A q-analogue of random-to-random shuffling

Ilani Axelrod-Freed, Judy Chiang, Veronica Lang

August 5, 2022

Abstract

We introduce and study the q-analog of random-to-random shuffling $\mathsf{R2R}_n^{(q)}$ on the space of complete flags $\mathbb{C}[G/B]$, its eigenvalues, and the $GL_n(\mathbb{F}_q)$-structure on its eigenspaces. This generalizes work by Dieker-Saliola, who determined the eigenvalues and eigenspace structure of random-to-random shuffling [5] on permutations.

We use the bimodule decomposition of $\mathbb{C}[G/B]$ by $GL_n(\mathbb{F}_q)$ and the Hecke algebra $\mathcal{H}_n(q)$ to give a reduced expression for $\mathsf{R2R}_n^{(q)}$ in $\mathcal{H}_n(q)$. We then conjecture an explicit formula for its eigenvalues as a polynomial in q with non-negative, integer coefficients; our conjecture recovers Dieker-Saliola's formula when $q = 1$ and implies the $GL_n(\mathbb{F}_q)$-representation structure on the eigenspaces of $\mathsf{R2R}_n^{(q)}$. We propose a proof strategy, which involves the action of the Hecke algebra on the algebra of words.

Contents

1 Introduction

Interested in the spectrum of the Tsetlin library (also called move-to-front scheme), Bidigare, Hanlon, and Rockmore (BHR) [1] studied random walks on hyperplane arrangements in $\mathbb{R}^n$. Examples of these walks include riffle shuffles and random-to-top shuffles. Inspired by their work, Reyes [12] refined the methods of BHR and of Brown and Diaconis [4] to bound the mixing rates of subsets of cards. His refinement also determines how the eigenvalues of symmetric BHR shuffles split among the irreducible representations of S_n.

Reyes additionally studied the random-to-random shuffle as the multiplicative symmetrization of the random-to-top shuffle. A multiplicative symmetrization is obtained by composing an operator with its transpose. In this case, the random-to-top and top-to-random shuffling operators are transposes, and composing them yields the random-to-random shuffle. The random-to-random operator shuffles a deck of n cards by sequentially moving a card from random position to the top (random-to-top) and the top to a random position (top-to-random). The question has a broad application in the theory of card shuffling, random walks on groups and semigroups, and representation theory.

The eigenvalues of shuffling operators are of interest as they help bound the number of shuffles required for a deck to be well-mixed. Phatarfod [9] was the first to discover the eigenvalues of the random-to-top and top-to-random operators in 1991. But the eigenvalues of random-to-random were not solved until 2018, by Dieker and Saliola [5].

Furthermore, the eigenspaces of these shuffling operators are all symmetric group representations, and it is interesting to see what their structure is. Reyes [12] and Reiner-Wachs (in unpublished work [11]) constructed an S_n-representation structure on eigenspaces of random-to-top shuffling. Through their work in finding the eigenvalues, Dieker-Saliola [5] presented a S_n-representation structure on eigenspaces of the random-to-random shuffling operator.

More genrerally, Reiner-Saliola-Welker [10] studied symmetrized BHR shuffles, and defined a family of symmetrized operators $\{\nu_{(k,1^{n-k})}\}$ related to random-to-random, which they showed pairwise commute. Lafreniére [7] found formulas for the eigenvalues of these operators in her PhD thesis which further showed the eigenvalues were integers, and gave an alternate proof that the operators pairwise commute. For our interests, the random-to-random shuffling operator is the special case of ν_k when $k = n - 1$ [12].

In Brauner-Commins-Reiner's (BCR) study of the invariant theory of the free left regular band [2], a natural q-analogue of the random-to-top operator (first defined by Brown in [3]) appears. The eigenvalues of this operator are implicit in Brown's work, and Brauner, Commins, and Reiner decompose the eigenspaces as $GL_n(\mathbb{F}_q)$-modules [2].

Given the complete understanding of the q-analog of the random-to-top operator, a natural next step is to study the q-analog of the random-to- random shuffling operator, and more generally a q-analogue of the ν_k family. Our report focuses on the following questions.

Question 1. *What are the eigenvalues of* $\mathsf{R2R}_n^{(q)}$*, and how can we interpret them combinatorially?*

The answer is conjectured in section 4.

Question 2. *What is the* $GL_n(\mathbb{F}_q)$ *representation structure on eigenspaces of* $\mathsf{R2R}_n^{(q)}$*?*

The conjectural answer to Question 2 comes along with our conjecture for question 1, explained in section 4.

Question 3. *Given that the* ν_k *operators pairwise commute, do the* $\nu_k^{(q)}$ *operators parwise commute? If so, how can we understand their commutativity conceptually?*

As an overview, in section 2, we review the definitions of the random-to-random operator on words and other useful definitions in [5] used to prove the eigenspace structures for the $q = 1$ case. Then, we define the q-analogs of the random-to-top, top-to-random, and random-to-random shuffling operators. Also, we review the definition of the defined ν_k family.

For the $q = 1$ case, Dieker-Saliola [5] consider the right and left action by position and by value respectively of $\mathbb{C}[S_n]$ by S_n to find eigenspaces for $R2R_n$. The analogous left action of $\mathbb{C}[G\backslash B]$ in $GL_n(\mathbb{F}_q)$ is nontrivial to understand, and the Hecke algebra $\mathcal{H}_n(q)$ of type A_{n-1} acts naturally as the q-deformation of $\mathbb{C}[S_n]$ as a right action. In section 3, we redefine the shuffling operators in Hecke algebra.

In section 4, we propose a conjectural q-analog of the recursion developed by Dieker and Saliola to relate eigenvalues of $\mathsf{R2R}_n^{(q)}$ to $\mathsf{R2R}_{n+1}^{(q)}$ given in Equation (7). From this conjectural recursion, we then derive an explicit conjectural formula for the eigenvalues of $\mathsf{R2R}_n^{(q)}$ based on the recursion which reduced to the case in Dieker and Saliola's [5] when $q = 1$. From our formula, we obtain a q-analog of the diagonal number of a horizontal strip, a statistic appearing in the eigenvalue formulas for $\mathsf{R2R_n^{(q)}}$ by Dieker-Saliola [5]). To prove the eigenvalue recursion, we also attempt to find a recursion in the algebra of words as in [5] and re-define other operators introduced in 2 based on Sage computational data.

2 Definitions and background

2.1 Shuffling operators

Definition (Shuffling operators for $q = 1$)**.** Consider a word $w = w_1 w_2 \dots w_n$, and let $\hat{w_i}$ denote the removal of the letter w_i from a word. For example, $w_1 \hat{w_2} w_3 = w_1 w_3$. Then

- $\mathsf{T2R}_n(w_1 w_2 \dots w_n) := \frac{1}{n} \sum_{i=1}^{n} w_2 w_3 \dots w_i w_1 w_{i+1} \dots w_n$,
- $\mathsf{R2T}_n(w_1 w_2 \dots w_n) := \frac{1}{n} \sum_{i=1}^{n} w_i w_1 \dots w_{i-1} \hat{w_i} w_{i+1} \dots w_n$,
- $\mathsf{R2R}_n(w_1 w_2 \dots w_n) := \mathsf{T2R}_n \circ \mathsf{R2T}_n(w_1 w_2 \dots w_n)$.

The eigenvalues of $\mathsf{R2R}_n$ are indexed by horizontal strips of Young diagrams, which we can understand using the definitions below.

Definition. Consider a partition λ, let (i, j) refer to the cell in row i and column j of the Young diagram of such partition λ. The **diagonal index** of (i, j) is $(j - i)$.

Definition. If λ and ν are two partitions such that the diagram of λ contains the diagram of ν, then the **skew partition** λ/ν contains the cells of λ that do not belong to ν. Specifically, a **horizontal strip** is a skew partition containing no more than one cell in each column.

Definition. $\mathsf{diag}(\lambda/\nu)$ denotes the sum of the diagonal indices of the cells in λ/ν:

$$\mathsf{diag}(\lambda/\nu) = \sum_{\text{cells } (i,j) \text{ in } \lambda/\nu} (j - i)$$

Theorem 2.1 computes the eigenvalue of $R2R_n$ when $q = 1$ [5].

Theorem 2.1 (Dieker and Saliola, [5])**.** *Every eigenvalue of the random-to-random shuffle acting on words of evaluation $\nu \vdash n$ is of the form*

$$\frac{1}{n^2} \cdot \mathrm{eig}(\lambda/\mu),$$

where

- *λ is a partition of n and λ/μ is a horizontal strip with $\lambda \trianglerighteq \nu$ such that $\mu \neq (n)$ and $\mu \neq (1, 1, \dots, 1)$ with an odd number of 1s;*
- eig *is the following combinatorial statistic defined on skew partitions:*

$$\mathrm{eig}(\lambda/\mu) = \binom{|\lambda| + 1}{2} - \binom{|\mu| + 1}{2} + \mathrm{diag}(\lambda/\mu). \tag{1}$$

Our main goal is to find an analogous description of the eigenvalues of $\mathsf{R2R}_n^{(q)}$. We provide a conjectured analog in section 4.

2.2 The ν_k family

We introduce the ν_k family arising from two sources: Proposition 112 in Lafreniére [7] and in Reiner-Saliola-Welker [10].

Definition.

$$\nu_k = \sum_{1\le a_1\le\cdots\le a_k\le n} \frac{1}{\prod_{j=1}^{n} \#\{\ell\in[k] : a_\ell = j\}!}\ sh_{a_1}\circ\cdots\circ sh_{a_k}\circ\partial a_1\circ\cdots\circ\partial a_k$$

Definition. $\nu_k \in \mathbb{C}[S_n]$ can also be defined as a right operator such that

$$\nu_k(\sigma) = \sum_{\tau\in S_n} \mathrm{noninv}_{n-k}(\tau)\cdot\sigma\tau,$$

where $\mathrm{noninv}_{n-k}(\tau)$ is the number of length $n-k$ increasing subsequences of $\tau = (\tau_1,\ldots,\tau_n)$ written in one-line notation.

2.3 The q-analog of shuffling operators in flags

Our shuffling operators so far can be expressed in terms of elements of the symmetric group acting on $\mathbb{C}[S_n]$, the vector space of words. We now want to find a q-analogue of both the words and the shuffling action on them.

Definition (complete flag of $\mathcal{F}$)**.** A **complete flag** in a vector space V is a sequence of $dim(V) = n$ nested linear subspaces

$$U_1 \subset U_2 \subset\cdots\subset U_{n-1}\subset U_n = V,$$

where U_i has dimension i. Let $\mathcal{F}(V)$ be the space of complete flags in V.

Remark. Another way to write the complete flag is $\mathbb{C}[G/B]$, where G is a group and B is the subgroup of upper triangular invertible matrices).

For our q-anologue of card shuffling, we will now be acting on linear combinations of flags in $\mathbb{C}[\mathcal{F}(\mathbb{F}_q^n)]$. Note that when $q = 1$, then $\mathbb{C}[\mathcal{F}(\mathbb{F}_q^n)] = \mathbb{C}[S_n]$, recovering our original space of words.

Definition (q-analog of shuffling operators)**.** Consider a complete flag $F = (F_1 \subset F_2\cdots\subset F_n = \mathbb{F}_q^n)$ with fixed basis vectors $f_1,\ldots,f_n\in\mathbb{F}_q^n$ such that $F_i = \langle f_1,\ldots,f_i\rangle$. We then define

- $\mathsf{R2T}_n^{(q)}(F)$=

$$:= \sum_{i=1}^{n}\sum_{L\in F_i\setminus F_{i-1}} \langle L\rangle\ \subset\ \langle L, F_1\rangle\ \subset\cdots\subset\ (\langle L, F_{i-1}\rangle = F_i)\ \subset\ F_{i+1}\ \subset\cdots\subset\ F_n.$$

- $\mathsf{T2R}_\mathsf{n}^{(\mathsf{q})}(F_1\ \subset\cdots\subset\ F_n)$

$$= \sum_{j=0}^{n-1}\sum_{\substack{a_1,\cdots,a_j\\ \in\mathbb{F}_q}} \langle f_2 + a_1 f_1\rangle\ \subset\cdots\subset\ \langle f_2+a_1f_1,\cdots,f_{j+1}+a_jf_1\rangle\ \subset\ F_{j+1}\ \subset\cdots\subset\ F_n$$

- $\mathsf{R2T}_{\mathsf{n}}^{(\mathsf{q})}(F_1 \subset \cdots \subset F_n) = \mathsf{T2R}_n^{(q)} \circ \mathsf{R2T}_n^{(q)}(F_1 \subset \cdots \subset F_n)$

$$= \sum_{\substack{0\leq i,j \\ \leq n-1}} \sum_{\substack{L\in F_i \\ \dim(L)=1 \\ L\notin F_{i-1}}} \sum_{\substack{a_j,\cdots,a_j \\ \in \mathbb{F}_q}} \langle f_1 + a_1 f_i \rangle \subset \cdots \subset \langle f_1 + a_1 f_i, \cdots, (\widehat{f_i + a_i} f_i), \cdots, f_j + a_j f_i \rangle$$
$$\subset \langle f_1, \cdots, f_j, f_i \rangle \subset \cdots \subset \langle f_1, \cdots, \widehat{f_i}, \cdots, f_n, f_i \rangle = F_n.$$

Remark. For $\mathsf{R2T}_{\mathsf{n}}^{(\mathsf{q})}$, it is sufficient to consider one arbitrary flag $\mathcal{F}$ to specify the definition. The reason is as follows: without loss of generality, consider $\mathsf{R2T}_{\mathsf{n}}^{(\mathsf{q})}$ acting on a flag $\mathcal{F}_1 \neq \mathcal{F}_0$, then there exists an invertible matrix $g \in GL_n(\mathbb{F}_q)$ such that $g\mathcal{F}_0 = \mathcal{F}_1$. Since $\mathsf{R2T}_{\mathsf{n}}^{(\mathsf{q})}$ commutes with matrices in $GL_n(\mathbb{F}_q)$, we have

$$\mathsf{R2T}_{\mathsf{n}}^{(\mathsf{q})}\mathcal{F}_1 = \mathsf{R2T}_{\mathsf{n}}^{(\mathsf{q})}(g\mathcal{F}_0) = g(\mathsf{R2T}_{\mathsf{n}}^{(\mathsf{q})}(\mathcal{F}_0)),$$

showing the action of $\mathsf{R2T}_n^{(q)}$ on all other flags in the flag space is specified.

Remark. For $\mathsf{T2R}_n^{(q)}$, we add the line f_1 back in at the $(j+1)^{\text{th}}$ subspace. Before that, we take a summation over all subspaces of F_2 which don't include f_1, then all subspaces of F_3 which don't include f_1 (but do include F_2), and so on.

Proposition 2.2. *The operators* $\mathsf{R2T}_n^{(q)}$ *and* $\mathsf{T2R}_n^{(q)}$ *are each other's transposes.*

Proof. Let $\mathsf{R2T}_{n(F,E)}^{(q)}$ be the coefficient of E in $\mathsf{R2T}_{n(F)}^{(q)}$. Then, $\mathsf{R2T}_{n(F,E)}^{(q)}$ counts the number of ways to transition from flag F to flag E with random-to-top operator, as an entry in the matrix $\mathsf{R2T}_n^{(q)}$.

Note that its entry can only equal zero or one from the argument below. Let

$$F = (F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{F}_n^q) \text{ and } E = (E_1 \subset E_2 \subset \cdots \subset E_n = \mathbb{F}_n^q)$$

If F_1 (the first entry in F) is not contained in E_2 (the second entry in E), then it is impossible to obtain E by bringing a random line in $\mathbb{F}_q^n$ to the top of F (i.e. cannot apply random-to-top). Otherwise, there is a unique way to obtain E through applying $\mathsf{R2T}_n^{(q)}$ to F, by bringing the line E_1 to the top of F. Since we define $\mathsf{T2R}_n^{(q)}{}_{(F,E)}$ analogously, the same reason applies showing that $\mathsf{T2R}_n^{(q)}{}_{(F,E)}$ must equal zero or one.

To show $\mathsf{R2T}_n^{(q)}$ and $\mathsf{T2R}_n^{(q)}$ are transposes, it suffices to prove

$$\mathsf{R2T}_{n(F,E)}^{(q)} = \mathsf{T2R}_{n(F,E)}^{(q)}.$$

Suppose $\mathsf{R2T}_n^{(q)}{}_{(F,E)} = 1$. Then E is of the form

$$E = L \subset \langle f_1, L \rangle \subset \cdots \subset \langle f_1, \ldots, f_{i-1}, L \rangle = \langle f_1, \ldots, f_i \rangle \subset \cdots \subset \langle f_1, \ldots, f_n \rangle = \mathbb{F}_q^n,$$

for some $i \in \{1, \ldots, n\}$ and line $L \in F_i$ where $L \notin F_{i-1}$. Therefore, $\mathsf{T2R}^{(q)}_{n(E)}$ contains the term

$$\begin{aligned}\langle f_1 + 0 \cdot L\rangle \subset \langle f_1 + 0 \cdot L, f_2 + 0 \cdot L\rangle \subset \cdots \subset \langle f_1 + 0 \cdot L, \ldots, f_{i-1} + 0 \cdot L\rangle \\ \subset \langle f_1, \ldots, f_{i-1}, L\rangle = \langle f_1, \ldots, f_i\rangle \subset \cdots \subset \langle f_1, \ldots, f_n\rangle = \mathbb{F}_q^n\end{aligned}$$

given by $j = i - 1$ and $a_1 = a_2 = \cdots = a_j = 0$. Hence, $\mathsf{R2T}^{(q)}_{n(F,E)} = 1$ implies $\mathsf{T2R}^{(q)}_{n(F,E)} = 1$.

Now suppose $\mathsf{T2R}^{(q)}_{n(F,E)} = 1$. Then E is of the form

$$\langle f_2 + a_1 f_1\rangle \subset \cdots \subset \langle f_2 + a_1 f_1, \ldots f_{j+1} + a_j f_1\rangle \subset \langle f_1, \ldots, f_{j+1}\rangle \subset \cdots \subset \langle f_1, \ldots, f_n\rangle = \mathbb{F}_q^n,$$

for some $j \in \{0, \ldots, n-1\}$ and constants $a_1, \ldots, a_j$. In this case, $\mathsf{R2T}^{(q)}_{n(E)}$ contains the term

$$\langle f_1\rangle \subset \langle f_2 + a_1 f_1, f_1\rangle = \langle f_1, f_2\rangle \subset \cdots \subset \langle f_1, \ldots, f_n\rangle = \mathbb{F}_q^n$$

given by $L = f_1$ and $i = j$. Therefore, $\mathsf{T2R}^{(q)}_{n(F,E)} = 1$ implies $\mathsf{R2T}^{(q)}_{n(F,E)} = 1$. Since $\mathsf{T2R}^{(q)}_{n(F,E)} = 1$ if and only if $\mathsf{R2T}^{(q)}_{n(F,E)} = 1$, we have

$$(\mathsf{T2R}^{(q)}_n)^T = \mathsf{R2T}^{(q)}_n$$

as desired. □

Remark. When $q = 1$, $(\mathsf{T2R}_n))^T = \mathsf{R2T}_n(w)$ in [5].

2.4 Flags as a quotient of $\mathbf{GL}_n(\mathbb{F}_q)$

To further describe the action of $\mathsf{R2T}^{(q)}_n, \mathsf{T2R}^{(q)}_n$ and $\mathsf{R2R}^{(q)}_n$ on flags, it will be useful to have a notion of flags written in terms of matrices, and the action of the operators written in terms of matrices as well.

In particular, we will work with $\mathrm{GL}_n(\mathbb{F}_q)$, the space of $n \times n$ invertible matrices with entries in $\mathbb{F}_q$.

We can represent flags with matrices in the following way: Let $f_1, \ldots, f_n \in \mathbb{F}_q^n$ be linearly independent vectors. Then the matrix $[f_1 \cdots f_n]$ represents the flag $F = \langle f_1\rangle \subset \langle f_1, f_2\rangle \subset \ldots \subset \langle f_1, f_2, \ldots, f_n\rangle = \mathbb{F}_q^n$. Note that two matrices M_1 and M_2 represent the same flag if and only if $M_1 = M_2 b$ for some invertible upper triangular matrix b.

Definition. The *Borel set* B is the group of $n \times n$ invertible upper triangular matrices.

Then

$$\mathcal{F}(\mathbb{F}_q^n) = \mathrm{GL}_n(\mathbb{F}_q)/B =: G/B.$$

We can now multiply a flag F by a matrix M simply by multiplying any matrix $[F]$ corresponding to the flag F by M. Importantly, matrix multiplication is well defined on flags in this way. In other words, given any matrix M as well as two matrices $[F]$ and $[F']$ which have the same image under quotienting by B (i.e. they correspond to the same flag), $M[F]$ and $M[F']$ have the same image under the quotient by B as well.

Definition. Let *base flag*, F_0 be the flag corresponding to matrices in the kernal of the quotient map by B.

Two useful ways we write F_0 in terms of matrices are as the identity matrix, or as

$$e_B = \frac{1}{|B|} \sum_{b \in B} b$$

the normalized sum over all possible ways to write F_0 as a matrix in B.

When $q = 1$, shuffling operators can be written as elements of $\mathbb{C}[S_n]$ acting on $\mathbb{C}[S_n]$ on the left by permuting the values of the letters, or acting on the right by permuting the positions of letters. It now becomes natural to ask in what algebra the q-analogs of shuffling operators live. We know we can act on flags on the left with elements of $\mathbb{C}\mathrm{GL}_n(\mathbb{F}_q)$. As shown in the figure below, these operators also exist in the Hecke algebra, which acts on the right on $\mathbb{C}[G/B]$. We introduce the Hecke algebra in the following section.

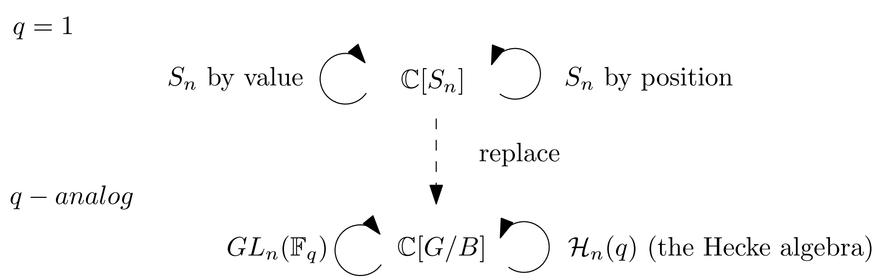


Figure 1: Motivation of using the Hecke algebra

3 Hecke algebra

Since action of $\mathsf{T2R}_n^{(q)}$, $\mathsf{R2T}_n^{(q)}$ and $\mathsf{R2R}_n^{(q)}$ commute with multiplication by matrices, they can be expressed in terms of elements of the Hecke algebra. Below, we define the Hecke algebra and express $\mathsf{R2R}_n^{(q)}$ in reduced form in terms of Hecke algebra generators. The definitions in this section give well-known properties of the Hecke algebra, and these definitions along with more information are given in [8].

3.1 Intro to Hecke Algebras

We begin by defining the Hecke algebra and will then define the action of the Hecke algebra on the $\mathbb{C}$-vector space generated over complete flags, $\mathbb{C}[G/B]$.

Definition (Hecke algebra $\mathcal{H}_n(q)$)**.** The Hecke algebra ($\mathcal{H}_n(q)$) is the associative $\mathbb{C}$-algebra with generators $T_{s_1}, \ldots, T_{s_{n-1}}$ satisfying

1. $(T_{s_i} - q)(T_{s_i} + 1) = 0$
2. $T_{s_i}T_{s_j} = T_{s_j}T_{s_i}$ for $|i - j| > 2$
3. $T_{s_i}T_{s_{i+1}}T_{s_i} = T_{s_{i+1}}T_{s_i}T_{s_{i+1}}$

Remark. When $q = 1$, then $\mathcal{H}_n(q) = \mathbb{C}[S_n]$, the associative $\mathbb{C}$-algebra with generators $s_1, \ldots, s_{n-1}$ satisfying

1. $s_i^2 - 1 = (s_i - 1)(s_i + 1) = 0$
2. $s_is_j = s_js_i$ for $|i - j| > 2$
3. $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$

Remark. For simplification, we usually replace T_{s_i} with T_i.

Definition. Let $\omega = s_{i_1} \ldots s_{i_k}$ be a reduced expression for the permutation $\omega \in S_n$. Then

$$T_\omega := T_{i_1} \ldots T_{i_k}.$$

Based on the above definitions, one can verify that given a reduced expression ω and transposition s_i,

$$T_\omega T_{s_i} = \begin{cases} T_{\omega s_i}, \text{ if } l(\omega s_i) = l(\omega) + 1 \\ qT_{\omega s_i} + (q-1)T_\omega, \text{ if } l(\omega s_i) = l(\omega) - 1, \end{cases} \tag{2}$$

where $l(\omega)$ is the number of inversions in ω, also called the Coxeter group length of ω. We will use this property later when acquiring a reduced expression for $\mathsf{R2R}_n^{(q)}$.

Proposition 3.1. *The elements $\{T_\omega : \omega \in S_n\}$ form a basis for $\mathcal{H}_n^{(q)}$ as a $\mathbb{C}$-vector space.*

More concretely, we can think of $\mathcal{H}_n(q)$ as the group of linear transformations on $\mathbb{C}[G/B]$, the vector space of complete flags, such that for all linear transformations $T \in \mathcal{H}_n(q)$, we have that $mT(F) = T(mF)$ for all F (a linear combination of flags) and matrix $m \in GL_n(\mathbb{F}_q)$.

We define the action of the T_ω on flags as follows.

Definition. For a reduced word ω, let

$$\phi(T_\omega) := \frac{1}{|B|} \sum_{x \in B\omega B} x.$$

We define the right action of T_ω on a flag $F \in \mathcal{C}[G/B]$ to be

$$T_\omega(F) := F \left(\frac{1}{|B|} \sum_{x \in B\omega B} x \right) = F \cdot \phi(T_\omega)$$

in $\mathbb{C}[G/B]$.

Importantly, T is well defined on cosets of B. That is to say, if we have two matrices $F, F' \in \mathrm{GL}_n(\mathbb{F}_q)$ corresponding to the same flag, then while it may be that $T_\omega(F) \neq T_\omega(F')$ in $\mathrm{GL}_n(\mathbb{F}_q)$, it will always be that $T_\omega(F)$ and $T_\omega(F')$ have the same image under the quotient by B, meaning they again correspond to the same flag.

For every flag $F_1 \in G/B$, there exists a matrix $g \in \mathrm{GL}_n(\mathbb{F}_q)$ such that $gF_0 = F_1$. Then shows

$$T_\omega(F_1) = T_\omega(gF_0) = g(T_\omega(F_0)),$$

which implies that the action of T_ω on $\mathbb{C}[G/B]$ is fully determined by where it sends F_0.

3.2 Shuffling operators in $\mathcal{H}_n(q)$

For our purposes, it will be useful to understand $T_\omega(F_0)$, the linear combination of flags which our basis flag F_0 is sent to under the action of T_ω, and which uniquely determines the map T_ω.

Proposition 3.2. *Let $\mathcal{F}_\omega$ be the set of flags which can be written in the form $b\omega$ for some $b \in B$. Then*

$$\sum_{x \in B\omega B} x = \sum_{\substack{F \in \mathcal{F}_\omega \\ b \in B}} [F]b,$$

where $[F]$ is some arbitrary way to represent the flag F in the matrix form.

Proof. Let F be a flag with matrix form $[F]$. Since right multiplication by invertible upper triangular matrices does not change the flag, for all $b \in B$, $[F]b \neq [F]$ is also a matrix form of the flag F. Therefore, $\frac{1}{|B|} \sum_{b \in B} [F]b$ is simply a linear combination of flags, representing the same flag as F. By the same reasoning, the number of distinct flags from $b_1\omega b_2$ is the same as the one from $b_1\omega$. Hence, $B\omega B$ contains every matrix way to write a flag that can be written in the form $b_1\omega$ without repeats. So, every $b_1\omega b_2$ for some $b_1, b_2 \in B$ can always be written as $b_1\omega$. $\square$

Proposition 3.3. *Let $\mathcal{F}_\omega$ be the set of flags which can be written in the form $b\omega$ for some $b \in B$. Then,*

$$T_\omega(F_0) = \sum_{F \in \mathcal{F}_\omega} F,$$

Proof. The identity matrix $I \in \mathrm{GL}_n(\mathbb{F}_q)$ corresponds to the base flag F_0. Then, (in the space $\mathbb{C}[G/B]$),

$$T_\omega(F_0) = I\left(\frac{1}{|B|}\sum_{x \in B\omega B} x\right) = \frac{1}{|B|}\sum_{x \in B\omega B} x.$$

By Proposition 3.2,

$$\frac{1}{|B|}\sum_{x \in B\omega B} x = \frac{1}{|B|}\sum_{\substack{F \in \mathcal{F}_\omega \\ b \in B}} [F]b = \sum_{F \in \mathcal{F}_\omega} [F]e_B = \sum_{F \in \mathcal{F}_\omega} F.$$

□

Proposition 3.4. *For all $1 \leq i \leq n$, we have $T_i = (T_i)^T$.*

Proof. Let $(T_i)_{F_0,F}$ be the entry in the adjacency matrix of T_i that corresponds to a transition from F_0 to F. In other words, $(T_i)_{F_0,F}$ is the coefficient of F in the sum $T_i(F_0)$. We claim that $(T_i)_{F_0,F}$ always equals 0 or 1. In both of these cases, we will show that $(T_i)_{F_0,F} = (T_i)_{F,F_0}$.

First, suppose $(T_i)_{F_0,F} = 1$. Then by Proposition 3.2, F can be written in the form bs_i for some $b \in B$. Then

$$T_i(F) = (bs_i)\frac{1}{|B|}\sum_{x \in Bs_iB} x$$

This sum includes the terms

$$\frac{1}{|B|}\sum_{b' \in B} s_i b'$$

and therefore $T_i(F)$ has the term

$$bs_i\frac{1}{|B|}\sum_{b' \in B} s_i b' = b(s_i)^2 e_B = e_B$$

Therefore, $(T_i)_{F,F_0} = 1$.

Now suppose $(T_i)_{F_0,F} = 0$. Then $F = b\omega$ for some $b \in B$ and $\omega \neq s_i$. And

$$T_i(F) = b\omega\sum_{x \in Bs_iB} x$$

Every term in this sum is of the form $b\omega b_1 s_i b_2$ where $b, b_1, b_2 \in B$. We claim that $b\omega b_1 s_i b_2 \notin B$ for any choice of b, b_1, b_2. Assume to the contrary that $b\omega b_1 s_i b_2 = b'$ for some $b' \in B$. Then

$$b\omega b_1 = (b' b_2^{-1}) s_i$$

and note that $b' b_2^{-1} \in B$. However, for $\omega \neq s_i$ we can never have $b\omega b_1 = b_3 s_i$ for any $b, b_1, b_3 \in B$. This gives a contradiction, and we conclude that $(T_\omega)_{F,F_0} = 0$.

$\square$

Based on Propositions 3.3 and 3.4, we can redefine the shuffling operators in Hecke algebra using the T_ω operators.

Theorem 1. *Given a word w,*

- $\mathsf{R2T}_n^{(q)} = \sum_{i=0}^{n-1} T_{i\ldots 21} = \sum_{i=0}^{n-1} T_i \ldots T_2 T_1$
- $\mathsf{T2R}_n^{(q)} = \sum_{j=0}^{n-1} T_{12\ldots j} = \sum_{j=0}^{n-1} T_1 T_2 \ldots T_j$
- $\mathsf{R2R}_n^{(q)} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} T_i \ldots T_1 T_1 \ldots T_j.$

Proof. Let $F_0 = (F_1 \subset F_2 \subset \cdots F_n)$ be the base flag, written by the identity matrix, and let $\omega_i = s_i \cdots s_2 s_1$. Since $\mathsf{R2T}_n^{(q)}$ and elements of the Hecke algebra commute with matrix multiplication, it is enough to show that

$$\mathsf{R2T}_n^{(q)} F_0 = \sum_{i=0}^{n-1} T_{\omega_i} F_0.$$

In particular we will show that $T_{\omega_i} F_0$ is equal to

$$\sum_{\substack{f_i \in F_i \\ f_i \notin F_{i-1}}} \langle f_i \rangle \ \subset\ \langle f_i, F_1 \rangle \ \subset \cdots \subset\ (\langle f_i, F_{i-1} \rangle = F_i) \ \subset\ F_{i+1} \ \subset \cdots \subset\ F_n,$$

where the i^{th} term of the summation for $\mathsf{R2T}_n^{(q)}$ given in Definition 2.3.

Proposition 3.3 tells us the action of T_{ω_i} on F_0. To understand this action, we must first understand the set $\mathcal{F}_{\omega_i}$. Flags $b\omega_i$ can be written in the form

$$b\omega_i = \begin{pmatrix} * & \circ & \circ & \cdots & \circ \\ & * & \circ & \cdots & \circ \\ & & * & \cdots & \circ \\ & & & \ddots & \vdots \\ & & & & * \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ 0 & & 1 & 0 & & \\ 1 & & 0 & 0 & & \\ 0 & & 0 & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} = \begin{pmatrix} \circ & * & \circ & \circ & \circ & & \circ \\ \circ & & \ddots & \circ & \circ & & \circ \\ \circ & & & * & \circ & & \circ \\ * & & & & \circ & & \circ \\ & & & & * & & \circ \\ & & & & & \ddots & \circ \\ & & & & & & * \end{pmatrix}.$$

Here, $\circ$ represents any entry, $*$ represents a nonzero entry, and empty spaces should have 0's. Picking a representative matrix for each equivalence class of flags, we find that

$$\mathcal{F}_\omega = \left\{ \begin{pmatrix} a_1 & 1 & & & & & \\ & & \ddots & & & & \\ a_{i-2} & & & & & & \\ a_{i-1} & & & 1 & 0 & & \\ 1 & & & 0 & 0 & & \\ 0 & & & 0 & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \text{ s.t. } a_1, \dots, a_{i-1} \in \mathbb{F}_q \right\}.$$

Note that

$$\{(a_1, \dots, a_{i-1}, 1, 0, \dots 0)^T \mid a_1, \dots, a_{i-1} \in \mathbb{F}_q\} = \{f_i \in F_i \mid f \notin F_{i-1}\}$$

. Then, by Proposition 3.3,

$$T_{\omega_i} = \sum_{\substack{f_i \in F_i \\ f_i \notin F_{i-1}}} \begin{pmatrix} | & 1 & & & & & \\ | & & \ddots & & & & \\ | & & & 1 & 0 & & \\ f & & & 0 & 0 & & \\ | & & & 0 & 1 & & \\ | & & & & & \ddots & \\ | & & & & & & 1 \end{pmatrix} \tag{3}$$

$$= \sum_{\substack{f_i \in F_i \\ f_i \notin F_{i-1}}} \langle f_i \rangle \subset \langle f_i, F_1 \rangle \subset \cdots \subset (\langle f_i, F_{i-1} \rangle = F_i) \subset F_{i+1} \subset \cdots \subset F_n. \tag{4}$$

One can make an analogues proof to the above for $\mathsf{T2R}_n^{(q)}$. However we will instead prove that

$$\mathsf{T2R}_n^{(q)} = \sum_{j=0}^{n-1} T_1 T_2 \dots T_j$$

by showing that this is equal to $(\mathsf{R2T}_n^{(q)})^T$. This follows from Proposition 3.4:

$$\mathsf{T2R}_n^{(q)} = \left(\sum_{i=0}^{n-1} T_i \cdots T_1 \right)^T = \sum_{i=0}^{n-1} (T_1)^T \cdots (T_i)^T = \sum_{i=0}^{n-1} (T_1) \cdots (T_i).$$

Finally, the expression for $\mathsf{R2R}_n^{(q)}$ follows by composing $\mathsf{R2T}_n^{(q)}$ and $\mathsf{T2R}_n^{(q)}$. $\square$

3.3 Reduced form for $\mathsf{R2R}_n^{(q)}$

To understand how $\mathsf{R2R}_n^{(q)}$ acts on words, we reduce the double sum in $\mathsf{R2R}_n^{(q)}$ defined in Theorem 1.

Lemma 3.5. *For positive integers $n > 1$ and $p < n-1$,*

$$\begin{aligned}&\sum_{i=p-1}^{n-1}\sum_{j=p-1}^{n-1} T_i \dots T_p T_p \dots T_j \\ &= 1 + \sum_{i=p}^{n-1} T_{i\dots p} + \sum_{j=p}^{n-1} T_{p\dots j} + (q-1)\sum_{i=p}^{n-1}\sum_{j=p}^{n-1} T_{i\dots(p+1)p(p+1)\dots j} \\ &\quad + q\sum_{i=p}^{n-1}\sum_{j=p}^{n-1} T_i \dots T_{p+1}T_{p+1}\dots T_j.\end{aligned}$$

Proof. First, we can replace the summation's lower limits of the equation's left-hand-side to $i = p$ and $j = p$ and consider the terms with $i = p-1$ or $j = p-1$ separately, resulting in

$$\sum_{i=p-1}^{n-1}\sum_{j=p-1}^{n-1} T_i \dots T_p T_p \dots T_j = 1 + \sum_{i=p}^{n-1} T_i \dots T_p + \sum_{j=p}^{n-1} T_p \dots T_j + \sum_{i=p}^{n-1}\sum_{j=p}^{n-1} T_i \dots T_p T_p \dots T_j.$$

Consider the term $T_i \dots T_p$. Since

$$\ell(s_i s_{i-1} \dots s_p) = i - p + 1 = \ell(s_i) + \ell(s_{i-1}) + \dots + \ell(s_p),$$

it can be reduced to

$$T_i \dots T_p = T_{i\dots p}.$$

Similarly, $T_p \dots T_j = T_{p\dots j}$, and therefore the right-hand-side equation simplifies to

$$1 + \sum_{i=p}^{n-1} T_{i\dots p} + \sum_{j=p}^{n-1} T_{p\dots j} + \sum_{i=p}^{n-1}\sum_{j=p}^{n-1} T_i \dots T_p T_p \dots T_j.$$

For the final term, we use definition of T_i to simplify T_pT_p.

$$\begin{aligned}
&\sum_{i=p}^{n-1}\sum_{j=p}^{n-1} T_i \dots T_pT_p \dots T_j \\
&\quad = \sum_{i=p}^{n-1}\sum_{j=p}^{n-1} \left(q(T_i \dots T_{p+1}T_{p+1} \dots T_j) + (q-1)(T_i \dots T_{p+1}T_pT_{p+1} \dots T_j)\right) \\
&\quad = (q-1)\sum_{i=p}^{n-1}\sum_{j=p}^{n-1} T_i \dots T_{p+1}T_pT_{p+1} \dots T_j + q\sum_{i=p}^{n-1}\sum_{j=p}^{n-1} T_i \dots T_{p+1}T_{p+1} \dots T_j \\
&\quad = (q-1)\sum_{i=p}^{n-1}\sum_{j=p}^{n-1} T_{i\dots(p+1)p(p+1)\dots j} + q\sum_{i=p}^{n-1}\sum_{j=p}^{n-1} T_i \dots T_{p+1}T_{p+1} \dots T_j
\end{aligned}$$

In the last line, we used the fact that

$$\ell(s_i \dots s_{p+1}s_ps_{p+1} \dots s_j) = \ell(s_i) + \ell(s_{i-1}) + \cdots + \ell(s_{p+1}) + \ell(s_p) + \ell(s_{p+1}) + \cdots + \ell(s_j).$$

□

Theorem 3.6.

$$\begin{aligned}
\mathsf{R2R}_n^{(q)} = [n]_q + \sum_{p=1}^{n-1} q^{p-1}\Bigg((q+1)T_p + q\sum_{i=p+1}^{n-1} T_{i\dots p} + q\sum_{j=p+1}^{n-1} T_{p\dots j} \\
+ (q-1)\sum_{i=p+1}^{n-1}\sum_{j=p+1}^{n-1} T_{i\dots(p+1)p(p+1)\dots j}\Bigg).
\end{aligned}$$

Proof. For convenience, let

$$L_{(p,n-1)} = 1 + \sum_{i=p}^{n-1} T_{i\dots p} + \sum_{j=p}^{n-1} T_{p\dots j} + (q-1)\sum_{i=p}^{n-1}\sum_{j=p}^{n-1} T_{i\dots(p+1)p(p+1)\dots j}$$

Then Lemma 3.5 states that

$$\sum_{i=p-1}^{n-1}\sum_{j=p-1}^{n-1} T_i \dots T_pT_p \dots T_j = L_{(p,n-1)} + q\sum_{i=p}^{n-1}\sum_{j=p}^{n-1} T_i \dots T_{p+1}T_{p+1} \dots T_j.$$

To prove Theorem 3.6, we will show by induction that for any $1 \le m \le n-1$,

$$\mathsf{R2R}_n^{(q)} = \sum_{p=1}^{m} q^{p-1}\left(L_{(p,n-1)}\right) + q^m\sum_{i=m}^{n-1}\sum_{j=m}^{n-1} T_i \dots T_{m+1}T_{m+1} \dots T_j \tag{5}$$

When $m = n-1$, this expression will give

$$\mathsf{R2R}_n^{(q)} = \sum_{p=1}^{n-1} q^{p-1} L_{(p,n-1)} \tag{6}$$

which is an equivalent formula to the one provided in Theorem 3.6.

For our base case of $m = 1$, we apply Lemma 3.5 once to the definition of $\mathsf{R2R}_n^{(q)}$:

$$\begin{aligned}
\mathsf{R2R}_n^{(q)} &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} T_i \dots T_1 T_1 \dots T_j \\
&= L_{(1,n-1)} + q \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} T_i \dots T_2 T_2 \dots T_j \\
&= \sum_{p=1}^{1} q^{p-1} L_{(1,n-1)} + q^1 \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} T_i \dots T_2 T_2 \dots T_j.
\end{aligned}$$

Now suppose that the inductive hypothesis given in Equation (5) is true for some $1 \leq m < n-1$. We will prove the statement to be true for $m = m+1$. By applying Lemma 3.5 once to Equation (5), we see that

$$\begin{aligned}
\mathsf{R2R}_n^{(q)} &= \sum_{p=1}^{m} q^{p-1} L_{(p,n-1)} + q^m \sum_{i=m}^{n-1}\sum_{j=m}^{n-1} T_i \dots T_{m+1} T_{m+1} \dots T_j \\
&= \sum_{p=1}^{m} q^{p-1} L_{(p,n-1)} + q^m \left(L_{(m+1,n-1)} + q \sum_{i=m+1}^{n-1} \sum_{j=m+1}^{n-1} T_i \dots T_{m+2} T_{m+2} \dots T_j \right) \\
&= \sum_{p=1}^{m+1} q^{p-1} L_{(p,n-1)} + q^{m+1} \sum_{i=m+1}^{n-1} \sum_{j=m+1}^{n-1} T_i \dots T_{m+2} T_{m+2} \dots T_j
\end{aligned}$$

This completes the inductive step of the proof of Equation (5), which allows us to conclude that in the case of $m = n-1$,

$$\begin{aligned}
\mathsf{R2R}_n^{(q)} &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} T_i \dots T_1 T_1 \dots T_j \\
&= \sum_{p=1}^{n-1} q^{p-1} L_{(p,n-1)} \\
&= \sum_{p=1}^{n} q^{p-1} \left(1 + \sum_{i=p}^{n-1} T_{i \dots p} + \sum_{j=p}^{n-1} T_{p \dots j} + (q-1) \sum_{i=p}^{n-1}\sum_{j=p}^{n-1} T_{i \dots (p+1) p (p+1) \dots j} \right).
\end{aligned}$$

We can now pull out an $[n]_q$ and combine terms in the summations which have $i = p$ and/or $j = p$ to obtain the reduced form given in Theorem 3.6.

□

4 Conjectural eigenvalues and eigenspaces

4.1 Conjectural eigenvalue recursion

When $q = 1$, Dieker-Saliola [5] describe eigenvalues through the recursion:

$$\epsilon_{\lambda+e_i} = \epsilon_\lambda + (n+1) + (\lambda_i + 1) - i, \tag{7}$$

where ϵ_λ is one of the eigenvalues associated with a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ which partitions n and $\lambda+e_i$ is the partition $(\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i+1, \lambda_{i+1}, \ldots, \lambda_k)$.

Conjecture 4.1. *The q-analog of recursion* (7) *is*

$$\begin{aligned} \epsilon_{\lambda+e_i} &= q\epsilon_\lambda + [(n+1) + (\lambda_i + 1) - i]_q \\ &= q\epsilon_\lambda + [(n+1) + \mathrm{diag}(\text{box } i)]_q. \end{aligned} \tag{8}$$

Remark. This has been tested for all partitions with $n \leq 6$.

Lemma 4.2. *For all integers a, b, and i with $b \geq i$,*

$$q^i[a]_q + [b]_q = [a+i]_q + q^i[b-i]_q. \tag{9}$$

Proof. By definition of $[a]_q$, we can simplify the left-hand side of Equation (9):

$$\begin{aligned} q^i[a]_q + [b]_q &= q^i(1 + q + \cdots + q^{a-1}) + (1 + q + \cdots + q^{b-1}) \\ &= (q^i + q^{i+1} + \cdots + q^{i+a-1}) + (1 + q + \cdots + q^{b-1}) \\ &= (1 + q + \cdots + q^{i+a-1}) + (q^i + q^{i+1} + \cdots + q^{b-1}) \\ &= [a+i]_q + q^i[b-i]_q. \end{aligned}$$

□

Theorem 4.3. *Assuming Equation* (8) *is true,*

$$\epsilon_{(\lambda+e_i)+e_j} = \epsilon_{(\lambda+e_j)+e_i}.$$

Proof. If $j = i$, the statement is true trivially. Let $j \neq i$, we can expand the right-hand side of equation and obtain

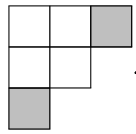
$$\begin{aligned} \epsilon_{(\lambda+e_i)+e_j} &= q\left(q\epsilon_\lambda + [n + \lambda_i + 2 - i]_q\right) + [(n+1) + \lambda_j + 2 - j]_q \\ &= q^2\epsilon_\lambda + q[n + \lambda_i + 2 - i]_q + [(n+1) + \lambda_j + 2 - j]_q \end{aligned}$$

Then, by Lemma 4.2,

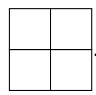
$$\begin{aligned} \epsilon_{(\lambda+e_i)+e_j} &= q^2\epsilon_\lambda + [n + \lambda_i + 2 - i + 1]_q + q[(n+1) + \lambda_j + 2 - j - 1]_q \\ &= q(q\epsilon_\lambda + [n + \lambda_j + 2 - j]_q) + [(n+1) + \lambda_i + 2 - i]_q \\ &= \epsilon_{(\lambda+e_j)+e_i}. \end{aligned}$$

□

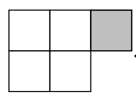
Example. We'll identify the eigenvalue associated with the following partition and horizontal strip:



The boxes of the horizontal strip are indicated in grey. To build this horizontal strip and its corresponding eigenvalue, we start with a horizontal strip of size 0, which is associated with the eigenvalue of $\epsilon_{(2,2)} = 0$:



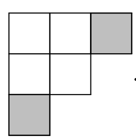
We can now add a box to the first row:



To find the new eigenvalue, we apply our recursion:

$$\begin{aligned}\epsilon_{(3,2)} = \epsilon_{(2,2)+e_1} &= q\epsilon_{2,2} + [(n+1) + (\lambda_i + 1) - i]_q \\ &= q(0) + [5 + 3 - 1]_q \\ &= [7]_q.\end{aligned}$$

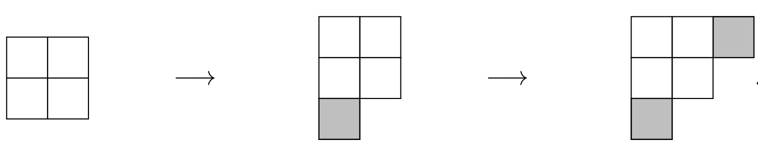
We now add another box to the third row of our horizontal strip:



Applying our recursion again, we obtain the new eigenvalue:

$$\begin{aligned}\epsilon_{(3,2,1)} = \epsilon_{(3,2)+e_3} &= q\epsilon_{(3,2)} + [(n+1) + (\lambda_i + 1) - i]_q \\ &= q[7]_q + [6 + 1 - 3]_q \\ &= q[7]_q + [4]_q.\end{aligned}$$

We can also obtain the same eigenvalue by adding boxes in the opposite order:



Then our recursion gives the following value for $\epsilon_{(3,2,1)}$:

$$\begin{aligned}\epsilon_{(3,2,1)} = \epsilon_{(2,2,1)+e_1} &= q\epsilon_{(2,2,1)} + [(n+1)+(\lambda_1+1)-1]_q \\ &= q(\epsilon_{(2,2)+e_3}) + [6+3-1]_q \\ &= q(q\epsilon_{(2,2)} + [5+1-3]_q) + [8]_q \\ &= q(0+[3]_q) + [8]_q \\ &= q[3]_q + [8]_q.\end{aligned}$$

By Lemma 4.2,

$$q[3]_q + [8]_q = [4]_q + q[7]_q$$

where the order we add boxes does not matter as desired.

4.1.1 Explicit formula for eigenvalues and eigenspaces

Assuming the conjectured eigenvalue recursion given in Equation (8) is true, we obtained the formula for all eigenvalues of $\mathsf{R2R}_n^{(q)}$ associated with a partition λ of n.

Conjecture 4.4. *Let λ and μ partitions such that λ/μ is a horizontal strip containing k boxes and μ does not equal $(|\mu|)$, or $(1,1,\ldots,1)$ for an odd number of 1's. We can construct this horizontal strip by adding k boxes to μ, and let box j be the j^{th} box we add to construct the horizontal strip. Then $\mathsf{R2R}_n^{(q)}$ has the eigenvalue*

$$\mathsf{eig}_q(\lambda/\mu) = \sum_{j=1}^{k} q^{k-j}[|\mu| + j + \mathsf{diag}(\text{box } j)]_q \tag{10}$$

associated to λ/μ.

Proof. By induction on k. To derive Equation (10), we repeatedly apply the eigenvalue recursion in Equation (8), starting from an eigenvalue of $\epsilon_\mu = 0$. For the base case, suppose $k = 0$. Then $\lambda = \mu$ and

$$\begin{aligned}\mathsf{eig}_q(\mu/\mu) &= \sum_{j=1}^{0} q^{k-j}[|\mu| + j + \mathsf{diag}(\text{box } j)]_q \\ &= 0 \\ &= \epsilon_\mu.\end{aligned}$$

Now suppose that Equation (10) holds for horizontal strips of size $k = p$. Consider a horizontal strip λ/μ of size $k = p+1$. We can obtain λ/μ and its corresponding eigenvalue by adding box $p+1$ to the horizontal strip λ'/μ containing all boxes of λ except box $p+1$. By the inductive hypothesis,

$$\mathsf{eig}_q(\lambda'/\mu) = \sum_{j=1}^{p} q^{p-j}[|\mu| + j + \mathsf{diag}(\text{box } j)]_q.$$

We obtain eig(λ/μ) by applying the recursion given in Equation (8) to $\mathsf{eig}(\lambda'/\mu)$, adding box $p+1$:

$$\begin{aligned}\mathsf{eig}_q(\lambda/\mu) &= q(\mathsf{eig}_q(\lambda'/\mu)) + [|\mu| + (p+1) + \mathsf{diag}(\text{box } p+1)]_q \\ &= q\left(\sum_{j=1}^{p} q^{p-j}[|\mu| + j + \mathsf{diag}(\text{box } j)]_q\right) + [|\mu| + p + 1 + \mathsf{diag}(\text{box } p+1)]_q \\ &= \sum_{j=1}^{p+1} q^{p+1-j}[|\mu| + j + \mathsf{diag}(\text{box } j)]_q.\end{aligned}$$

Therefore, Equation (10) holds for all values of k. □

Our explicit conjectural formula for the eigenvalues of $\mathsf{R2R}_n^{(q)}$ encodes a conjectural $GL_n(\mathbb{F}_q)$-representation theoretic structure on the eigenspaces, one which has the same q-Frobenius characteristic as the $\mathsf{R2R}_n^{(q)}$ eigenspaces given by Dieker and Saliola [5, Theorem 29].

Definition. The *ascents* of a standard young tableau of size n are the entries i such that $i+1$ appears weakly northeast of i or $i = n$.

A tableau is a *desarrangement tableau* if its smallest ascent is even.

Conjecture 4.5. *The $\epsilon(q)$-eigenspace has Frobenius characteristic*

$$\sum_{\substack{\lambda/\mu \\ eig_q(\lambda/\mu)=\epsilon}} d^{\mu} s_{\lambda},$$

for s_λ the Schur function associated to λ, and d^μ the number of desarrangement tableaux of shape μ.

In Theorem 5 of their paper, Dieker and Saliola give the following formula for the eigenvalue associated with a horizontal strip λ/μ, where diag(λ/μ) is the sum of the diagonal numbers of the cells in λ/μ:

$$\mathsf{eig}_q(\lambda/\mu) = \binom{|\lambda|+1}{2} - \binom{|\mu|+1}{2} + \mathsf{diag}(\lambda/\mu). \tag{11}$$

When $q = 1$, our formula reduces to the following:

$$\begin{aligned}\mathsf{eig}_q(\lambda/\mu) &= \sum_{j=1}^{k}(|\mu| + j + \mathsf{diag}(\text{box } j)) \\ &= \sum_{j=1}^{|\mu|+k} j - \sum_{j=1}^{|\mu|} j + \sum_{j=1}^{k}(\mathsf{diag}(\text{box } j)).\end{aligned}$$

Using the definition of $\mathsf{diag}(\lambda/\mu)$ and the the fact that $1+2+\cdots+n = \frac{n(n+1)}{2} = \binom{n+1}{2}$, we can rewrite the above as

$$\mathsf{eig}_q(\lambda/\mu) = \binom{|\mu|+k+1}{2} - \binom{|\mu|+1}{2} + \mathsf{diag}(\lambda/\mu) \tag{12}$$

to obtain the formula Dieker and Saliola provide. This is identical to Equation (11) because $|\lambda| = |\mu| + k$.

4.1.2 Alternative formula for eigenvalues

We'll now describe a way to simplify our eigenvalue formula that is somewhat analogous to Dieker and Saliola's process. If $\mathsf{diag}(\text{box } j) \geq 0$, then by Lemma 4.2,

$$\begin{aligned} q^{k-j}[|\nu| + j + \mathsf{diag}(\text{box } j)]_q &= q^{k-j}[|\mu|+j]_q + q^{k-j}q^{|\mu|+j}[\mathsf{diag}(\text{box } j)]_q &(13)\\ &= q^{k-j}[|\mu|+j]_q + q^{|\mu|+k}[\mathsf{diag}(\text{box } j)]_q. &(14)\end{aligned}$$

If $\mathsf{diag}(\text{box } j) < 0$, then let $\mathsf{diag}(\text{box } j) = -\ell$. Then

$$\begin{aligned} q^{k-j}[|\mu| + j + \mathsf{diag}(\text{box } j)]_q &= q^{k-j}[|\mu|+j-\ell]_q \\ &= q^{k-j}(1+q+q^2\cdots+q^{|\mu|+j-\ell-1}) \\ &= q^{k-j}\left((1+q+q^2+\cdots+q^{|\mu|+j-1}) - (q^{|\mu|+j-\ell}+\cdots+q^{|\mu|+j-1})\right) \\ &= q^{k-j}\left([|\mu|+j]_q - q^{|\mu|+j}(q^{-1}+q^{-2}+\cdots+q^{-\ell})\right) \\ &= q^{k-j}[|\mu|+j]_q - q^{|\mu|+k}(q^{-1}+q^{-2}+\cdots+q^{-\ell}). \end{aligned}$$

If we define

$$[-\ell]_q := -(q^{-1}+q^{-2}\ldots q^{-\ell})$$

then Equation (14) still holds because this definition of $[\ell]_q$ preserves the property

$$[a+b]_q = [a]_q + q^a[b]_q$$

when $b < 0$. We can also arrive at this definition by observing that for positive integers n, $[n]_q$ is defined to be

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\cdots+q^{n-1}.$$

It is also the case that

$$[-\ell]_q := -(q^{-1}+q^{-2}+\cdots+q^{-\ell}) = \frac{1-q^{-\ell}}{1-q}.$$

Using this definition of $[-\ell]_q$ can therefore write

$$\mathsf{eig}_q(\lambda/\mu) = \sum_{j=1}^{k}\left(q^{k-j}[|\mu|+j]_q + q^{|\mu|+k}[\mathsf{diag}(\text{box } j)]_q\right) \tag{15}$$

We can now define

$$\mathsf{diag}_q(\lambda/\mu)_q := \sum_{j=1}^{k} q^{|\mu|+k}[\mathsf{diag}(\text{box } j)]_q.$$

If $q = 1$, then the coefficient $q^{|\mu|+k}$ becomes 1, and the diagonal statistic reduces to $\mathsf{diag}_1(\lambda/\mu)$. With this statistic, we obtain the following:

$$\mathsf{eig}_q(\lambda/\mu) = \sum_{j=1}^{k} q^{k-j}[|\mu| + j]_q + \mathsf{diag}(\lambda/\mu)_q \tag{16}$$

Now observe that

$$\begin{aligned}\sum_{j=1}^{k} q^{k-j}[|\mu| + j]_q &= \sum_{j=1}^{k} \left(q^{k-j}[|\mu|]_q + q^{|\mu|+k-j}[j]_q\right) \\ &= [k]_q[|\mu|]_q + q^{|\mu|} \sum_{j=1}^{k} q^{k-j}[j]_q.\end{aligned}$$

The last summation evaluates to the following:

$$\sum_{j=1}^{k} q^{k-j}[j]_q = 1 + 2q + 3q^2 + \cdots + kq^{k-1}.$$

This cannot be easily simplified further. So we are left with the following formula for eig(λ/μ):

$$\mathsf{eig}_q(\lambda/\mu) = [k]_q[|\mu|]_q + q^{|\mu|} \sum_{j=1}^{k} q^{k-j}[j]_q + \mathsf{diag}(\lambda/\mu)_q. \tag{17}$$

4.2 Known eigenvalues of $\mathsf{R2R}_n^{(q)}$

Theorem 4.6. *The eigenvalue of* $\mathsf{R2R}_n^{(q)}$ *on* $G^{(n)} \otimes H^{(n)}$ *is* $([n]_q)^2$*. This is the eigenvalue corresponding to the horizontal strip* λ/λ *where* $\lambda = (n)$*.*

Proof. We have that $\chi_{H^{(n)}}(T_\omega) = q^{\ell(\omega)}$.

The operator $\mathsf{R2T}_n^{(q)}$ on $G^{(n)}$ scales by $\chi_{H^{(n)}}(\mathsf{R2T}_n^{(q)})$. Using our definition of $\mathsf{R2T}_n^{(q)}$ in terms of the Hecke algebra generators, we therefore have:

$$\chi_{H^{(n)}}(\mathsf{R2T}_n^{(q)}) = \chi_{H^{(n)}}\left(\sum_{i=0}^{n-1} T_{i\ldots1}\right) = \sum_{i=0}^{n-1} q^i = [n]_q.$$

Using a similar process, we can find the eigenvalue of $\mathsf{T2R}_n^{(q)}$ on $G^{(n)} \otimes H^{(n)}$:

$$\chi_{H^{(n)}}(\mathsf{T2R}_n^{(q)}) = \chi_{H^{(n)}}\left(\sum_{i=0}^{n-1} T_{1\ldots i}\right) = \sum_{i=0}^{n-1} q^i = [n]_q.$$

The space $G^{(n)} \otimes H^{(n)}$ is one-dimensional, so it is an eigenvector of any element of the Hecke algebra, including $\mathsf{R2T}_n^{(q)}$ and $\mathsf{T2R}_n^{(q)}$. Given that $\mathsf{R2T}_n^{(q)}$ and $\mathsf{T2R}_n^{(q)}$ share an eigenvector, that eigenvector is also an eigenvector of $\mathsf{R2R}_n^{(q)} = \mathsf{T2R}_n^{(q)} \circ \mathsf{R2T}_n^{(q)}$ with eigenvalue $[n]_q \cdot [n]_q = ([n]_q)^2$.

□

Theorem 4.7. *The operator* $\mathsf{R2R}_n^{(q)}$ *acting on* G^λ *for* $\lambda = (1, 1, \dots 1)$ *has eigenvalue*

$$\epsilon_{(1,1,\dots 1)} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Proof. The other one-dimensional irreducible representation is given by $\lambda = (1, 1, \dots, 1)$. Here, we have $\chi_{H^\lambda}(T_\omega) = (-1)^{\ell(\omega)}$. Then

$$\begin{aligned}
\chi_{H^\lambda}(\mathsf{R2R}_n^{(q)}) &= \chi_{H^\lambda}\left(\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} T_i \dots T_1 T_1 \dots T_j\right) \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \chi_{H^\lambda}(T_i) \dots \chi_{H^\lambda}(T_1)\chi_{H^\lambda}(T_1) \dots \chi_{H^\lambda}(T_j) \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} (-1)^{i+j}
\end{aligned}$$

If n is even, this summation evaluates to 0, and if n is odd then it evaluates to 1. □

5 Conjectured recursion in the Algebra of Words

5.1 From Flags to Algebra of Words

As explained on page 254 of [6], there is a $GL_n(\mathcal{F}_q) \times \mathcal{H}_n(q)$-bimodule decomposition

$$\mathbb{C}[G/B] \cong \bigoplus_{\lambda \vdash n} G^\lambda \otimes H^\lambda,$$

where G^λ is the irreducible unipotent $GL_n(\mathcal{F}_q)$ representation corresponding to λ and H^λ is the irreducible Hecke algebra representation corresponding to λ. Given this decomposition and our formula for $\mathsf{R2T}_n^{(q)}$ in $\mathcal{H}_n(q)$, it suffices to find the eigenvalues of $\mathsf{R2R}_n^{(q)}$ on each irreducible $\mathcal{H}_n(q)$-representation H^λ. In [5], Dieker and Saliola found the eigenvalue of $\mathsf{R2R}_n^{(q)}$ on each irreducible S_n-representation S^λ by considering the single copy of S^λ within the permutation representations M^λ in the algebra of words, which is comprised of words of content λ. There is a q-analog of M^λ in the algebra of words which contains exactly one copy of H^λ (see Mathas [8] exercise 4.19). The operators T_i act on a word $w = w_1 w_2 \dots w_n$ according to the following rule:

$$(w_1 w_2 \dots w_n)T_i = \begin{cases} qw & \text{if } w_i = w_{i+1} \\ w_1 w_2 \dots w_{i-1} w_{i+1} w_i w_{i+2} \dots w_n & \text{if } w_i < w_{i+1} \\ q(w_{i-1} w_{i+1} w_i w_{i+2} \dots w_n) + (q-1)w & \text{if } w_i > w_{i+1}. \end{cases}$$

When $q = 1$, the operator T_i always switches the i^{th} and $(i+1)^{th}$ characters in w, so its action on w is the same as s_i.

5.2 Algebra of Words Recursion

In order to prove their recursion relating eigenvalues of $\mathsf{R2R}_n$ to $\mathsf{R2R}_{\mathsf{n+1}}$, Dieker-Saliola [5] introduce the following operators, defined below for $q = 1$.

Definition. For a word w on the ordered alphabet $A = \{a_1, a_2, a_3, \dots\}$,

- $sh_{a_i}(w)$ is the sum of all words obtained from w by inserting the i-th letter of A; explicitly,

$$sh_{a_i}(w_1, \dots, w_n) = \sum_{0 \le j \le n} w_1 \dots w_j a_i w_{j+1} \dots w_n$$

- ∂_{a_i} is the linear operator that maps a word to the sum of all words obtained by removing exactly one occurrence of the letter a_i; explicitly,

$$\partial_{a_i}(w_1, \dots, w_n) = \sum_{\substack{1 \le j \le n \\ w_j = a_i}} w_1 \dots w_{j-1} w_{j+1} \dots w_n.$$

- Θ_{a_i, a_j} (the replacement operator) is the linear transformation that maps a word $w = w_1 w_2 \dots w_n$ to the sum of all the words that can be obtained from w by replacing exactly one occurrence of a_i in w by a_j; explicitly,

$$\Theta_{a_i, a_j}(w) = \sum_{\substack{1 \le k \le n \\ w_k = a_i}} w_1 \dots w_{k-1} \cdot a_j \cdot w_{k+1} \dots w_n.$$

Let w be a word of length n with the alphabet $A = \{1, 2, \dots, n\}$. To prove the eigenvalues recursion in 4.1, Dieker-Saliola [5] proved the following recurrence when $q = 1$:

$$\mathsf{R2R}_{\mathsf{n+1}} \circ sh_a(w) - sh_a \circ \mathsf{R2R}_n(w) = (n+1) sh_a(w) + \sum_{1 \le b \le n} sh_b \circ \Theta_{b,a}(w). \quad (18)$$

We hope to find a q-analog of this recursion to mimic Dieker and Saliola's proof of the eigenvalues of $\mathsf{R2R}_n$.

Definition. Let $a \cdot w$ be the word obtained by appending the letter a to the beginning of a word w. Similarly, let $w_1 w_2 \cdot a \cdot w_3$ be word obtained by inserting a between the letters w_2 and w_3 in the word $w_1 w_2 w_3$. To find the q-analog of Equation (18), we define the operators $\mathsf{sh_a}^{(q)}$ and $\Theta_{a,b}^{(q)}$ as follows:

- $\mathsf{sh_a}^{(q)}(w) := \mathsf{T2R}^{(q)}(a \cdot w)$
- $\Theta_{a,b}^{(q)}(w) :=$

$$\text{for } a \neq b \begin{cases} \sum\limits_{\substack{1 \leq k \leq n \\ w_k = a_i}} w_1 \dots w_{k-1} \cdot a_j \cdot w_{k+1} \dots w_n, \text{ if } w \text{ is in increasing order} \\ \left(\sum\limits_{\substack{1 \leq k \leq n \\ w_k = a_i}} w_1 \dots w_{k-1} \cdot a_j \cdot w_{k+1} \dots w_n \right) \circ T_i \dots T_j, \text{ if } w \circ T_i \dots T_j = w' \text{ is ordered.} \end{cases}$$

$$\text{for } a = b \begin{cases} \sum\limits_{\substack{1 \leq k \leq n \\ w_k = a_i}} [\text{mult}(a)]_q w_1 \dots w_{k-1} \cdot a_j \cdot w_{k+1} \dots w_n, \text{ if } w \text{ is in increasing order} \\ \left(\sum\limits_{\substack{1 \leq k \leq n \\ w_k = a_i}} [\text{mult}(a)]_q w_1 \dots w_{k-1} \cdot a_j \cdot w_{k+1} \dots w_n \right) \circ T_i \dots T_j, \text{ if } w \circ T_i \dots T_j = w' \text{ is ordered.} \end{cases}$$

Remark. Dieker and Saliola define the "top" of the word to be the end of the word, so in their notation, $\mathsf{T2R}_n((w_1 w_2 \cdots w_{n-1}) \cdot w_n) = \mathsf{sh}_{w_n}(w_1 \dots w_{n-1})$ when $q = 1$.

Conjecture 5.1. *Based on the definitions above, for $a = 1$, a given word w of length n,*

$$\mathsf{R2R}_{n+1}^{(q)} \mathsf{sh}_1^{(q)}(w) = q \cdot \mathsf{sh}_1^{(q)} \mathsf{R2R}_n^{(q)}(w) + [n+1]_q \mathsf{sh}_1^{(q)}(w) + \sum_{j=1}^{n} q^{n+2-j} \cdot \mathsf{sh}_j^{(q)} \Theta_{j,1}^{(q)}(w).$$

Remark. Conjecture 5.1 has been checked computationally with Sage for w with content (n), $(n-1,1)$, $(n-1,1,1)$, and $(n-2,2)$. However, it does not apply to $a = 2$ and this is expected similar to the argument of $q = 1$ case in [5].

Conjecture 5.2. *For content (n) with $a = 2$ and starting word $w = (11 \dots 1)$,*

$$\mathsf{R2R}_{n+1}^{(q)} \mathsf{sh}_a^{(q)}(w) = q \cdot \mathsf{sh}_a^{(q)} \mathsf{R2R}_n^{(q)}(w) + \mathsf{sh}_a^{(q)}(w) + q^{2n-1} \cdot \mathsf{sh}_1^{(q)}(\Theta_{1,2}(w)).$$

Remark. Conjecture 5.2 is not in general true for all content when $a = 2$.

5.3 Data and code summary

We attempted to interpret the eigenvalues given λ, where λ comes from the irreducible representation H^λ of the Hecke algebra, or unipotent representations G^λ of $GL_n(\mathbb{F}_q)$.

5.3.1 Further data for algebra word recurrence

One example that Conjecture 5.1 failed for $a > 1$ is the following. Take $w = (1112)$, $a = 2$, we calculate the remainder of

$$\mathsf{R2R}_{n+1}^{(q)}\mathsf{sh}_1^{(q)}(w) - q \cdot \mathsf{sh}_1^{(q)}\mathsf{R2R}_n^{(q)}(w) - [n+1]_q\mathsf{sh}_1^{(q)}(w)$$

in hopes it can be expressed as $\sum_{j=1}^{n} q^{n+2-j} \cdot \mathsf{sh}_j^{(q)}\Theta_{j,1}^{(q)}(w)$. However, the remainder here is

$$q^8 \cdot (\mathsf{sh_1}^{(q)}(\Theta_{1,2}{}^{(q)}(1112))) + q^6 \cdot \mathsf{sh_2}^{(q)}(\Theta_{2,2}{}^{(q)}(1112)) + r', \tag{19}$$

where

$$\begin{aligned} r' =&(q^8 - q^7) \cdot (12121) + (q^8 - q^7) \cdot (21121) + (q^{11} - q^{10}) \cdot (11122) \\ &+ (q^8 - q^7) \cdot (11221) + (q^9 - q^7) \cdot (12211) + (q^9 - q^7) \cdot (21211) + (q^9 - q^7) \cdot (22111) \\ =&(q-1)\Big(q^7 \cdot (12121) + q^7 \cdot (21121) + (q^{10}) \cdot (11122) \\ &+ (q^7) \cdot (11221) + (q+1)(q^7) \cdot (12211) + (q+1)(q^7) \cdot (21211) + (q+1)(q^7) \cdot (22111)\Big). \end{aligned}$$

We expect as $a > 1$, there will be more terms. For example, when $a = 2$, we might have a term with $q - 1$ coefficient.

5.3.2 Code Summary

Our computations were completed in SageMath.

To compute the data in Section 5.2, we coded the right action of the Hecke algebra on words, which we used to code the operators $\mathsf{R2T}^{(\mathsf{q})}$, $\mathsf{T2R}^{(\mathsf{q})}$, $\mathsf{R2R}^{(\mathsf{q})}$, $\mathrm{sh}^{(q)}$, and $\Theta^{(q)}$ on the algebra of words. For working with words, we used an extensive algebra of words code developed by Franco Saliola and Nadia Lafreniére which included the analogous operators in the $q = 1$ case, and other useful operators on words.

To compute the eigenvalues of $\mathsf{R2R}^{(\mathsf{q})}$, we coded, for a general i and $\lambda \vdash n$, a matrix representation for the action of T_i on the $\mathcal{H}_n(q)$-irreducible representation corresponding to λ: H^λ. We used the Kazhdan-Lusztig cell representation of H^λ, which can be built using RSK, descents of tableaux, and top coefficients of the Kazhdan Lusztig polynomial (see p604 of Stembridge [13]). Then, using our formula for $\mathsf{R2R}_n^{(q)}$ in $\mathcal{H}_n(q)$, we could obtain the eigenvalues of $\mathsf{R2R}_n^{(q)}$'s action on H^λ.

Acknowledgements

This project was partially supported by RTG grant NSF/DMS-1148634. It was supervised as part of the University of Minnesota School of Mathematics Summer 2022 REU program. The authors would like to thank Sarah Brauner, Patty Commins, and Vic Reiner for helpful mentoring, and Nadia Lafreniére for coding support.

References

[1] Pat Bidigare, Phil Hanlon, and Dan Rockmore, *A combinatorial description of the spectrum for the tsetlin library and its generalization to hyperplane arrangements*, Duke Mathematical Journal **99** (1999), no. 1, 135–174.

[2] Sarah Brauner, Patricia Commins, and Victor Reiner, *Invariant theory for the free left-regular band and a q-analogue,* `arxiv:2206.11406`.

[3] Kenneth S. Brown, *Semigroups, rings, and Markov chains*, J. Theoret. Probab. **13** (2000), no. 3, 871–938. MR 1785534

[4] Kenneth S. Brown and Persi Diaconis, *Random walks and hyperplane arrangements*, Ann. Probab.

[5] A. B. Dieker and F. V. Saliola, *Spectral analysis of random-to-random Markov chains*, Adv. Math. **323** (2018), 427–485. MR 3725883

[6] Tom Halverson and Arun Ram, *Bitraces for $gl_n(_q)$ and the iwahori-hecke algebra of type a_{n-1}*, Indag. Mathem. **10** (1999), no. 2, 247–268.

[7] Nadia Lafrenière, *Valeurs propres des opérateurs de mélanges symétrisés,* `arxiv:1912.07718`, (2019).

[8] Andrew Mathas, *Iwahori hecke algebras and schur algebras of the symmetric group*, AMS University Lecture Series **15** (1999).

[9] Ravindra Madhav Phatarfod, *On the matrix occurring in a linear search problem*, Journal of Applied Probability **28** (1991), 336–346.

[10] Victor Reiner, Franco Saliola, and Volkmar Welker, *Spectra of symmetrized shuffling operators*, Memoirs of the American Mathematical Society (2011).

[11] Victor Reiner and Michelle Wachs, *Unpublished notes on eigenspaces of the random-to-top shuffling operator*, (2002).

[12] J.C.U. Reyes and Stanford University. Department of Mathematics, *Random walk, semi-direct products, and card shuffling*, Stanford University, (2002).

[13] John Stembridge, *Canonical bases and self-evacuating tableaux*, Duke Mathematical Journal **82** (1996), no. 3, 585–606.

Appendix: Eigenvalues of $\mathsf{R2R}_n^{(q)}$ for $n \leq 6$

The following are eigenvalues and their corresponding partitions that we found computationally. Note that "multiplicity" here refers to the eigenvalue's multiplicity within a single $H_n(q)$ irreducible H^λ. To get the multiplicity of that eigenvalue within the entire flag space $\mathbb{C}[G/B]$, one should also multiply by the dimension of the $GL_n(\mathbb{F}_q)$ irreducible G^λ (using the q-hook formula). This data confirms our explicit conjectures, both for the eigenvalues and q-Frobenius image of the eigenspaces.

$n = 2$

$H_n(q)$ irrep	Eigenvalues	Multiplicities
(2)	$([2]_q)^2$	1
$(1,1)$	$[0]_q$	1

$n = 3$

$H_n(q)$ irrep	Eigenvalues	Multiplicities
(3)	$([3]_q)^2 = q([2]_q)^2 + [5]_q$	1
$(2,1)$	$[0]_q$	1
	$[4]_q$	1
$(1,1,1)$	$[1]_q$	1

$n = 4$

$H_n(q)$ irrep	Eigenvalues	Multiplicities
(4)	$([4]_q)^2 = q([3]_q)^2 + [7]_q$	1
$(3,1)$	$[0]_q$	1
	$[6]_q$	1
	$[6]_q + q[4]_q$	1
$(2,2)$	$[0]_q$	1
	$[4]_q$	1
$(2,1,1)$	$[0]_q$	1
	$[2]_q$	1
	$[2]_q + q[4]_q$	1
$(1,1,1,1)$	$[0]_q$	1

$n = 5$

$H_n(q)$ irrep	Eigenvalues	Multiplicities
(5)	$([5]_q)^2 = q([4]_q)^2 + [9]_q$	1
$(4,1)$	$[0]_q$	1
	$[8]_q + q[6]_q + q^2[4]_q$	1
	$[8]_q + q[6]_q$	1
	$[8]_q$	1
$(3,2)$	$[0]_q$	2
	$[5]_q$	1
	$[7]_q$	1
	$[7]_q + q[4]_q = [5]_q + q[6]_q$	1
$(3,1,1)$	$[0]_q$	2
	$[3]_q$	1
	$[7]_q$	1
	$[7]_q + q[2]_q = [3]_q + q[6]_q$	1
	$[7]_q + q[2]_q + q^2[4]_q$	1
$(2,2,1)$	$[0]_q$	2
	$[3]_q$	1
	$[5]$	1
	$[5] + q[2]_q$	1
$(2,1,1,1)$	$[0]_q$	2
	$[2]_q$	1
	$[6]_q$	1
$(1,1,1,1,1,1)$	$[1]_q$	1

$n = 6$

$H_n(q)$ irrep	Eigenvalues	Multiplicities
(6)	$([6]_q)^2 = q([5]_q)^2 + [11]_q$	1
$(5,1)$	$[0]_q$	1
	$[10]_q$	1
	$[10]_q + q[8]_q$	1
	$[10]_q + q[8]_q + q^2[6]_q$	1
	$[10]_q + q[8]_q + q^2[6]_q + q^3[4]_q$	1
$(4,2)$	$[0]_q$	3
	$[6]_q$	1
	$[9]_q$	2
	$[9]_q + q[5]_q$	1
	$[9]_q + q[7]_q$	1
	$[9]_q + q[7]_q + q^2[4]_q$	1
$(4,1,1)$	$[0]_q$	3
	$[4]_q$	1
	$[9]_q$	2
	$[9]_q + q[3]_q$	1
	$[9]_q + q[7]_q$	1
	$[9]_q + q[7]_q + q^2[2]_q$	1
	$[9]_q + q[7]_q + q^2[5]_q + q^3[1]_q$	1
$(3,3)$	$[0]_q$	2
	$[7]_q$	2
	$[7]_q + q[5]_q$	1
$(3,2,1)$	$[0]_q$	6
	$[4]_q$	2
	$[6]_q$	2
	$[8]_q$	2
	$[8]_q + q[3]_q$	1
	$[8]_q + q[5]_q$	1
	$[8]_q + q[5]_q + q^2[2]_q$	1
	$[6]_q + q[3]_q$	1
$(3,1,1,1)$	$[0]_q$	4
	$[3]_q$	2
	$[8]_q$	2
	$[8]_q + q[2]_q$	1
	$[8]_q + q[6]_q$	1
$(2,2,2)$	$[0]_q$	2
	$[5]_q$	2
	$[5]_q + q[3]_q$	1
$(2,2,1,1)$	$[0]_q$	4
	$[3]_q$	2
	$[6]_q$	2
	$[6]_q + q[2]_q$	1
$(2,1,1,1)$	$[0]_q$	2
	$[2]_q$	2
	$[2]_q + q[6]_q$	1
$(1,1,1,1,1,1)$	$[0]_q$	1