Polarizations of Strongly Stable Ideals

UMN Algebra and Combinatorics REU 2022

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4 August 2022

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In general,

$$\mathbf{x}^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \mapsto m_1(\mathbf{a}) m_2(\mathbf{a}) \cdots m_n(\mathbf{a}) = m(\mathbf{a})$$

where each $m_i(\mathbf{a})$ is a squarefree monomial of degree a_i in the variables of the set $\{x_{i1}, \ldots, x_{i,d_i}\}$.

Strongly Stable Ideals

Strongly stable ideals

Definition

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. A monomial ideal $I \subset S$ is *strongly stable* if for any generator m of I, I contains every monomial that is reachable from m by swapping x_j's out for x_i's when $i < j$.

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So our strongly stable ideal I is $I = (x_1x_3^2, x_1x_2x_3, x_1^2x_3, x_1x_2^2, x_1^3)$

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Example

Notice that $I = (x_2, x_3^2) \subset S$ is not strongly stable since we can swap x_2 for x_1, but $x_1 \notin I$.

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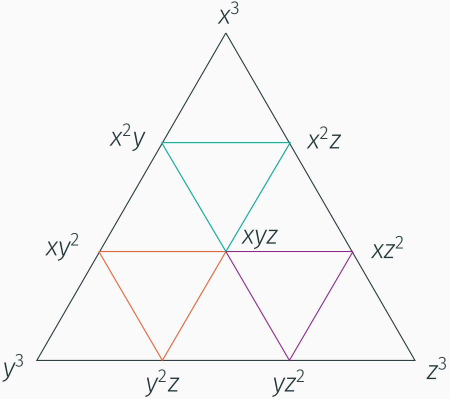
- Polarization replaces a monomial ideal with a square-free monomial ideal sharing the same homological data, giving access to combinatorial tools such as Stanley-Reisner theory.
- Strongly stable ideals arise naturally in algebraic combinatorics; e.g. generic initial ideals are always strongly stable in characteristic 0.

Results

Graph of Linear Syzygies

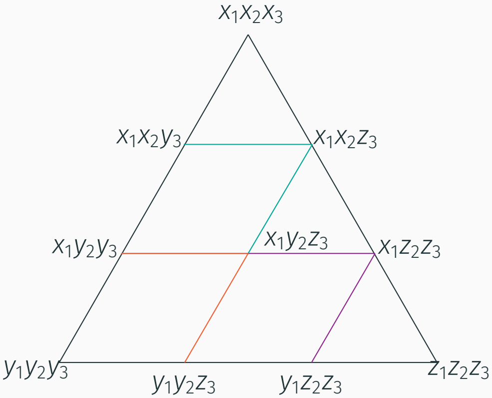
Suppose we have a strongly stable ideal I with a polarization $\tilde{I}$.

- Let $\Delta^{\mathbb{Z}}(n, d)$ denote be the lattice simplex with vertices given by all $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}_0^n$ with $\sum a_i = d$. Its vertices are in one-one correspondence with minimal generator of $(x_1, \ldots, x_n)^d$.



Graph of Linear Syzygies (cont.)

- We form the linear syzygy graph G_I by taking vertices as generators of I and edges when there is a linear relation between two generators. This graph can be superimposed on $\Delta^{\mathbb{Z}}(n, d)$.
- Suppose $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$. We refer to the induced subgraph on all vertices of the form $\mathbf{c} - e_i$ as a down-graph $D(\mathbf{c})$.



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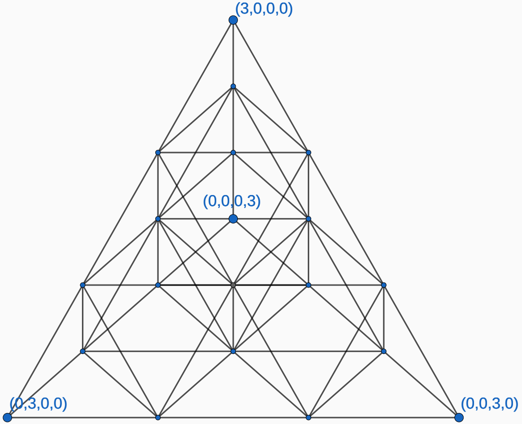

Specifying the labeling

Put partial orders $\geq_i$ on $\Delta^{\mathbb{Z}}(n,d)$ defined by $\mathsf{a} \geq_i \mathsf{b}$ if $a_i \geq b_i$ and $a_j \leq b_j$ for all $j \neq i$.

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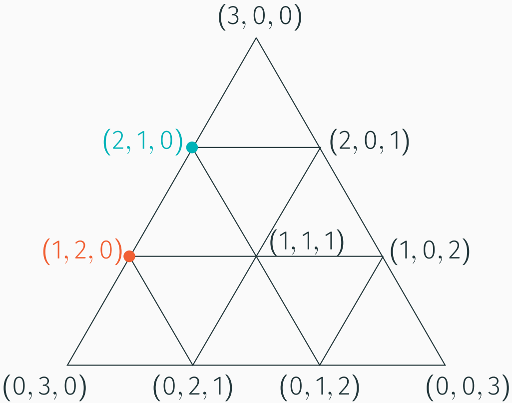


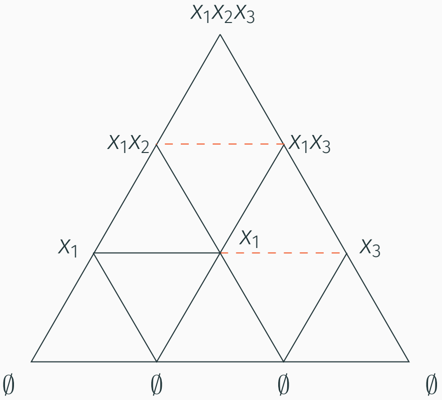
For example, $(2, 1, 0) \geq_1 (1, 2, 0)$, but $(2, 1, 0) \leq_2 (1, 2, 0)$ and they are incomparable with respect to $\geq_3$.

Specifying the Labeling

Lemma (AFL '22)
Let $\mathbf{x}^{\mathbf{a}}$ *and* $\mathbf{x}^{\mathbf{b}}$ *be minimal generators of a monomial ideal* I *and* $m(\mathbf{a})$ *and* $m(\mathbf{b})$ *the corresponding generators in a polarization of* I*. Fix an index* i*. If* $\mathbf{a} \leq_i \mathbf{b}$*, then the i'th part* $m_i(\mathbf{a})$ *divides* $m_i(\mathbf{b})$*.*

Specifying the labeling

Let $B(\check{X}_i)$ be the Boolean Poset on $\check{X}_i = \{x_{i1}, \ldots, x_{id}\}$. Polarizations give rise to isotone maps $X_i : \Delta^{\mathbb{Z}}(n,d) \to B(\check{X}_i)$ with respect to $\geq_i$.



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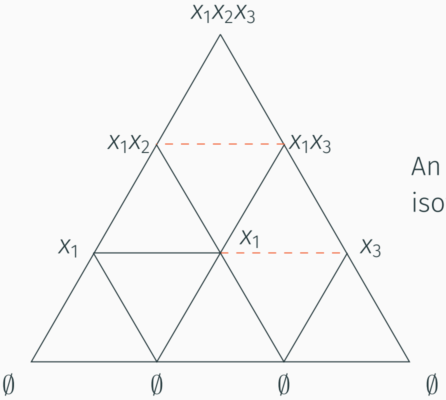
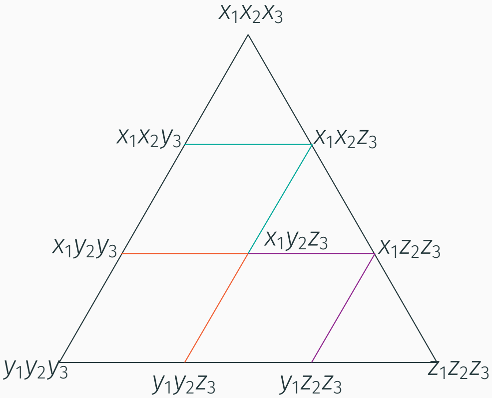
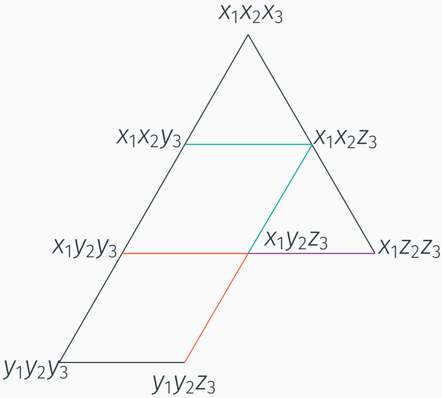


An example of the image of an isotone X_1 map on $\Delta^{\mathbb{Z}}(3, 3)$.

Theorem (Almousa–Fløystad–Lohne (2022))
A set of isotone maps $X_1, \dots X_n$ determines a polarization of $I = (x_1, \dots, x_n)^d$ if and only if for every $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$, the graph of linear syzygy edges contains a spanning tree for $D(\mathbf{c})$.



Theorem
A set of isotone maps $X_1, \dots X_n$ determines a polarization of a strongly stable ideal I if and only if for every $\mathbf{c} \in \Delta^{\mathbb{Z}}(n, d+1)$, the graph of linear syzygy edges contains a spanning tree for $D(\mathbf{c}) \cap G_I$.



Applications

Definition (Associated prime)
Let R be a Noetherian ring and M a finitely generated R-module. A prime ideal $P \subset R$ is an *associated prime ideal* of M, if there exists an element $x \in M$ such that $P = \mathrm{ann}(x)$, where $\mathrm{ann}(x) = \{a \in R : ax = 0\}$. The set of all associated primes of I is denoted $\mathrm{ass}(R/I)$.

Alexander Duals and Associated Primes

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Remark 1: For I a squarefree monomial ideal in a polynomial ring R:

- I is an intersection of its associated primes.
- The associated prime ideals of I are a finite number of prime ideals generated by variables.

Definition (Alexander Dual)

Let I be a square-free monomial ideal in a polynomial ring S. The Alexander dual ideal $I^{\vee}$ of I is the monomial ideal in S whose monomials are precisely those that have a nontrivial common divisor with every generator of I.

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Remark 2: Given a square-free monomial ideal I, the minimal generators of the Alexander dual of I correspond to the product of the variables that generate the associated primes of I.

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Then by Remark 1, we know its associated primes are the following: $\text{ass}(R/I) = \{(x_1, x_6), (x_3, x_4, x_6), (x_3, x_5, x_6)\}$.

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Then by Remark 2, we know that the Alexander dual of I is: $I^\vee = (x_1x_6, x_3x_4x_6, x_3x_5x_6)$.

Definition (Color Classes, Rainbow Monomials)
We call the set of all variables sharing their first index $\{x_{i,1}, \ldots, x_{i,m}\}$ the $i-$color class. We call a monomial in degree d a *rainbow* monomial when it is of the form $x_{1,j_1} \ldots x_{d,j_d}$, a product of *exactly* one variable from each color class.

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$x_{1,1}x_{2,1}$ is a weakly-rainbow monomial

$x_{1,1}x_{1,2}x_{3,1}$ is neither rainbow, nor weakly-rainbow

Our main result for this section gives us a helpful fact regarding polarizations of *any* monomial ideal $J \subset k[x_1, ..., x_n]$.

Theorem
If $I \subset S = k[x_{1,1}, \ldots, x_{1,d}, \ldots, x_{n,1}, \ldots, x_{n,d}]$ is a polarization of any monomial ideal $J \subset k[x_1, ..., x_n]$, then the generators of $I^\vee$ are weakly-rainbow.

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Hence $I^\vee = (x_1y_1, x_1x_2z_1, x_2y_1z_1, x_2y_2z_1)$.

Then since $x_1x_2z_1$ is not weakly-rainbow, by our theorem I is *not* a polarization of J.

Stanley-Reisner Complex and Shellability

Definition

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Definition
An ordering $F_1, \ldots F_t$ of the facets of a simplicial complex Δ is a *shelling* if, for each j with $1 < j \leq t$, the intersection

$$\left(\bigcup_{i=1}^{j-1} F_i \right) \cap F_j$$

is a nonempty union of facets of ∂F_j. If there exists a shelling of Δ, then Δ is called *shellable*.

Shellability when $I = (x_1, \ldots, x_n)^d$

Question. For $I = (x_1, \ldots, x_n)^d$, is $\Delta_{\tilde{I}}$ shellable?

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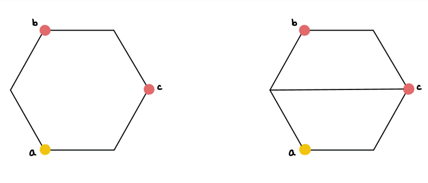
Theorem (Almousa–Fløystad–Lohne (2022))
$\Delta_{\tilde{I}}$ is shellable when $n = 3$ or $d = 2$.

Definition
A graph is *well-connected* if for any vertices a, b, c, there exists a shortest path from b to c such that the distance from a to anything on the path is $\leq \max(d(a, b), d(a, c))$.

Example
The left graph is not well-connected, but the right graph is.



Well-Connected Graphs

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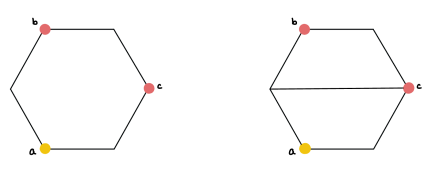


Question Is the linear syzygy graph of $\tilde{I}^\vee$ is well-connected for any Artinian I?

Shellability given Well-Connected

Theorem
Let $\tilde{I}$ be a polarization of an Artinian I. If the linear syzygy graph G on the Alexander dual $\tilde{I}^\vee$ is well-connected, $\Delta_{\tilde{I}}$ is shellable.

Smooth Points on Hilbert Schemes

- The *Hilbert scheme* $H = H_{\mathbb{P}^n}^{p(z)}$ is a space "parameterizing" closed subschemes (projective varieties) with Hilbert polynomial $p(z)$ inside $\mathbb{P}^n$.
- It is not well understood (can have terrible singularities, don't know the number of irreducible components, etc.).

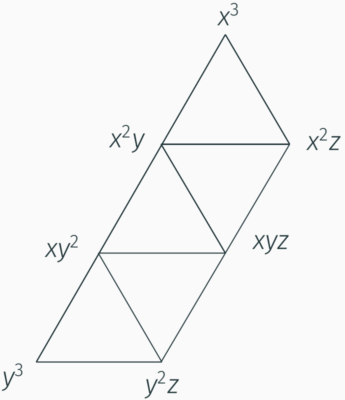
Overarching question: When are the points on H corresponding to the varieties cut out by polarizations of strongly stable ideals smooth?

Theorem (Lohne, 2013)
If J is either the standard or box polarization of $(x_1, \ldots, x_n)^d \subset k[x_1, \ldots, x_n]$, then the point on H corresponding to J is smooth.

$I_{3,3}$:

Definition

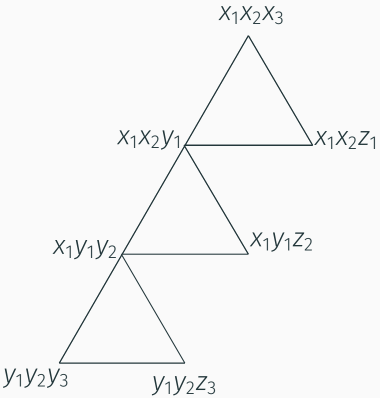
For each $m, n \geq 2$, define $I_{m,n}$ to be the strongly stable closure of $\{x_2^{n-1}x_m\}$.



$J_{3,3}$:

Definition

Define the *pyramidal polarization* $J_{m,n}$ of $I_{m,n}$ as follows: take the standard polarization of $I_{m,n}$ and separate the $x_3, \ldots, x_m$ variables as much as possible.



How to show a point on H is smooth? In general finding the dimension of its component is hard.

Our strategy: the T^2 functor.

Associated to any ideal $J \subset S$ is the *cotangent complex* of S/J-modules

$$L_2 \to L_1 \to L_0.$$

A Case Study

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$$\mathrm{Hom}(L_0, S/J) \to \mathrm{Hom}(L_1, S/J) \to \mathrm{Hom}(L_2, S/J).$$

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T^2 tells us how "obstructed" our ideal J is. It provides an "upper bound" on how bad a possible singularity at J can be.

A Case Study

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For $m, n \geq 2$, $T^2_{J_{m,n}} = 0$.

Hence $J_{m,n}$ defines a smooth point $x_{m,n}$ on its Hilbert scheme.

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Theorem
$\dim_{k(x_{m,n})} T_{x_{m,n}} = n(n-1)(m^2 + m - 1)$.

One More Word on Tangent Spaces

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For instance, this would imply that the tangent space of the standard polarization of $I_{m,n}$ is also dimension $n(n-1)(m^2+m-1)$.

Acknowledgements

- Ayah Almousa, John O'Brien
- Gregg Musiker
- The UMN Math Dept. Faculty & Staff
- This project was partially supported by RTG grant NSF/DMS-1745638.