-4, 2-6, 3-5$
d	$1-6, 2-3, 4-5$	$1-3, 2-6, 4-5$
e	$1-5, 2-3, 4-6$	$1-6, 2-3, 4-5$
f	$1-5, 2-4, 3-6$	$1-5, 2-3, 4-6$

Table 1: Universal covering for each case



$$wt = \frac{x_5^3 x_6^5}{x_2^3 x_3^3 x_4}$$

Theorem (Chiang, N.)

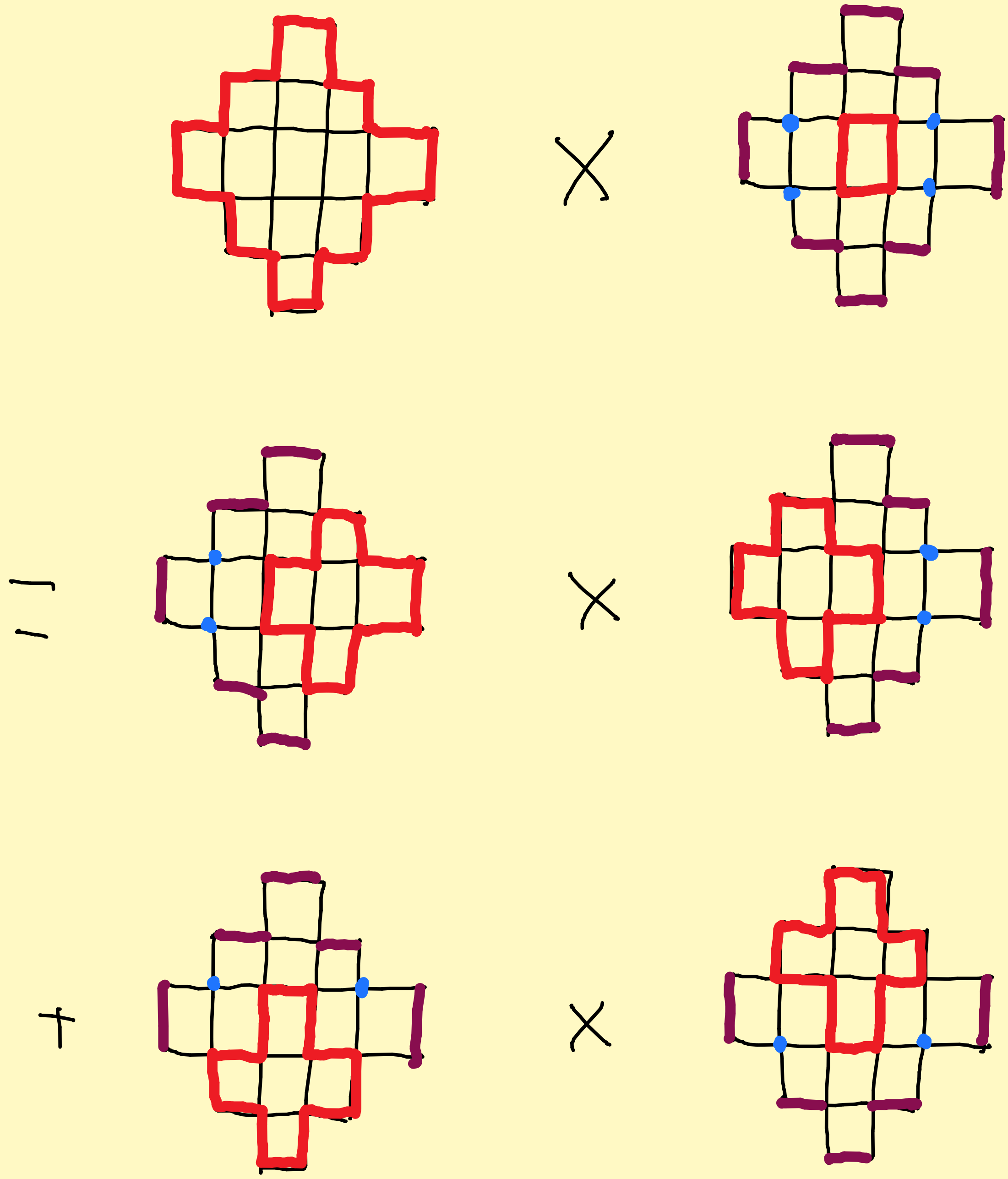
The above construction gives the minimal matching for all non-self-intersecting Aztec Castles.

4. Proof sketch:

Kuo condensation:

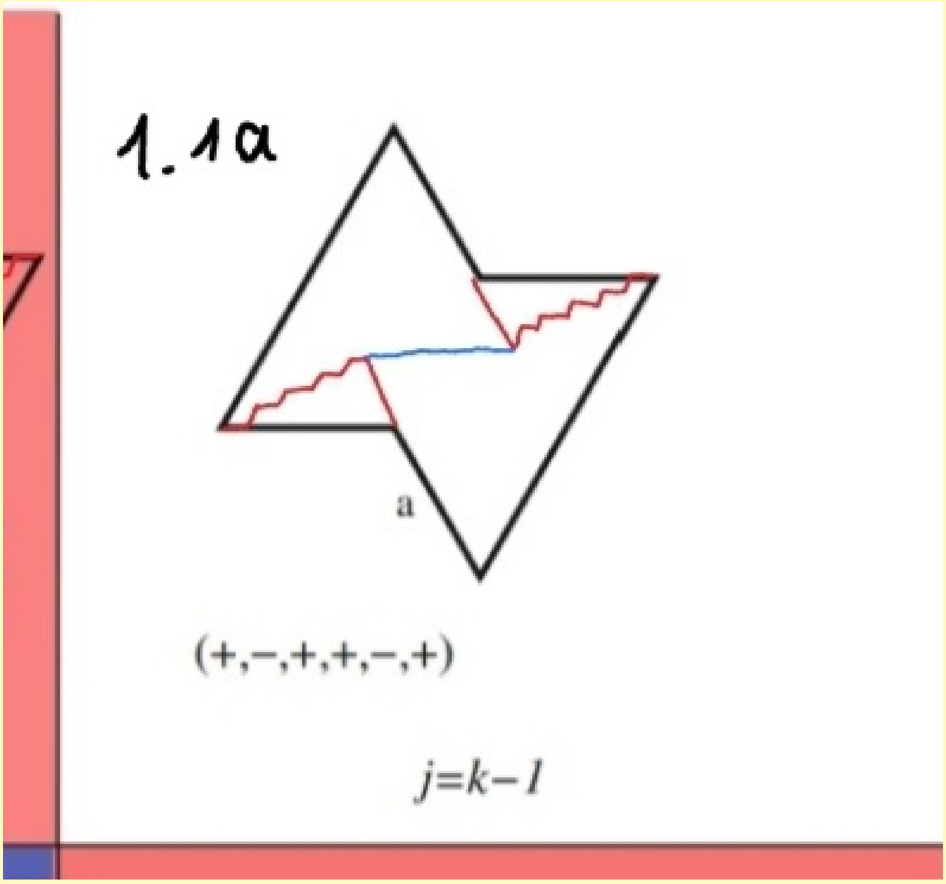
Lemma 2.3 (Balanced Kuo Condensation; Theorem 5.1 in [11]). *Let $G = (V_1, V_2, E)$ be a (weighted) planar bipartite graph with $|V_1| = |V_2|$. Assume that p_1, p_2, p_3, p_4 are four vertices appearing in a cyclic order on a face of G. Assume in addition that $p_1, p_3 \in V_1$ and $p_2, p_4 \in V_2$. Then*

$$\begin{aligned} w(G)w(G - \{p_1, p_2, p_3, p_4\}) =& w(G - \{p_1, p_2\})w(G - \{p_3, p_4\}) \\ &+ w(G - \{p_1, p_4\})w(G - \{p_2, p_3\}). \end{aligned} \tag{1}$$



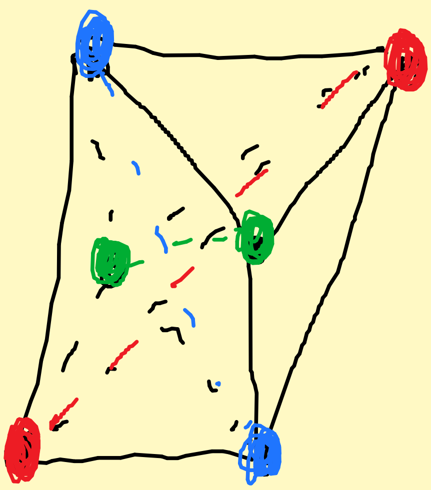
Example: region 1.1a

$k > 0,\ i, j > k-1$



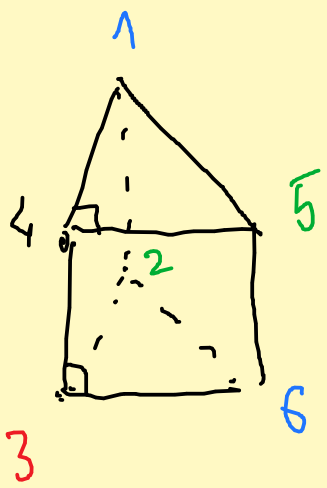
Recurrence:

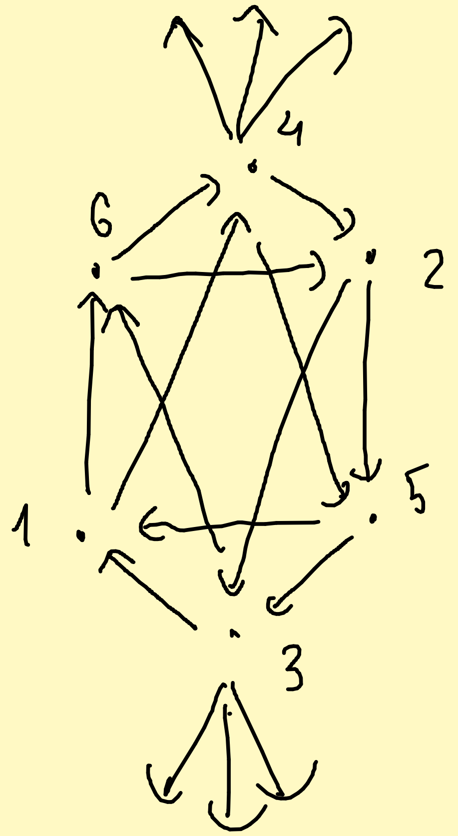
$$z_{i-1,j-1,k-1}\, z_{i,j,k} = z_{i-1,j,k}\, z_{i,j-1,k-1} + z_{i,j-1,k}\, z_{i-1,j,k-1}$$



Lemma: When the prism is in the down triangle position, the square angle vertices have outgoing arrows to the framed vertices.

Eg:

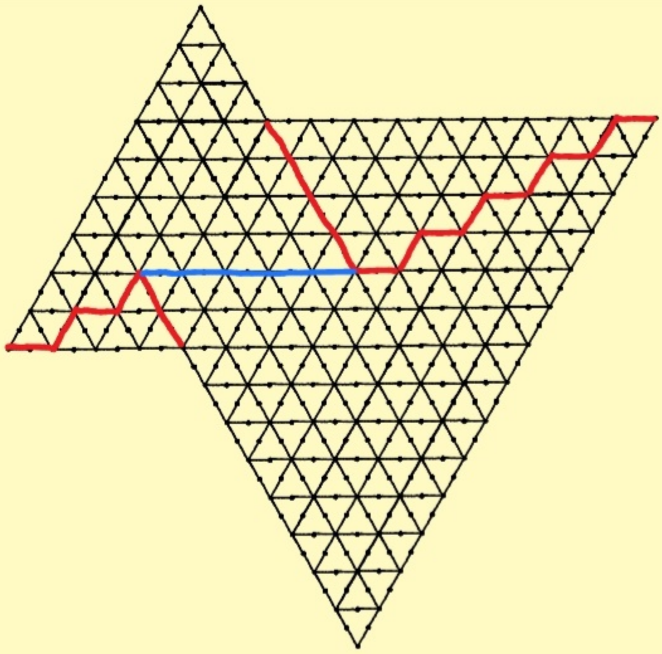




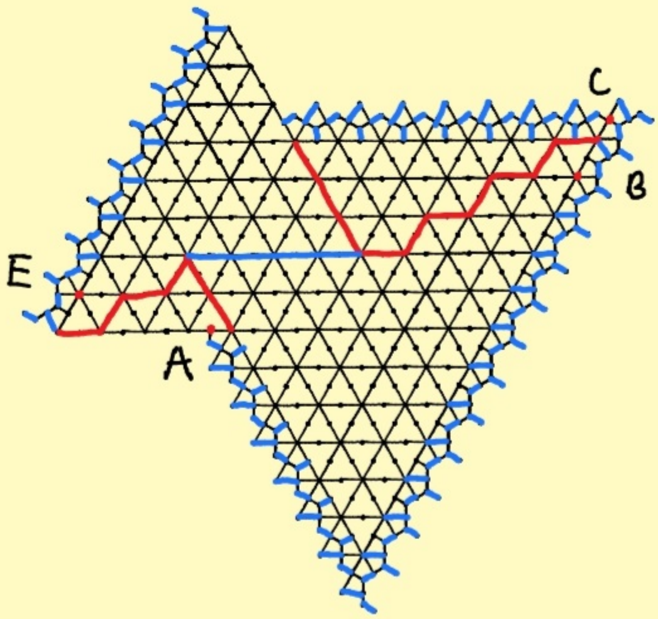
$$z_{i-1,j-1,k-1}\, z_{i,j,k} = \prod y_i \; z_{i-1,j,k}\, z_{i,j-1,k-1} + z_{i,j-1,k}\, z_{i-1,j,k-1}$$

$\Rightarrow$ Only need to compare

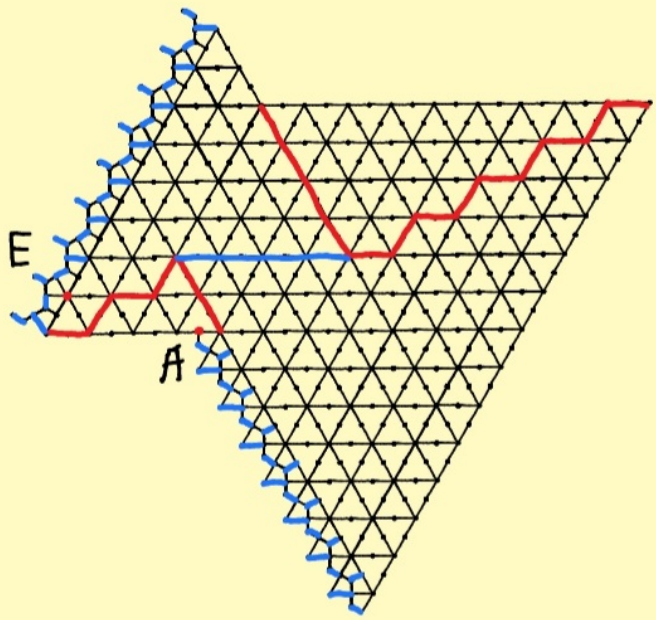
$$C_{i-1,j-1,k-1}\, C_{i,j,k} \quad \text{with} \quad C_{i,j-1,k}\, C_{i-1,j,k-1}$$



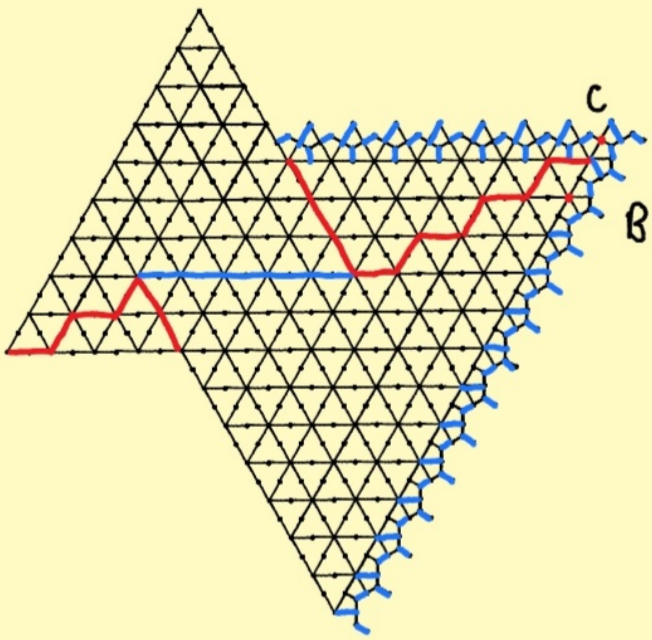
$G = C_{i,j,k}$



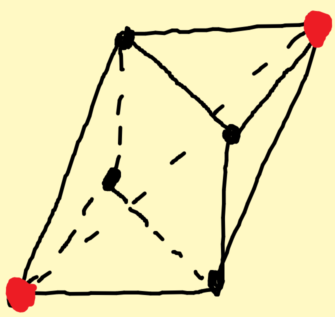
$G - \{A,B,C,E\} = C_{i-1,j-1,k-1}$

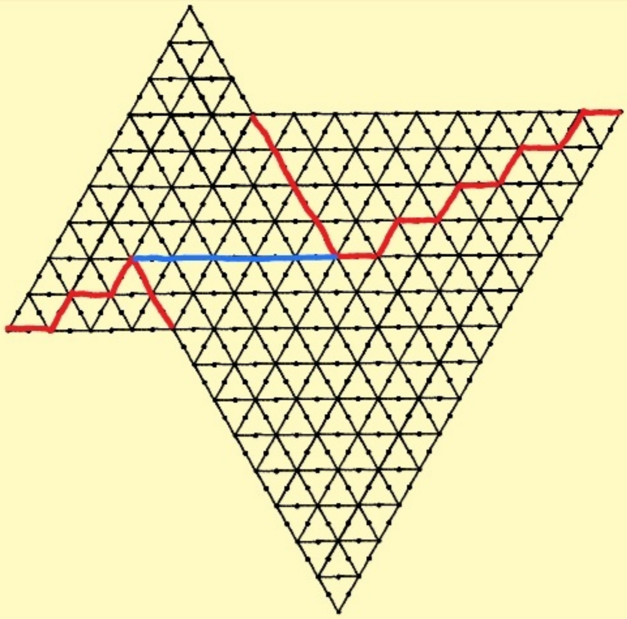


$G - \{A,E\} = C_{i,j-1,k}$

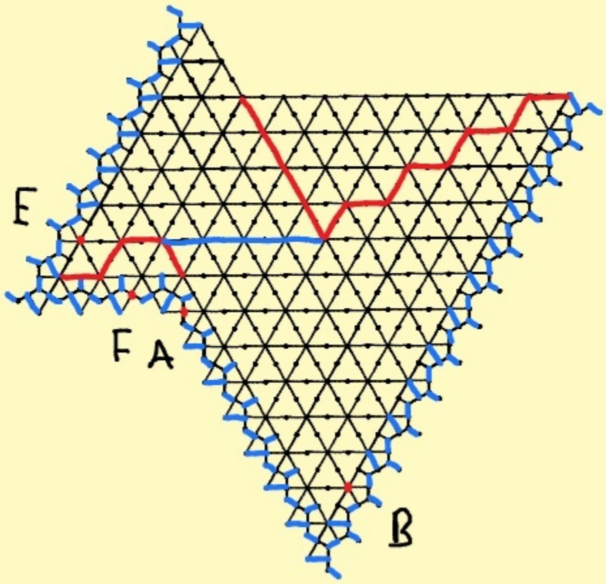


$G - \{B,C\} = C_{i-1,j,k-1}$

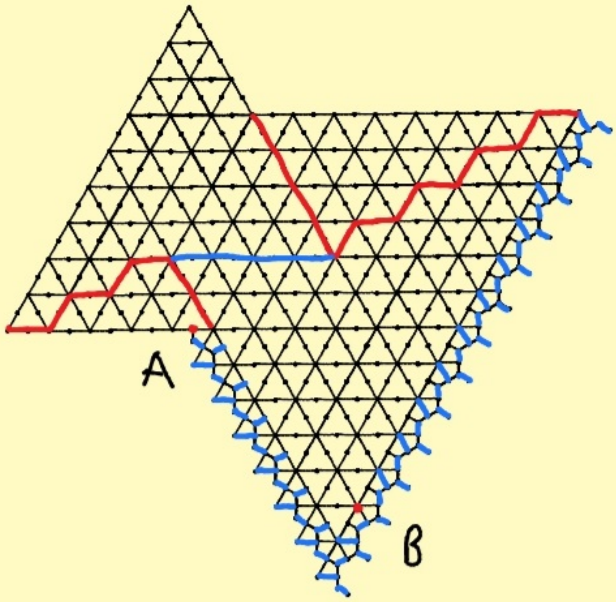




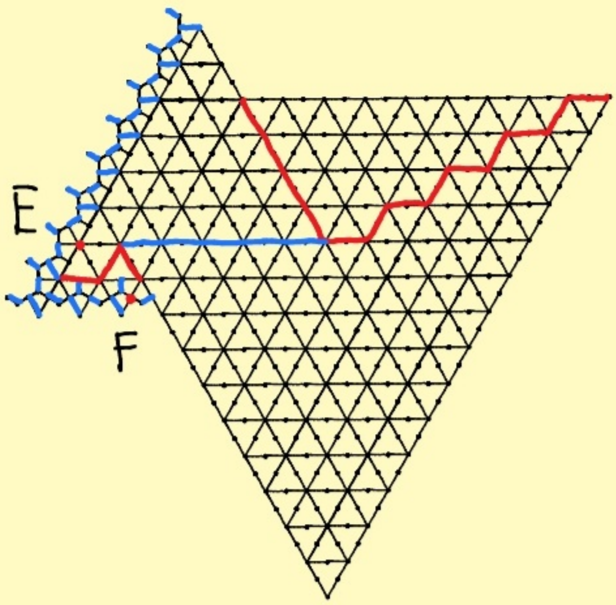
$G = C_{i+1, j, k-1}$



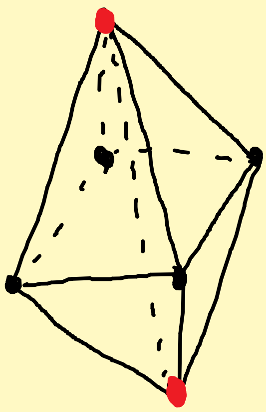
$G - \{A, B, E, F\} = C_{i, j-1, k-1}$

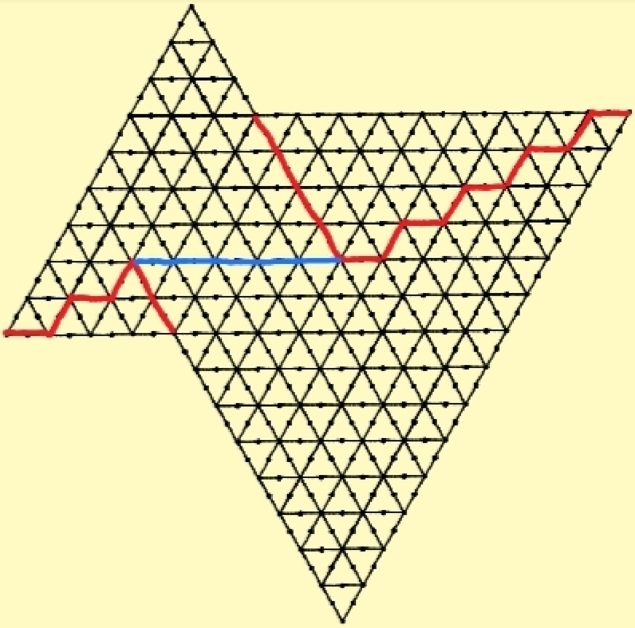


$G - \{A, B\} = C_{i+1, j-1, k-2}$

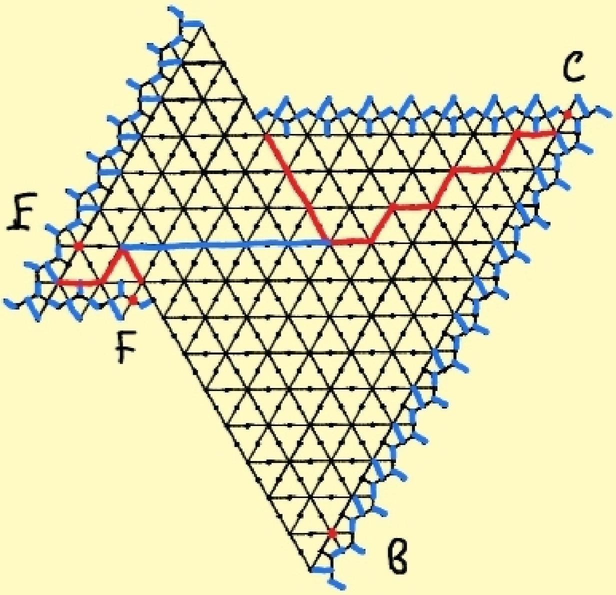


$G - \{E, F\} = C_{i,j,k}$

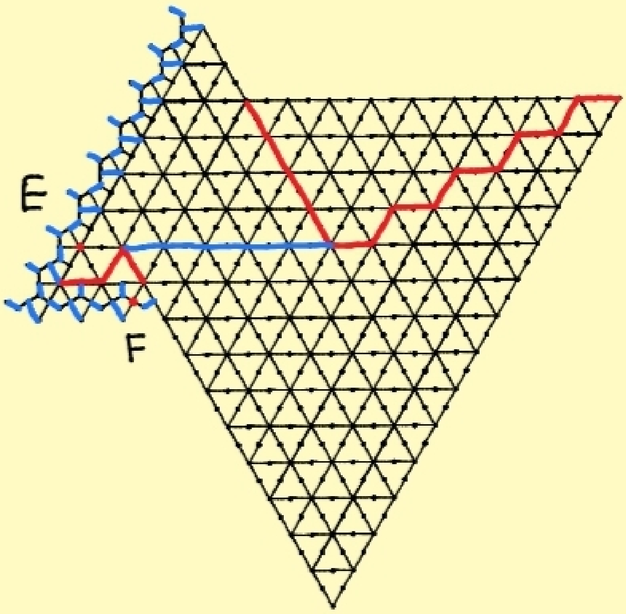




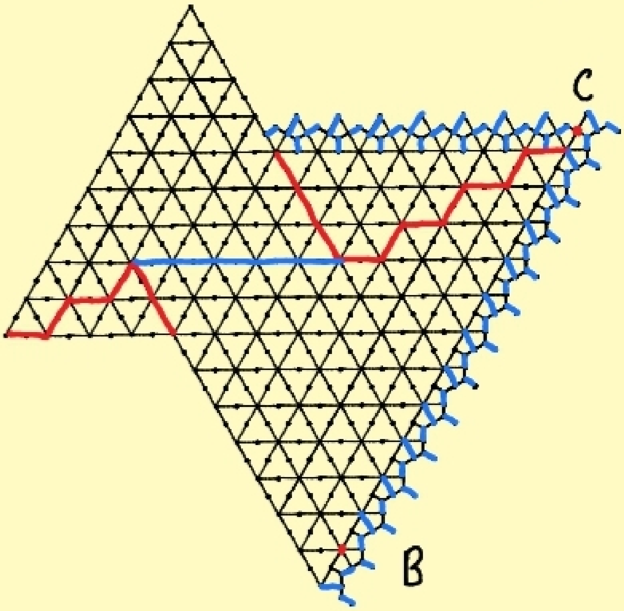
$G = C_{i,j,k}$



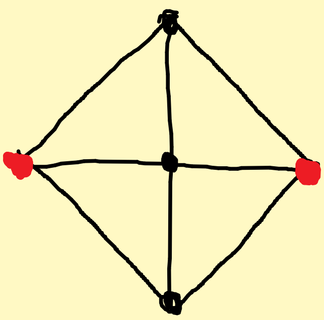
$G - \{B, C, E, F\} = C_{i-2,j,k}$



$G - \{E, F\} = C_{i-1,j,k+1}$



$G - \{B, C\} = C_{i-1,j,k-1}$



Thank

You