A q-analog of Random-to-Random Shuffling

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Outline

Random-to-Top, Top-to-Random Shuffling

Definition (top-to-random, random-to-top shuffling)

Given a deck of n cards, uniformly distributed,

- top-to-random: puts top card in any position.
- random-to-top: brings any card to the front.
- random-to-random: inserts any card in any position.

Note that

$$\begin{aligned}\{\text{deck of } n \text{ cards}\} &\longleftrightarrow \{\text{words in } \{1,\ldots,n\} \text{ length } n, \text{ without repeats}\}\\ &\longleftrightarrow \{\text{permutation in the symmetric group}\}.\end{aligned}$$

Random-to-Top, Top-to-Random Shuffling

Definition (top-to-random, random-to-top shuffling)

Given a word $w = w_1 w_2 \dots w_n$,

$$\mathrm{T2R}_n(w) := \frac{1}{n} \sum_{i=1}^{n} w_2 w_3 \dots w_i w_1 w_{i+1} \dots w_n,$$

$$\mathrm{R2T}_n(w) := \frac{1}{n} \sum_{i=1}^{n} w_i w_1 \dots w_{i-1} w_{i+1} \dots w_n.$$

Example: for $w = 123$,

$$\mathrm{T2R}_3(123) = \frac{1}{3}(123 + 213 + 231),$$

$$\mathrm{R2T}_3(123) = \frac{1}{3}(123 + 213 + 312).$$

Random-to-Random Shuffling

Definition (random-to-random)

Given a word $w = w_1 w_2 \dots w_n$,

$$\mathrm{R2R}_n(w_1 w_2, \dots, w_n) := \mathrm{T2R}_n \circ \mathrm{R2T}_n(w_1 w_2, \dots, w_n)$$

Example: for $w = 123$,

$$\begin{aligned}\mathrm{R2R}_3(123) &= \frac{1}{3} \cdot \mathrm{T2R}_3 \circ ((123) + (213) + (231)) \\ &= \frac{1}{3} \cdot (123) + \frac{2}{9} \cdot (213) + \frac{2}{9} \cdot (132) + \frac{1}{9} \cdot (231) + \frac{1}{9} \cdot (312) + 0 \cdot (321).\end{aligned}$$

Transition matrices

Idea: operators are linear maps $\mathbb{C}[S_n] \to \mathbb{C}[S_n]$, so we can write them as *transition matrices*.

Example: the transition matrix of $\mathrm{R2T}_3$ is

	(123)	(132)	(213)	(231)	(312)	(321)
(123)	1	0	1	1	0	0
(132)	0	1	0	0	1	1
(213)	1	0	1	0	0	0
(231)	0	1	0	1	1	1
(312)	1	0	0	0	1	0
(321)	0	1	1	1	0	1

Specifically, $\mathrm{R2T}_3(123) = \frac{1}{3}(123 + 213 + 312)$.

Convention: normalize the operators to have integer coefficients.

Motivating question

Question:

How many times do we need to shuffle our deck before the deck is "well-mixed"?

Solution:

This is determined by the eigenvalues of the transition matrix.

Question, rephrased:

What are the eigenvalues of $R2R_n$?

Eigenvalues of $\mathrm{R2R_n}$

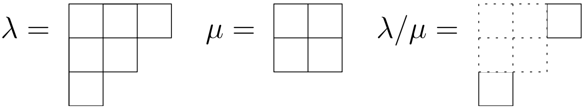
Eigenvalues of $\mathrm{R2R}_n$ are indexed by horizontal strips of Young diagrams.

Definition

If λ, μ are two partitions such that diagram of λ contains the one of μ, then

- **skew partition**: λ/μ contains the cells of λ that do not belong to μ.

Example: $\lambda = (3, 2, 1)$ and $\mu = (2, 2)$



Eigenvalues of $R2R_n$

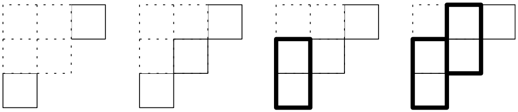
Eigenvalues of $R2R_n$ are indexed by horizontal strips of Young diagrams.

Definition

If λ, μ are two partitions such that diagram of λ contains the one of μ, then

- **skew partition**: λ/μ contains the cells of λ that do not belong to μ.
- **horizontal strip** is a skew partition containing no more than one cell in each column.

Example: $\lambda = (3, 2, 1)$ and $\mu = (2, 2)$



horizontal strips non-horizontal strips

Eigenvalues of $\mathrm{R2R_n}$

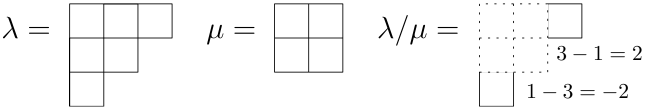
Eigenvalues of $\mathrm{R2R}_n$ are indexed by horizontal strips of Young diagrams.

Definition

If λ and μ are two partitions such that the diagram of λ contains the diagram of μ, then

- $\mathrm{diag}\,(\lambda/\mu) = \sum_{\text{cells}(i,j) \text{ in } \lambda/\mu} (j - i)$.

Example: $\lambda = (3,2,1)$ and $\mu = (2,2)$, then $\mathrm{diag}(\lambda/\mu) = 0$.



Eigenvalues of $\mathrm{R2R}_n$

Theorem (Dieker-Saliola, 2018)

Given λ a partition of n and μ a horizontal strip[a], $\mathrm{R2R}_n$ *has an eigenvalue*

$$eig(\lambda/\mu) = \mathrm{diag}(\lambda/\mu) + \sum_{j=1}^{|\lambda/\mu|} j + |\mu|.$$

All of the eigenvalues of $\mathrm{R2R}_n$ *are of this form and* $eig(\lambda/\mu) \in \mathbb{Z}_{\geq 0}$.

[a] Special cases: $\mu \neq (k)$, $\mu \neq (1^{\ell})$ for ℓ odd

Example: $\lambda = (3,2,1)$ and $\mu = (2,2)$, then

$$eig(\lambda/\mu) = 0 + \sum_{j=1}^{2}(j+4) = 0 + 5 + 6 = 11.$$

Eigenvalues of $\mathrm{R2R_n}$

The key to finding explicit eigenvalue formulas is the following:

Theorem (Dieker-Saliola, 2018)

If ϵ_μ is an eigenvalue of $\mathrm{R2R}_n$, then an eigenvalue of $\mathrm{R2R}_{n+1}$ is

$$\epsilon_{\mu+e_i} = \epsilon_\mu + (n+1) + (\mu_i + 1) - i.$$

Intuition: We want to build a horizontal strip (λ/μ) from μ

Example: Consider $\lambda = (3,2,1)$, $\mu = (2,2)$:

1. Suppose $\epsilon_\mu = \epsilon_{(2,2)} = 0$.
2. $i = 1$, $\mu_1 = 2$: gives $\epsilon_{\mu+e_1} = \epsilon_{(3,2)} = \epsilon_{(2,2)} + (4+1) + (2+1) - 1 = 7$.
3. $i = 3$, $\mu_3 = 0$ gives:

$$\epsilon_\lambda = \epsilon_{\mu+e_1+e_3} = \epsilon_{(3,2,1)} = \epsilon_{(3,2)} + (5+1) + (0+1) - 3 = 11.$$

Definitions in q-analog

Assume q is a power of a prime number,

classical objects	**q-analog**
$k \in \mathbb{Z}_{>0}$	$[k]_q = 1 + q + q^2 + \cdots + q^{k-1}$
set $\{1, 2, \ldots, n\}$	vector space $\mathbb{F}_q^n$
S_n	$GL_n(\mathbb{F}_q) = \{$ invertible $n \times n$ matrices with entries in $\mathbb{F}_q$ $\}$
words (permutation)	complete flags

Table: Comparison between classical object and q-analog

Statement of Problem

Question

- How to define $\mathrm{R2R}_n^{(q)}$?
- What are the eigenvalues of $\mathrm{R2R}_n^{(q)}$?
- What is the $GL_n(\mathbb{F}_q)$ representation structure on $\mathrm{R2R}_n^{(q)}$ eigenspaces?

Previous work:

- $\mathrm{T2R}_n$ and $\mathrm{R2T}_n$:
  - Phatarfod (1991), Uyemura-Reyes (2002), Reiner-Wachs (2002).
- $\mathrm{R2R}_n$:
  - Reiner-Saliola-Welker (2011), Dieker-Saliola (2017), Lafreniére (2019).
- $\mathrm{R2T}_n^{(q)}$:
  - Brown (1999), Brauner-Commins-Reiner (2022).

Definition (complete flag)

A **complete flag** in a vector space V is a sequence of $dim(V) = n$ nested linear subspaces

$$U_1 \subset U_2 \subset \cdots \subset U_{n-1} \subset U_n = V,$$

where U_i has dimension i.

Example: given $n = 2, q = 2,\ V = \mathbb{F}_2^2 = \langle e_1, e_2 \rangle = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle$.

The complete flags are

- $f_1 = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle = V$
- $f_2 = \langle e_2 \rangle \subset \langle e_1, e_2 \rangle = V$
- $f_3 = \langle e_1 + e_2 \rangle \subset \langle e_1, e_2 \rangle = V$

Definition ($\mathrm{R2T}_n^{(q)}$)

Consider a flag $F = (F_1 \subset F_2 \cdots \subset F_n = \mathbb{F}_q^n)$:

- $\mathrm{R2T}_n^{(q)}(F)$

$$:= \sum_{i=1}^{n} \sum_{\substack{L \in F_i \\ \dim(L)=1 \\ L \notin F_{i-1}}} \langle L \rangle \subset \langle L, F_1 \rangle \subset \cdots \subset (\langle L, F_{i-1} \rangle = F_i) \subset F_{i+1} \subset \cdots \subset F_n.$$

Example: for $n = 2$, $q = 2$,

$$\begin{aligned} \mathrm{R2T}_2^{(2)}(\langle e_1 \rangle \subset \langle e_1, e_2 \rangle) &= (\langle e_1 \rangle \subset \langle e_1, e_2 \rangle) \\ &+ (\langle e_2 \rangle \subset \langle e_1, e_2 \rangle) \\ &+ (\langle e_1 + e_2 \rangle \subset \langle e_1, e_2 \rangle). \end{aligned}$$

Definition ($\mathrm{T2R}_n^{(q)}$)

Consider a flag $F = (F_1 \subset F_2 \cdots \subset F_n = \mathbb{F}_q^n)$

$\mathrm{T2R}_n^{(q)}(F)$

$$= \sum_{j=0}^{n-1} \sum_{\substack{a_1,\cdots,a_j \\ \in \mathbb{F}_q}} f_2 + a_1 f_1 \subset \cdots \subset f_2 + a_2 f_1, \cdots, f_{j+1} + a_j f_1 \subset F_{j+1} \subset \cdots \subset F_n.$$

Definition ($\mathrm{R2R}_n^{(q)}$)

$\mathrm{R2R}_n^{(q)}(F)$

$$= \sum_{i,j} \sum_{\substack{f_i \in F_i \\ f_i \notin F_{i-1}}} \sum_{a_j,\cdots,a_j \in \mathbb{F}_q} \langle f_1 + a_1 f_i \rangle \subset \cdots \subset \langle f_1 + a_1 f_i, \cdots, (\widehat{f_i + a_i f_i}), \cdots, f_j + a_j f_i \rangle$$

$$\subset \langle f_1, \cdots, f_j, f_i \rangle \subset \cdots \subset \langle f_1, \cdots, \widehat{f_i}, \cdots, f_n, f_i \rangle = F_n.$$

A different approach

The operators defined in terms of flags are quite complicated...

Question: Is there a better way to understand them?

Approach: Follow what Dieker-Saliola do when $q = 1$

R2R action

Dieker-Saliola considers R2R$_n$ as a right action on words.

$$S_n \text{ by value} \circlearrowright \mathbb{C}[S_n] \circlearrowleft S_n \text{ by position}$$

Example: Let $s_i = (i, i+1) \in S_n$.

- left action (by value): $s_1(312) = (321)$.
- right action (by position): $(312)s_1 = (132)$

BIG IDEA:
$\mathbb{C}[S_n]$ decomposes as a $(S_n \times S_n)$ bimodule:

$$\mathbb{C}[S_n] = \bigoplus_{\lambda \vdash n} (S^\lambda)^* \otimes_{\mathbb{C}} S^\lambda,$$

where S^λ is an irreducible representation of S_n.

Motivation of using the Hecke algebra

$q = 1$

S_n by value $\circlearrowleft$ $\mathbb{C}[S_n]$ $\circlearrowright$ S_n by position

$\downarrow$ replace

$q - analog$

$GL_n(\mathbb{F}_q)$ $\circlearrowleft$ $\mathbb{C}[G/B]$ $\circlearrowright$ $\mathcal{H}_n(q)$ (the Hecke algebra)

Here, $\mathbb{C}[G/B]$ is the $\mathbb{C}$-vector space on complete flags.

Motivation of using the Hecke algebra

A $q-$analog of BIG IDEA:

$$GL_n(\mathbb{F}_q) \circlearrowleft \mathbb{C}[G/B] \circlearrowright \mathcal{H}_n(q) \text{ (the Hecke algebra)}$$

Note: $\mathbb{C}[G/B]$ also decomposes as a bimodule.

$$\mathbb{C}[G/B] = \bigoplus_{\lambda \vdash n} G^\lambda \otimes_{\mathbb{C}} H^\lambda,$$

where

- G^λ is an irreducible unipotent representation of $GL_n(\mathbb{F}_q)$
- H^λ is an irreducible representation of $\mathcal{H}_n(q)$.

Motivation of using the Hecke algebra

A $q-$analog of BIG IDEA:

$$GL_n(\mathbb{F}_q) \circlearrowleft \mathbb{C}[G/B] \circlearrowright \mathcal{H}_n(q) \text{ (the Hecke algebra)}$$

Note: $\mathbb{C}[G/B]$ also decomposes as a bimodule.

$$\mathbb{C}[G/B] = \bigoplus_{\lambda \vdash n} G^\lambda \otimes_{\mathbb{C}} H^\lambda,$$

Upshot:

- Understanding how $\mathrm{R2R}_n^{(q)}$ acts on H^λ shows how $\mathrm{R2R}_n^{(q)}$ acts on G^λ.
- **Only** need to understand the eigenvalues of $\mathrm{R2R}_n^{(q)}$ on the various H^λ's!

Hecke algebra

Definition (Hecke algebra)

The Hecke algebra $\mathcal{H}_n(q)$ is the associative $\mathbb{C}$-algebra with generators $T_1, \ldots, T_{n-1}$ satisfying

1. $(T_i - q)(T_i + 1) = 0$,
2. $T_i T_j = T_j T_i$ for $|i - j| \geq 2$,
3. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$.

Note: The Hecke algebra $\mathcal{H}_n(q)$ is a q-deformation of $\mathbb{C}[S_n]$.

Setting $q = 1$ recovers $\mathbb{C}[S_n]$, with generators $s_1, \ldots, s_{n-1}$ and relations

1. $(s_i - 1)(s_i + 1) = s_i^2 - 1 = 0$,
2. $s_i s_j = s_j s_i$ for $|i - j| \geq 2$,
3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

Writing the q-Shuffling Operators in $\mathcal{H}_n(q)$

Theorem (C-L-AF)

The q-shuffling operators have the following expression in $\mathcal{H}_n(q)$:

- $\mathrm{R2T}_n^{(q)} = \sum_{i=0}^{n-1} T_i \ldots T_2 T_1,$
- $\mathrm{T2R}_n^{(q)} = \sum_{j=0}^{n-1} T_1 T_2 \ldots T_j,$
- $\mathrm{R2R}_n^{(q)} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} T_i \ldots T_1 T_1 \ldots T_j.$

When $q = 1$,

$$\mathrm{R2T}_n^{(1)} = \mathrm{R2T}_n, \quad \mathrm{T2R}_n^{(1)} = \mathrm{T2R}_n, \quad \mathrm{R2R}_n^{(1)} = \mathrm{R2R}_n.$$

There is a $\mathbb{C}$-basis for $\mathcal{H}_n(q)$ indexed by the reduced words of S_n.

Theorem (C-L-AF)

A reduced expression for $\mathrm{R2R}_n^{(q)} \in \mathcal{H}_n(q)$ *is*

$$\mathrm{R2R}_n^{(q)} = [n]_q + \sum_{p=1}^{n-1} q^{p-1}\Bigg((q+1)T_p + q\sum_{i=p+1}^{n-1} T_i \dots T_p + q\sum_{j=p+1}^{n-1} T_p \dots T_j + (q-1)\sum_{k=p+1}^{n-1}\sum_{\ell=p+1}^{n-1} T_k \dots T_{(p+1)} T_p T_{(p+1)} \dots T_\ell\Bigg).$$

Advantage of Using the Hecke algebra

1. **More data!** Once we proved the reduced expression for $\mathrm{R2R}_n^{(q)}$ in $\mathcal{H}_n(q)$, we could compute eigenvalues for larger n and any q.

2. **Words over flags**: We will discuss a way that $\mathcal{H}_n(q)$ acts on words, rather than flags. This simplifies computations significantly!

Conjectures

Goal: understand the eigenvalues and eigenspaces of $\mathrm{R2R}_n^{(q)}$.
Approach: the q-analog of eigenvalue recursion:

Eigenvalue Recursion Conjecture (C-L-AF)

If $\epsilon_\mu(q)$ is an eigenvalue of $\mathrm{R2R}_n^{(q)}$, then

$$\epsilon_{\mu+e_i}(q)$$

is an eigenvalue for $\mathrm{R2R}_{n+1}^{(q)}$ such that

$$\epsilon_{\mu+e_i}(q) = q\epsilon_\mu(q) + [(n+1) + (\lambda_i + 1) - i]_q.$$

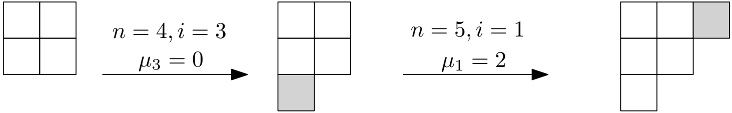
Recall when $q = 1$,

$$\epsilon_{\mu+e_i} = \epsilon_\mu + (n+1) + (\mu_i + 1) - i.$$

Eigenvalue recursion example

Recursion: $\epsilon_{\mu+e_i}(q) = q\epsilon_\mu(q) + [(n+1) + (\lambda_i + 1) - i]_q$.

For $\mu = (2,2)$, $\lambda = (3,2,1)$



$\epsilon^{(q)}_{(2,2)} = 0$

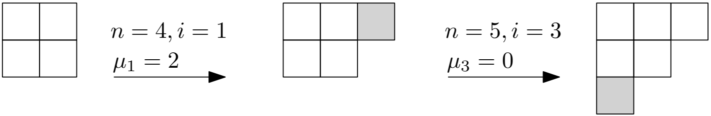
$$\epsilon^{(q)}_{(2,2,1)} = \epsilon^{(q)}_{(2,2)} + e_3$$
$$= q[0]_q + [(4+1) + (0+1) - 3]_q$$
$$= [3]_q$$

$$\epsilon^{(q)}_{(3,2,1)} = \epsilon^{(q)}_{(2,2,1)} + e_1$$
$$= q[3]_q + [(5+1) + (2+1) - 1]_q$$
$$= [8]_q + q[3]_q$$

Eigenvalue recursion example

Recursion: $\epsilon_{\mu+e_i}(q) = q\epsilon_\mu(q) + [(n+1) + (\lambda_i + 1) - i]_q.$

Lemma: (C-L-AF) The order we add boxes does not matter!



$\epsilon^{(q)}_{(2,2)} = 0$

$$\epsilon^{(q)}_{(3,2)} = \epsilon^{(q)}_{(2,2)} + e_1$$
$$= q[0]_q + [(4+1) + (2+1) - 1]_q$$
$$= [7]_q$$

$$\epsilon^{(q)}_{(3,2,1)} = \epsilon^{(q)}_{(3,2)} + e_3$$
$$= q[7]_q + [(5+1) + (0+1) - 3]_q$$
$$= q[7]_q + [4]_q = [8]_q + q[3]_q$$

q-analog of the diag statistic

Definition

For λ/μ a horizontal strip, we define

$$\mathrm{diag}(\lambda/\mu)_q := \sum_{\text{cells}(i,j) \text{ in } \lambda/\mu} q^{|\lambda|}[(j-i)]_q$$

But: $j - i$ might be negative....

Recall that for we defined $[k]_q$ for $k \in \mathbb{Z}_{>0}$:

$$[k]_q = 1 + q + \cdots + q^{k-1} = \frac{1-q^k}{1-q}.$$

Solution: More generally, for any $k \in \mathbb{Z}$, define

$$[k]_q := \frac{1 - q^k}{1 - q}.$$

Example:

$$[-2]_q = -q^{-1} - q^{-2}.$$

Lemma (C-L-AF)

For any (i, j) in λ/μ

$$q^{|\lambda|}[(j - i)]_q \in \mathbb{Z}[q].$$

Thus

$$\text{diag}(\lambda/\mu)_q := \sum_{\text{cells}(i,j) \text{ in } \lambda/\mu} q^{|\lambda|}[(j - i)]_q \in \mathbb{Z}[q].$$

In other words, $\text{diag}(\lambda/\mu)_q$ is a polynomial in q.

Explicit formula for eigenvalues

Then the **Eigenvalue Recursion Conjecture** implies the following:

Big Conjecture (C-L-AF)

Let λ/μ be a horizontal strip[a]. Then $\mathrm{R2R}_n^{(q)}$ has an eigenvalue

$$eig(\lambda/\mu)_q = \mathrm{diag}(\lambda/\mu)_q + \sum_{j=1}^{|\lambda/\mu|} q^{|\lambda/\mu|-j}[j+|\mu|]_q \in \mathbb{Z}_{\geq 0}[q].$$

All eigenvalues for $\mathrm{R2R}_n^{(q)}$ are obtained in this way.

[a] Special cases: $\mu \neq (k)$, $\mu \neq (1^\ell)$ for ℓ odd

Recall our questions:

Question

- ☑ How to define $\mathrm{R2R}_n^{(q)}$?
- ☐ What are the eigenvalues of $\mathrm{R2R}_n^{(q)}$?
- ☐ What is the $GL_n(\mathbb{F}_q)$ representation structure on $\mathrm{R2R}_n^{(q)}$ eigenspaces?

Upshot: Big Conjecture would answer all of our questions!

For experts: it implies that for any eigenvalue $\epsilon(q)$ of $\mathrm{R2R}_n^{(q)}$, the $\epsilon(q)$-eigenspace has q-Frobenius characteristic

$$\sum_{\substack{(\lambda/\mu) \\ eig(\lambda/\mu)_q = \epsilon(q)}} d^{\mu} s_{\lambda},$$

where d^{μ} is the number of desarrangement tableaux of shape μ.

Algebra of words

Our proof strategy involves the **Algebra of words**.

Definition (Algebra of words)

Let $M^{\langle n \rangle}$ be the $\mathbb{C}$-vector space generated by words of length n in the alphabet $\{1, 2, \ldots, n\}$ (repeated letters allowed!).

There is a right action on $M^{\langle n \rangle}$:

- by S_n, which acts by position, e.g. $(112) \cdot s_2 = (121)$
- More surprisingly, by $\mathcal{H}_n(q)$!

Action of $\mathcal{H}_n(q)$ on the algebra of words

Action: There is a right action by $\mathcal{H}_n(q)$ on $M^{\langle n \rangle}$ by

$$w_1 w_2 \dots w_n \cdot T_i = \begin{cases} q(w_1 w_2 \dots w_n) & w_i = w_{i+1}, \\ (w_1 \dots w_{i+1} w_i \dots w_n) & w_i < w_{i+1}, \\ q(w_1 \dots w_{i+1} w_i \dots w_n) + (q-1)(w_1 \dots w_i w_{i+1} \dots w_n) & w_i > w_{i+1}. \end{cases}$$

Example:

$$\begin{aligned} (112) \cdot T_1 &= q(112) \\ (112) \cdot T_2 &= (121) \\ (211) \cdot T_1 &= q(121) + (q-1)(211) \end{aligned}$$

Upshot: $\mathrm{R2R}_n^{(q)}$ acts on $M^{\langle n \rangle}$

Proof strategy

Dieker-Saliola use properties of $M^{\langle n \rangle}$ to prove their recursions:

Theorem (Dieker-Saliola, 2018)

From any $w \in M^{\langle n \rangle}$:

$$\mathrm{R2R}_{n+1} \circ sh_i(w) = sh_i \circ \mathrm{R2R}_n(w) + (n+1) sh_i(w) + \sum_{1 \leq j \leq n} sh_j \circ \Theta_{j,i}(w).$$

- $sh_i(w) := \sum_{0 \leq j \leq n} w_1 \dots w_j \cdot i \cdot w_{j+1} \dots w_n.$
- $\Theta_{j,i}(w) := \sum_{\substack{1 \leq k \leq n \\ w_k = j}} w_1 \dots w_{k-1} \cdot i \cdot w_{k+1} \dots w_n.$

Example:

- $sh_1(123) = (1123) + (1123) + (1213) + (1231).$
- $\Theta_{2,1}(122) = (112) + (121).$

Proof strategy

We came up with definition for $\mathrm{sh}_i^{(q)}$ and $\Theta_{j,i}^{(q)}$.

Conjecture (C-L-AF)

For $i = 1$, and any $w \in M^{\langle n \rangle}$:

$$\mathrm{R2R}_{n+1}^{(q)}\mathrm{sh}_1^{(q)} = q \cdot \mathrm{sh}_1^{(q)}\mathrm{R2R}_n^{(q)} + [n+1]_q \mathrm{sh}_1^{(q)} + \sum_{j=1}^{n} q^{n+2-j} \cdot \mathrm{sh}_j^{(q)}\Theta_{j,1}^{(q)}.$$

Big idea: Following Dieker-Saliola, we hope to

- Figure out the recursion for $i > 1$
- Prove recursion in the algebra of words
- Project onto irreducible representations to prove eigenvalue recursion

Recap

Questions:

- How to define $R2R_n^{(q)}$?
- What are its eigenvalues and eigenspaces?

Our Results:

- $R2R_n^{(q)}$ definition in the Hecke algebra
  - Gave us more data and allowed us to work with words instead of flags.
- Full conjecture for eigenvalues and eigenspaces
  - Proof Strategy:

$$\text{Conj. recurrence on words} \implies \text{Conj. eigenvalue recurrence} \implies \text{Big conjecture}$$

Thank you so much!