Mutation-Finite Doodles

Pranjal Dangwal, Ryan Lynch, Son Van Thanh Nguyen, **Ethan Pesikoff**;
Mentor: Pavlo Pylyavskyy; TA: Sylvester Zhang

August 3, 2022

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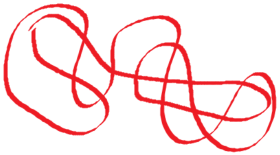

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Quivers

Definition

A *quiver* is an oriented graph which can have multiple edges between two vertices but 1-cycles (loops) and 2-cycles are disallowed.

Definition

Let Q be a quiver containing a vertex j.The *quiver mutation* μ_j transforms Q into a new quiver $Q' = \mu_j(Q)$ via a sequence of three steps:

1. For each oriented two-arrow path $k \to j \to i$, introduce a new arrow $k \to i$.
2. Reverse the direction of all arrows incident to the vertex j.
3. Remove all oriented 2-cycles.

Example

The *quiver mutation* μ_j transforms Q into a new quiver $Q' = \mu_j(Q)$

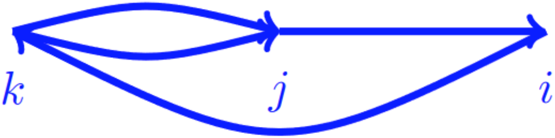

Figure: Original quiver

Example

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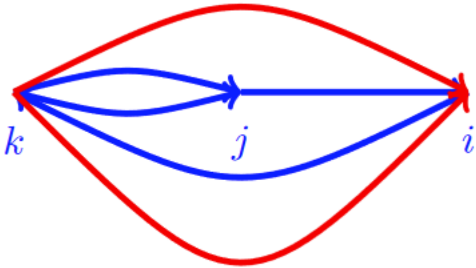

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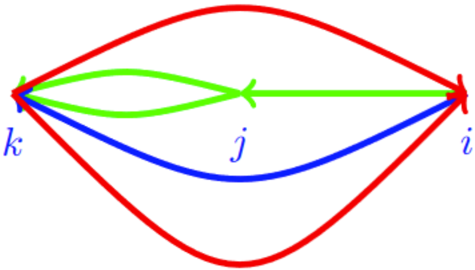

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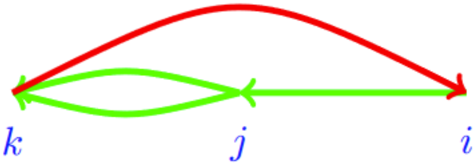


Figure: Mutated quiver

Quivers

Definition

If two quivers are related to each other by a sequence of mutations, we call them *mutation equivalent*. The set of all quivers mutation equivalent to a given quiver is known as its *mutation class*.

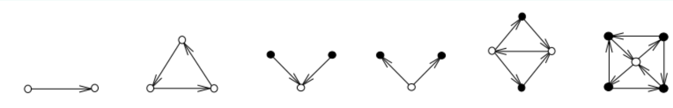
Definition

A quiver is *mutation (in)finite* if its mutation equivalence class is (in)finite.

Block Decomposability

Definition

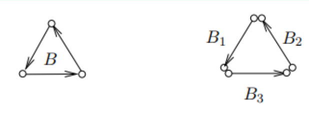
A *block* is one of the following six quivers.



Definition

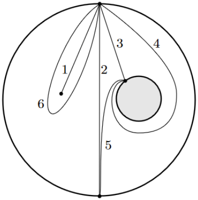
A quiver is *block* decomposable if it can be obtained by gluing together blocks at their white vertices, with each vertex part of at most two blocks.

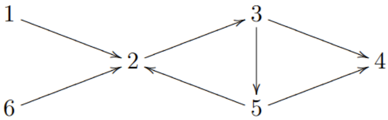
Example (Two block decompositions of an oriented triangle.)



Quivers from Surfaces

We can get quivers from triangulations of surfaces.





Theorem

Any quiver from a surface is of finite mutation type.

Theorem

A quiver arises from a surface if and only if it is block decomposable. Furthermore, if a quiver arises from a surface, so do all elements of its mutation class.

Mutation Finite Quivers

Theorem (Felikson, Shapiro, Tumarkin; 2009)

A quiver is mutation finite iff it is either block decomposable or mutation equivalent to one of the following exceptional quivers.

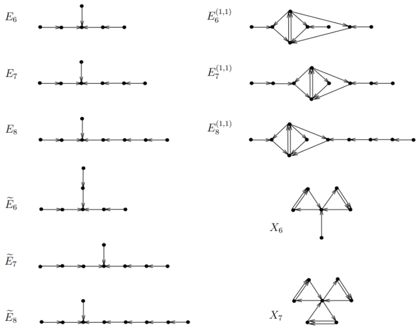


Figure: Exceptional Case Mutation Finite Quivers [FST12]

Singularities and Morsifications

Definition

An isolated singularity $(C, z) \in \mathbb{C}^2$ is *real* if $C \subset \mathbb{C}^2$ is an analytic curve invariant under conjugations, and z is its real singular point.

Theorem (Gusein-Zade, 1974)

Any plane curve singularity is topologically equivalent to a real one.

Definition

A *nodal deformation* of a singularity (C, z) inside the Milnor ball $\mathbf{B}$ is an analytic family of curves $C_t \cup \mathbf{B}$ such that

- the complex parameter t varies in a (small) disk centered at $0 \in \mathbb{C}$;
- for $t = 0$, we recover the original curve $C_0 = C$;
- each curve C_t is smooth along $\partial\mathbf{B}$, and intersects $\partial\mathbf{B}$ transversally;
- for any $t \neq 0$, the curve C_t has only ordinary nodes inside $\mathbf{B}$;
- the number of these nodes does not depend on t.

Singularities and Morsifications

Definition

a *real morsification* of a real singularity (C, z) is obtained by taking a nodal deformation $(C_t \cap \mathbf{B})$ which is equivariant with respect to complex conjugation, and restricting t to a (small) interval $[0, \tau) \in \mathbb{R}$. We also impose non-degeneracy conditions.

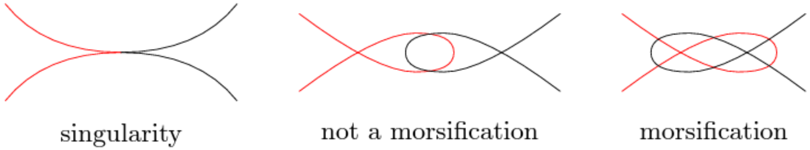


Figure: A Singularity and its morsification [FPST22]

A *divide* is a an immersion of lines and circles into a disk with only simple intersections. Basically, a doodle without tangency or triple intersections!

Quivers from Doodles

From a morsification, we draw a quiver.

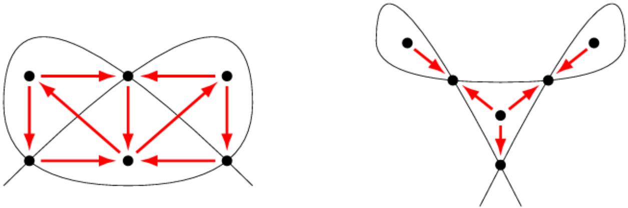


Figure: Obtaining quivers from divides [FPST22].

For a divide D we shall denote the quiver obtained from D by $Q(D)$.

Conjecture ([FPST22], 2022)

Given two real morsifications of real isolated plane curve singularities:
same complex topological type $\Longleftrightarrow$ *quivers are mutation equivalent.*

Results

Question

Can we classify which divides yield mutation finite quivers?

Not yet, but...

Question

*Can we classify which **scannable** divides yield mutation finite quivers?*

Theorem (UMNREU, 2022)

YES! *(We have a list.)*

As for the general case, we have some partial progress...

Theorem (UMNREU, 2022)

Any divide has an associated 'Gauss diagram', which is a discrete object that allows us to analyse divides combinatorially.

Contents

Scannable divides

Definition

A *scannable divide* on n strands consists of n parallel lines and finitely-many crossings between adjacent strands. We may also "cap" adjacent strands by connecting them together on the left or the right.

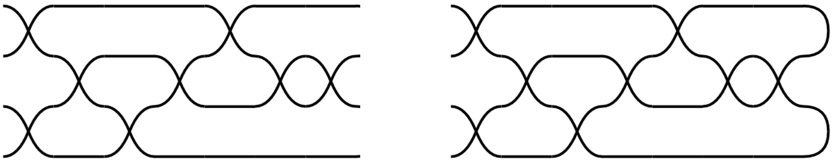


Figure: A scannable divide on 4 strands [FPST22].

Scannable Divides to Words

Definition

Given an uncapped scannable divide D on n strands, we number the gaps between then from 1 to $n-1$. The *word* associated to D is the sequence of crossings, from left to right.

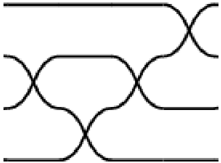


Figure: A scannable divide on 4 strands with word 2321 [FPST22].

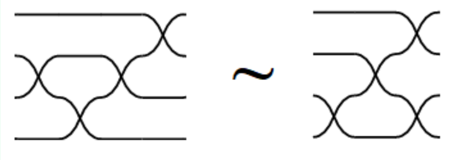
We obtain a *sub-word* by removing any subset of the letters of a word. Ex. 231 and 22 are sub-words of 2321.

Scannable Divides to Words

Definition

We say two words are *equivalent* if one may be transformed into the other by horizontal or vertical reflection, by a braid move, or by commutation of non-consecutive letters. A word is *reducible* if it is equivalent to one which may be decomposed into parts on disjoint intervals of letters.

Example



The word 2321 is equivalent to 1232, 2123, 3231, 3213, etc....
All these words are reducible.

The Uncrossing Lemmas

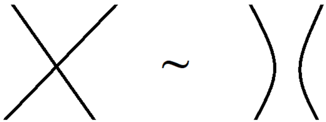


Figure: Uncrossing

Conjecture (Fomin, Pylyavskyy)

Given any divide D uncrossing an intersection gives a quiver which is a subtype of the original quiver.

Theorem (UMNREU, 2022)

If a divide D has block decomposable quiver, then 'uncrossing' one intersection preserves block decomposability.

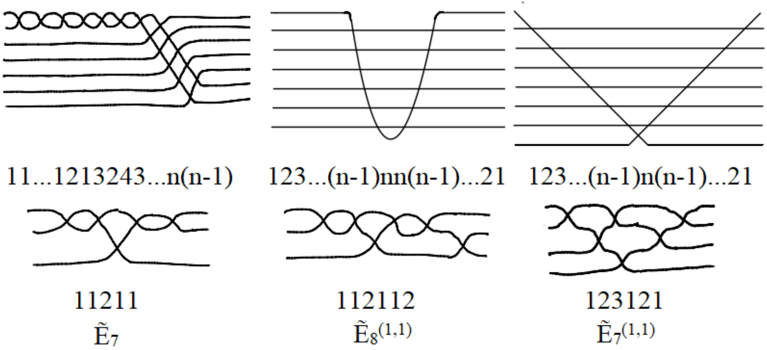
Corollary

If a scannable word is block decomposable, so is any sub-word.

The Uncapped Classification

Theorem (UMNREU 2022)

Any uncapped scannable divide D which is mutation finite is composed of irreducible words, each of which are of one of the following forms, up to word equivalence.



The Complete Classification

Theorem (UMNREU 2022)

Any scannable divide D which is mutation finite is obtained from an uncapped divide by capping in one of finitely many allowable ways.

Example

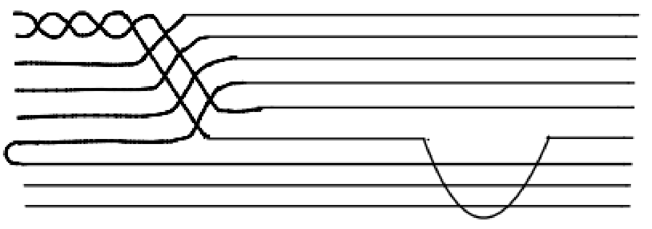
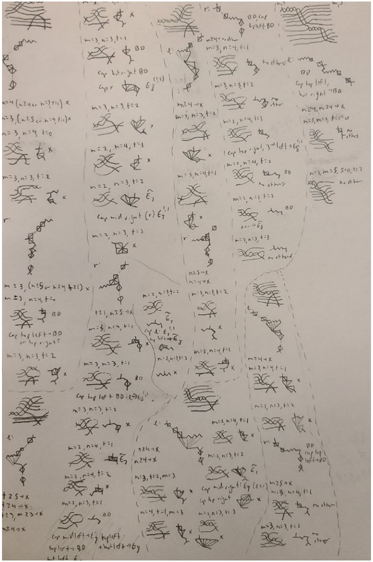


Figure: Capping a reducible scannable divide

The Complete Classification

The capping cases are **long**.



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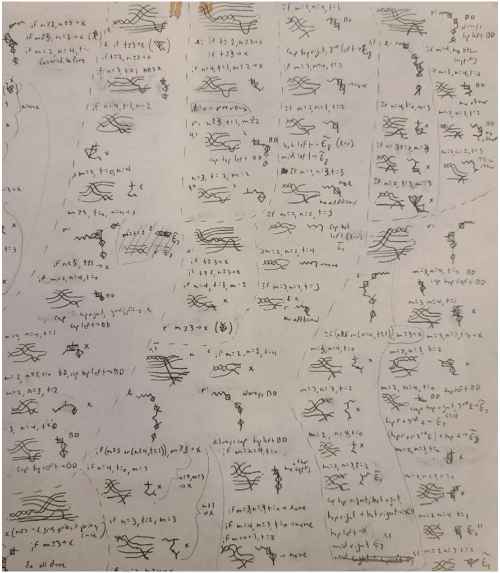

The Complete Classification

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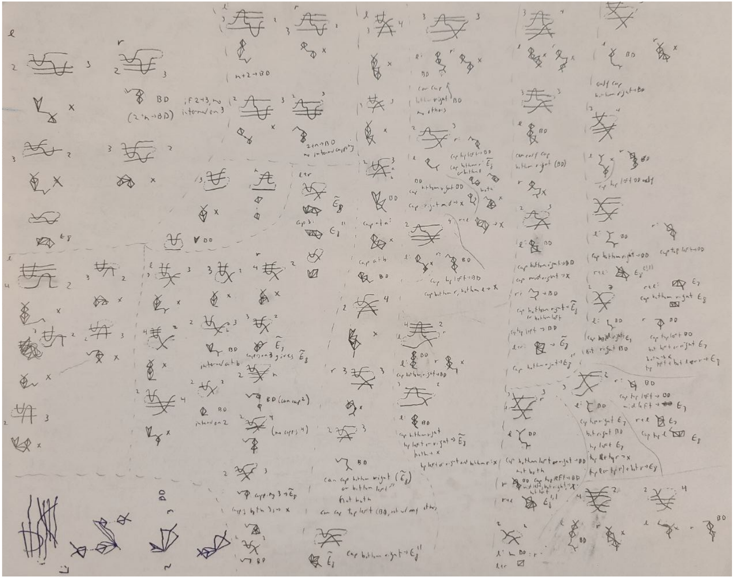

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Gauss Diagrams

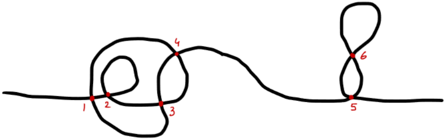
A *generalized Gauss diagram* is a combinatorial picture which describes a general divide.

Theorem (UMN REU 2022)

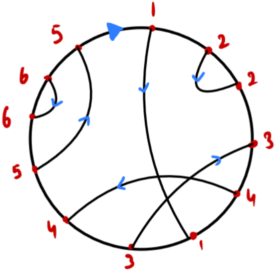
There is a one-to-one correspondence between valid Gauss diagrams and general divides under certain combinatorial and topological equivalences respectively.

We know how to get from Gauss diagrams to quivers!

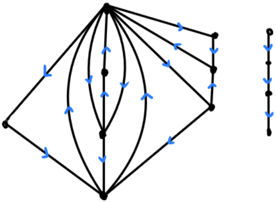
Example



General divides



Generalized Gauss diagrams



Quivers

Rules of Evolution

We define *Reidemeister moves* for divides in the natural way.

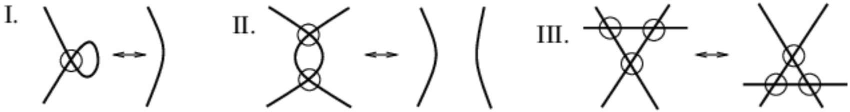


Figure: The three Reidemeister moves

Lemma

Any 1-strand unbounded divide is equivalent to any other through Reidemeister moves.

The Reidemeister moves along with the newly defined operations of *gluing* and *splitting* allow us to recursively construct all divides from the trivial divide!

Final steps?

More excitingly, we have a complete list of equivalent rules of evolution for Gauss diagrams.

Gauss diagrams retain all the topological information from a divide in a completely discrete structure. We can now analyze divides with a computer.

Question

Can we use these two properties together to find a finite set of 'minimal infinite' subdiagrams?

Answer

Hopefully!

Contents

Conclusion & Thanks

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