A q-analog of Random-to-Random Shuffling

Judy Chiang joint with Ilani Axelrod-Freed and Veronica Lang
Mentor: Sarah Brauner, TA: Patty Commins

2022 University of Minnesota, Twin Cities REU

Aug 4, 2022

Outline

Random-to-Top, Top-to-Random Shuffling

Definition (top-to-random, random-to-top shuffling)

Given a deck of n cards, uniformly distributed,

- top-to-random: puts top card in any position.
- random-to-top: brings any card to the front.
- random-to-random: inserts any card in any position.

Note that

$$\{\text{deck of } n \text{ cards}\} \to \{\text{words in } \{1, \ldots, n\} \text{ length } n, \text{ without repeats}\}$$
$$\to \{\text{permutation in the symmetric group}\}.$$

Random-to-Top, Top-to-Random Shuffling

Definition (top-to-random, random-to-top shuffling)

Given a word $w = w_1 w_2 \dots w_n$,

$$\mathrm{T2R}_n(w) := \frac{1}{n} \sum_{i=1}^{n} w_2 w_3 \dots w_i w_1 w_{i+1} \dots w_n;$$

$$\mathrm{R2T}_n(w) := \frac{1}{n} \sum_{i=1}^{n} w_i w_1 \dots w_{i-1} w_{i+1} \dots w_n.$$

Example: for $w = 123$,

$$\mathrm{T2R}_3(123) = \frac{1}{3}(123 + 213 + 231);$$

$$\mathrm{R2T}_3(123) = \frac{1}{3}(123 + 213 + 312).$$

Random-to-Random Shu ing

[redacted]

Given a word $w = w_1 w_2 \ldots w_n$,

$$R2R_n(w_1 w_2, \ldots, w_n) := T2R_n \quad R2T_n(w_1 w_2, \ldots, w_n)$$

Example: for $w = 123$,

$$\begin{aligned} R2R_3(123) &= \frac{1}{3} \quad T2R_3 \quad ((123) + (213) + (231)) \\ &= \frac{1}{3} \quad (123) + \frac{2}{9} \quad (213) + \frac{2}{9} \quad (132) + \frac{1}{9} \quad (231) + \frac{1}{9} \quad (312) + 0 \quad (321). \end{aligned}$$

Transition matrices

Idea: operators are linear maps $\mathbb{C}[S_n] \to \mathbb{C}[S_n]$, so we can write them as *transition matrices*.

Example: the transition matrix of $\mathrm{R2T}_3$ is

	(123)	(132)	(213)	(231)	(312)	(321)
(123)	1	0	1	1	0	0
(132)	0	1	0	0	1	1
(213)	1	0	1	0	0	0
(231)	0	1	0	1	1	1
(312)	1	0	0	0	1	0
(321)	0	1	1	1	0	1

Specifically, $\mathrm{R2T}_3(123) = \frac{1}{3}(123 + 213 + 312)$.

Convention: normalize the operators to have integer coefficients.

Motivating question

Question:

How many times do we need to shuffle our deck before the deck is "well-mixed"?

Solution:

This is determined by the eigenvalues of the transition matrix.

Question, rephrased:

What are the eigenvalues of $\mathrm{R2R}_n$?

Eigenvalues of $R2R_n$

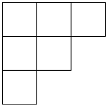
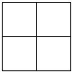
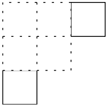
Eigenvalues of $R2R_n$ are indexed by horizontal strips of Young diagrams.

Definition

If , are two partitions such that diagram of contains the one of , then

skew partition: = contains the cells of that do not belong to .

Example: $= (3; 2; 1)$ and $= (2; 2)$

$\lambda =$  $\mu =$  $\lambda/\mu =$ 

Eigenvalues of $\mathrm{R2R_n}$

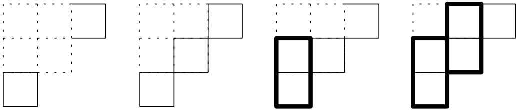
Eigenvalues of $\mathrm{R2R_n}$ are indexed by horizontal strips of Young diagrams.

Definition

If , are two partitions such that diagram of contains the one of , then

- **skew partition**: $=$ contains the cells of that do not belong to .
- **horizontal strip** is a skew partition containing no more than one cell in each column.

Example: $= (3;2;1)$ and $= (2;2)$



horizontal strips non-horizontal strips

Eigenvalues of $\mathrm{R2R_n}$

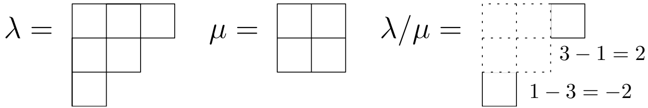
Eigenvalues of $\mathrm{R2R}_n$ are indexed by horizontal strips of Young diagrams.

Definition

If and are two partitions such that the diagram of contains the diagram of , then

$$\mathrm{diag}(\ =\ ) = \sum_{\text{cells}(i;j)\text{ in }\ =} (j \quad i).$$

Example: $= (3;2;1)$ and $= (2;2)$, then $\mathrm{diag}(\ =\ ) = 0$:



Eigenvalues of $R2R_n$

[redacted]

Given a partition of n and a horizontal strip[a], $R2R_n$ has an eigenvalue

$$\mathrm{eig}(= \;) = \mathrm{diag}(= \;) + \sum_{j=1}^{j \; j} j + j \; j :$$

All of the eigenvalues of $R2R_n$ are of this form and $\mathrm{eig}(= \;) \; 2 \; Z_0$.

[a] Special cases: 6 (k), 6 (1 ` ) for ` odd

Example: = (3 ; 2; 1) and = (2 ; 2), then

$$\mathrm{eig}(= \;) = 0 + \sum_{j=1}^{2} (j + 4) = 0 + 5 + 6 = 11 \; :$$

Eigenvalues of $R2R_n$

The key to nding explicit eigenvalue formulas is the following:

[redacted]

If is an eigenvalue of $R2R_n$, then an eigenvalue of $R2R_{n+1}$ is

$$_{+e_i} = \quad + (n+1) + (\quad _i + 1) \quad i:$$

Intuition: We want to build a horizontal strip ($=$) from

Example: Consider $= (3;2;1)$, $= (2;2)$:

1. Suppose $= \ _{(2;2)} = 0$.
2. $i = 1$, $_1 = 2$: gives $_{+e_1} = \ _{(3;2)} = \ _{(2;2)} + (4+1) + (2+1) \quad 1 = 7:$
3. $i = 3$, $_3 = 0$ gives:

$$= \ _{+e_1+e_3} = \ _{(3;2;1)} = \ _{(3;2)} + (5+1) + (0+1) \quad 3 = 11:$$

Definitions in q-analog

Assume q is a power of a prime number,

classical objects	**q-analog**
$k\ 2\ \mathbb{Z}_{>0}$	$[k]_q = 1 + q + q^2 + \quad + q^{k \quad 1}$
set $f1;2;:::;ng$	vector space $\mathbb{F}_q^n$
S_n	$GL_n(\mathbb{F}_q) = f$ invertible $n \quad n$ matrices with entries in $\mathbb{F}_q\ g$
words (permutation)	complete flags

Table: Comparison between classical object and q-analog

Statement of Problem

Question

How to define $\mathrm{R2R}_n^{(q)}$?

What are the eigenvalues of $\mathrm{R2R}_n^{(q)}$?

What is the $GL_n(\mathrm{F}_q)$ representation structure on $\mathrm{R2R}_n^{(q)}$ eigenspaces?

Previous work:

$\mathrm{T2R}_n$ and $\mathrm{R2T}_n$:

Phatarfod (1991), Uyemura-Reyes (2002), Reiner-Wachs (2002).

$\mathrm{R2R}_n$:

Reiner-Saliola-Welker (2011), Dieker-Saliola (2017), Lafreniére (2019).

$\mathrm{R2T}_n^{(q)}$:

Brown (1999), Brauner-Commins-Reiner (2022).

Definition (complete flag)

A **complete flag** in a vector space V is a sequence of $dim(V) = n$ nested linear subspaces

$$U_1 \subset U_2 \subset \cdots \subset U_{n-1} \subset U_n = V,$$

where U_i has dimension i.

Example: given $n = 2, q = 2$, $V = \mathbb{F}_2^2 = \langle e_1, e_2 \rangle = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$.

The complete flags are

$$f_1 = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle = V$$

$$f_2 = \langle e_2 \rangle \subset \langle e_1, e_2 \rangle = V$$

$$f_3 = \langle e_1 + e_2 \rangle \subset \langle e_1, e_2 \rangle = V$$

Definition ($\mathrm{R2T}_n^{(q)}$)

Consider a flag $F = (F_1 \quad F_2 \qquad F_n = \mathbb{F}_q^n)$:

$$\mathrm{R2T}_n^{(q)}(F)$$

$$:= \sum_{i=1}^{n} \sum_{\substack{L \in F_i \\ \dim(L)=1 \\ L \notin F_{i-1}}} \langle L \rangle \quad \langle L; F_1 \rangle \qquad (\langle L; F_{i-1} \rangle = F_i) \quad F_{i+1} \qquad F_n.$$

Example: for $n = 2$, $q = 2$,

$$\begin{aligned} \mathrm{R2T}_2^{(2)}(\langle e_1 \rangle \quad \langle e_1; e_2 \rangle) = {} & (\langle e_1 \rangle \quad \langle e_1; e_2 \rangle) \\ & + (\langle e_2 \rangle \quad \langle e_1; e_2 \rangle) \\ & + (\langle e_1 + e_2 \rangle \quad \langle e_1; e_2 \rangle). \end{aligned}$$

[redacted]

Consider a agF = (F_1 F_2 F_n = F_q^n)

$T2R_n^{(q)}(F)$

$$= \sum_{j=0}^{n-1} \sum_{\substack{a_1; \quad ;a_j \\ 2\, F_q}} f_2 + a_1 f_1 \qquad f_2 + a_2 f_1; \quad ; f_{j+1} + a_j f_1 \qquad F_{j+1} \qquad F_n:$$

[redacted]

$R2R_n^{(q)}(F)$

$$= \sum_{i;j} \sum_{\substack{f_i\, 2\, F_i \\ f_i\, 2\!\!= F_{i\ \ 1}}} \sum_{a_j; \quad ;a_j\, 2\, F_q} h f_1 + a_1 f_i\, i \qquad h \quad f_1 + a_1 f_i; \quad ;(f_i \backslash + a_i f_i); \quad ; f_j + a_j f_i\, i$$

$$h\ f_1; \quad ; f_j ; f_i\, i \qquad h \quad f_1; \quad ; \hat{f_i}; \quad ; f_n ; f_i\, i = F_n:$$

A di erent approach

The operators de ned in terms of ags are quite complicated...

Question: Is there a better way to understand them?

Approach: Follow what Dieker-Saliola do when $q = 1$

R2R action

Dieker-Saliola considers $R2R_n$ as a right action on words.

Example: Let $s_i = (i, i+1) \in S_n$.

left action (by value): $s_1(312) = (321)$.

right action (by position): $(312)s_1 = (132)$

BIG IDEA :

$\mathbb{C}[S_n]$ decomposes as a $(S_n \times S_n)$ bimodule:

$$\mathbb{C}[S_n] = \bigoplus_{\lambda \vdash n} (S^\lambda)^* \otimes_{\mathbb{C}} S^\lambda ,$$

where S^λ is an irreducible representation of S_n.

Motivation of using the Hecke algebra

Here, $\mathbb{C}[G/B]$ is the $\mathbb{C}$-vector space on complete ags.

Motivation of using the Hecke algebra

A q analog of BIG IDEA:

Note: $\mathbb{C}[G/B]$ also decomposes as a bimodule.

$$\mathbb{C}[G/B] = \bigoplus_{\vdash n} G \quad \otimes_{\mathbb{C}} H \; ;$$

where

- G is an irreducible unipotent representation of $GL_n(\mathbb{F}_q)$
- H is an irreducible representation of $H_n(q)$.

Motivation of using the Hecke algebra

A q analog of BIG IDEA:

Note: $\mathbb{C}[G/B]$ also decomposes as a bimodule.

$$\mathbb{C}[G/B] = \bigoplus_{\lambda \vdash n} G \quad \otimes_{\mathbb{C}} H \quad ;$$

Upshot:

Understanding how $\mathrm{R2R}_n^{(q)}$ acts on H shows how $\mathrm{R2R}_n^{(q)}$ acts on G .

Only need to understand the eigenvalues of $\mathrm{R2R}_n^{(q)}$ on the various H 's!

Hecke algebra

Definition (Hecke algebra)

The Hecke algebra $H_n(q)$ is the associative $\mathbb{C}$-algebra with generators $T_1, \ldots, T_{n-1}$ satisfying

1. $(T_i - q)(T_i + 1) = 0$,
2. $T_i T_j = T_j T_i$ for $|i - j| \geq 2$,
3. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$.

Note: The Hecke algebra $H_n(q)$ is a q-deformation of $\mathbb{C}[S_n]$.

Setting $q = 1$ recovers $\mathbb{C}[S_n]$, with generators $s_1, \ldots, s_{n-1}$ and relations

1. $(s_i - 1)(s_i + 1) = s_i^2 - 1 = 0$,
2. $s_i s_j = s_j s_i$ for $|i - j| \geq 2$,
3. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

Writing the q-Shuffling Operators in $H_n(q)$

Theorem (C-L-AF)

The q-shuffling operators have the following expression in $H_n(q)$:

$$\mathrm{R2T}_n^{(q)} = \sum_{i=0}^{n-1} T_i \cdots T_2 T_1,$$

$$\mathrm{T2R}_n^{(q)} = \sum_{j=0}^{n-1} T_1 T_2 \cdots T_j,$$

$$\mathrm{R2R}_n^{(q)} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} T_i \cdots T_1 T_1 \cdots T_j.$$

When $q = 1$,

$$\mathrm{R2T}_n^{(1)} = \mathrm{R2T}_n, \quad \mathrm{T2R}_n^{(1)} = \mathrm{T2R}_n, \quad \mathrm{R2R}_n^{(1)} = \mathrm{R2R}_n.$$

There is a $\mathbb{C}$-basis for $H_n(q)$ indexed by the reduced words of S_n.

Theorem (C-L-AF)

A reduced expression for $\mathrm{R2R}_n^{(q)} \in H_n(q)$ *is*

$$\mathrm{R2R}_n^{(q)} = [n]_q + \sum_{p=1}^{n-1} q^{p-1} \Big((q+1) T_p + q \sum_{i=p+1}^{n-1} T_i \cdots T_p$$

$$+ q \sum_{j=p+1}^{n-1} T_p \cdots T_j + (q-1) \sum_{k=p+1}^{n-1} \sum_{\ell=p+1}^{n-1} T_k \cdots T_{(p+1)} T_p T_{(p+1)} \cdots T_\ell \Big).$$

Advantage of Using the Hecke algebra

1. **More data!** Once we proved the reduced expression for $\mathrm{R2R}_n^{(q)}$ in $H_n(q)$, we could compute eigenvalues for larger n and any q.

2. **Words over ags**: We will discuss a way that $H_n(q)$ acts on words, rather than flags. This simplifies computations significantly!

Conjectures

Goal: understand the eigenvalues and eigenspaces of $R2R_n^{(q)}$

Approach: the q-analog of eigenvalue recursion:

[redacted]

If (q) is an eigenvalue of $R2R_n^{(q)}$, then

$$+e_i(q)$$

is an eigenvalue for $R2R_{n+1}^{(q)}$ such that

$$+e_i(q) = q \quad (q) + [(n+1) + (\quad _i + 1) \quad i]_q :$$

Recall when $q = 1$,

$$+e_i = \quad + (n+1) + (\quad _i + 1) \quad i :$$

Eigenvalue recursion example

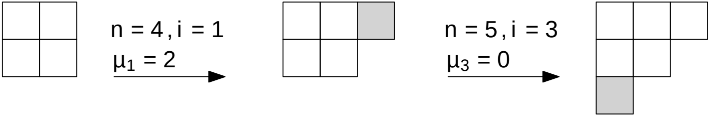
Recursion: $_{+e_i}(q) = q \quad (q) + [(n+1) + (\quad _i + 1) \quad i]_q$:

For $= (2;2)$, $= (3;2;1)$

Eigenvalue recursion example

Recursion: $\ _{+e_i}(q) = q \quad (q) + [(n+1) + (\ _i + 1) \quad i]_q.$

Lemma: (C-L-AF) The order we add boxes does not matter!



$\ ^{(q)}_{(2,2)} = 0$

$\ ^{(q)}_{(3,2)} = \ ^{(q)}_{(2,2)} + e_1$
$= q[0]_q + [(4+1) + (2+1) \quad 1]_q$
$= [7]_q$

$\ ^{(q)}_{(3,2,1)} = \ ^{(q)}_{(3,2)} + e_3$
$= q[7]_q + [(5+1) + (0+1) \quad 3]_q$
$= q[7]_q + [4]_q = [8]_q + q[3]_q$

q-analog of the diag statistic

Definition

For = a horizontal strip, we define

$$\mathrm{diag}(\; = \;)_q := \sum_{\text{cells}(i;j) \text{ in } \; = } q^{j \;\; j}[(j \quad i)]_q$$

But: $j \quad i$ might be negative....

Recall that for we defined $[k]_q$ for $k \; 2 \; \mathbb{Z}_{>0}$:

$$[k]_q = 1 + q + \quad + q^{k \;\; 1} = \frac{1 \quad q^k}{1 \quad q}:$$

Solution: More generally, for any $k\ 2\ Z;$ define

$$[k]_q := \frac{1 \quad q^k}{1 \quad q}:$$

Example:

$$[\quad 2]_q = \quad q^{\quad 1} \quad q^{\quad 2}:$$

Lemma (C-L-AF)

For any $(i;j)$ in $\ =$

$$q^{j \quad j}[(j \quad i)]_q\ 2\ Z[q]:$$

Thus

$$\text{diag}(\quad = \quad)_q := \underset{\text{cells}(i;j) \text{ in } \ =}{X} q^{j \quad j}[(j \quad i)]_q\ 2\ Z[q]:$$

In other words, $\text{diag}(\quad = \quad)_q$ is a polynomial in q.

Explicit formula for eigenvalues

Then the **Eigenvalue Recursion Conjecture** implies the following:

Big Conjecture (C-L-AF)

Let = be a horizontal strip[a]. Then $R2R_n^{(q)}$ has an eigenvalue

$$eig(\ =\ )_q = \mathrm{diag}(\ =\ )_q + \sum_{j=1}^{j\ \ j} q^{j\ =\ j\ \ j}[j + j\ \ j]_q\ 2\ Z\ _0[q]:$$

All eigenvalues for $R2R_n^{(q)}$ are obtained in this way.

[a] Special cases: 6= (k), 6= (1`) for ` odd

Recall our questions:

Question

- ☑ How to define $\mathrm{R2R}_n^{(q)}$?
- ☐ What are the eigenvalues of $\mathrm{R2R}_n^{(q)}$?
- ☐ What is the $GL_n(\mathbb{F}_q)$ representation structure on $\mathrm{R2R}_n^{(q)}$ eigenspaces?

Upshot: Big Conjecture would answer all of our questions!

For experts: it implies that for any eigenvalue $\ (q)$ of $\mathrm{R2R}_n^{(q)}$, the $\ (q)$-eigenspace has q-Frobenius characteristic

$$\sum_{\substack{(\ = \ ) \\ eig(\ = \ )_q = \ (q)}} d \quad s \;;$$

where d is the number of desarrangement tableaux of shape .

Algebra of words

Our proof strategy involves the Algebra of words.

[redacted]

Let $M^{\langle n \rangle}$ be the $\mathbb{C}$-vector space generated by words of length n in the alphabet $\{1, 2, \ldots, n\}$ (repeated letters allowed!).

There is a right action on $M^{\langle n \rangle}$:

- by S_n, which acts by position, e.g. $(112)s_2 = (121)$
- More surprisingly, by $H_n(q)$!

Action of $H_n(q)$ on the algebra of words

Action: There is a right action by $H_n(q)$ on $M^{\langle n\rangle}$ by

$$w_1 w_2 \cdots w_n \quad T_i = \begin{cases} q(w_1 w_2 \cdots w_n) & w_i = w_{i+1}, \\ (w_1 \cdots w_{i+1} w_i \cdots w_n) & w_i < w_{i+1}, \\ q(w_1 \cdots w_{i+1} w_i \cdots w_n) + (q-1)(w_1 \cdots w_i w_{i+1} \cdots w_n) & w_i > w_{i+1}. \end{cases}$$

Example:

$$\begin{aligned} (\textcolor{red}{11}2) \quad T_1 &= q(112) \\ (1\textcolor{red}{12}) \quad T_2 &= (121) \\ (\textcolor{red}{21}1) \quad T_1 &= q(121) + (q-1)(211) \end{aligned}$$

Upshot: $\mathrm{R2R}_n^{(q)}$ acts on $M^{\langle n\rangle}$

Proof strategy

Dieker-Saliola use properties of $M^{\langle n\rangle}$ to prove their recursions:

Theorem (Dieker-Saliola, 2018)

From any $w \in M^{\langle n\rangle}$:

$$\mathrm{R2R}_{n+1} \circ sh_i(w) = sh_i \circ \mathrm{R2R}_n(w) + (n+1)sh_i(w) + \sum_{1 \le j \le n} sh_j \circ \Theta_{j,i}(w).$$

$$sh_i(w) := \sum_{0 \le j \le n} w_1 \cdots w_j \; i \; w_{j+1} \cdots w_n.$$

$$\Theta_{j,i}(w) := \sum_{\substack{1 \le k \le n \\ w_k = j}} w_1 \cdots w_{k-1} \; i \; w_{k+1} \cdots w_n.$$

Example:

$$sh_1(123) = (1123) + (1123) + (1213) + (1231).$$

$$\Theta_{2,1}(122) = (112) + (121).$$

Proof strategy

We came up with definition for $\mathrm{sh}_i^{(q)}$ and $\Theta_{j;i}^{(q)}$.

Conjecture (C-L-AF)

For $i = 1$, and any $w \in M^{\langle n \rangle}$:

$$\mathrm{R2R}_{n+1}^{(q)}\mathrm{sh}_1^{(q)} = q\left(\mathrm{sh}_1^{(q)}\mathrm{R2R}_n^{(q)} + [n+1]_q\mathrm{sh}_1^{(q)} + \sum_{j=1}^{n} q^{n+2-j}\,\mathrm{sh}_j^{(q)}\Theta_{j;1}^{(q)}\right).$$

Big idea: Following Dieker-Saliola, we hope to

- Figure out the recursion for $i > 1$
- Prove recursion in the algebra of words
- Project onto irreducible representations to prove eigenvalue recursion

Recap

Questions:

How to define $\mathrm{R2R}_n^{(q)}$?

What are its eigenvalues and eigenspaces?

Our Results:

$\mathrm{R2R}_n^{(q)}$ definition in the Hecke algebra

Gave us more data and allowed us to work with words instead of flags.

Full conjecture for eigenvalues and eigenspaces

Proof Strategy:

$$\begin{aligned} \text{Conj. recurrence on words} &\implies \text{Conj. eigenvalue recurrence} \\ &\implies \text{Big conjecture} \end{aligned}$$