SCANNABLE DIVIDES OF FINITE MUTATION TYPE

PRANJAL DANGWAL, RYAN LYNCH, SON VAN THANH NGUYEN, ETHAN PESIKOFF;
MENTOR: PAVLO PYLYAVSKYY; TA: SYLVESTER ZHANG

CONTENTS

1. INTRODUCTION

Ever since the inception of cluster algebras by S. Fomin and A. Zelevinsky [FZ01], the combinatorial theory of quiver mutations has been found to bridge seemingly unrelated areas of mathematics. Recent work by S. Fomin, P. Pylyavskyy, E. Shustin, and D. Thurston elucidates a novel and remarkable connection between the combinatorial theory of quiver mutations and the topology of complex plane curve singularities. Specifically, the authors describe a method to extract a quiver from a real morsification of a plane curve and relate topological properties of the curve with combinatorial properties of the associated quiver. They prove the following theorem.

Theorem 1.1. [FPST22] *Let Q be the quiver obtained from a real morsification of some singularity of a complex plane curve. Then Q uniquely determines the complex topological type of the singularity.*

Furthermore, the authors postulate an even stronger statement in their main conjecture.

Conjecture 1.2. *Given two real morsifications of real isolated plane curve singularities, the following are equivalent:*

(1) the two singularities have the same complex topological type;
(2) the quivers associated with the two morsifications are mutation equivalent.

This conjecture would allow both a complete topological classification of singularities via purely combinatorial and algebraic means. Moreover, further investigation into the relationship between invariants of the quiver under mutation and topological invariants of the divide may have deep implications to the topology of plane curve singularities.

In light of this recent work, we investigate the mutation equivalence classes of quivers arising from algebraic divides. (An algebraic divide, loosely speaking, is the set of real points in some ball

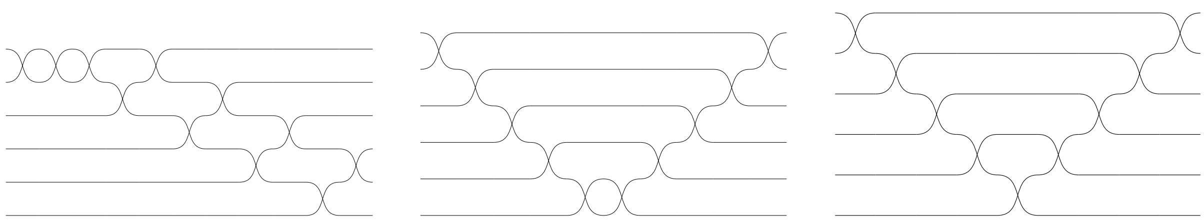
Date: June 2022.

around a singularity, viewed as an immersion of segments and circles in a disk. See Definition 4.5.) Specifically, we begin a work towards a classification of which such quivers are of finite mutation type. Pending a proof on Conjecture 4.9, this work then provides as a corollary a classification of which underlying plane curve singularities have finite topological equivalence class.

We first examine *scannable divides* (Definition 5.1), a well a well-behaved subset of algebraic divides. In particular, significant progress towards Conjecture 4.9 has been made in the case of scannable divides. We discover the basic building block of a scannable divide to be an irreducible uncapped scannable divide (Definitions 5.2 and 5.8), in the sense that any scannable divide may be constructed in a natural way from irrecudible uncapped divide and a set of peripheral so-called cappings. We prove the following theorem.

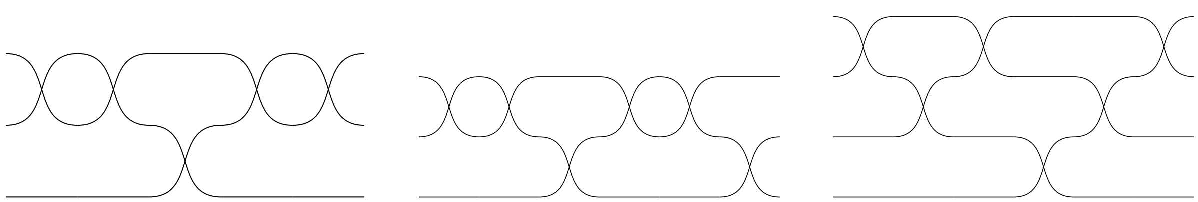
Theorem 1.3. *Any uncapped scannable divide whose quiver is connected is one of the following six types, expressed in word-form up to equivalence (as in Definition 5.3) for some positive integer* n*.*

- $111\ldots121324354\ldots n(n-1)$;
- $123\ldots(n-1)nn(n-1)\ldots21$;
- $123\ldots(n-1)n(n-1)\ldots21$;
- 11211;
- 112112;
- 121321.



(A) 1,1,1,2,1,3,2,4,3,5,4 (B) 1,2,3,4,5,5,4,3,2,1 (C) 1,2,3,4,5,4,3,2,1

FIGURE 1. The infinite families.



(A) 1,1,2,1,1 (B) 1,1,2,1,1,2 (C) 1,2,1,3,2,1

FIGURE 2. The exceptional cases.

Together with the finitely many ways to cap a scannable divide and preserve finiteness, Theorem 1.3 gives a complete classification of which scannable divides yield mutation finite quivers.

Along with the classification of scannable divides, we also show results about all divides. In particular, we have the following theorem.

Theorem 1.4. *If a divide* D *yields a mutation-finite divide, then uncrossing into a quiver* D' *yields a divide whose quiver is also mutation finite. Furthermore, if* D *is block-decomposable, the mutation* *class of D' is a subtype of the mutation class of D (meaning that some representative of the former is an induced subquiver of some representative of the latter).*

In particular, Theorem 1.4 allows us to equip the set of divides with a partial ordering which one divide is dominates another if it can be obtained by a series of uncrossings. Then any divide dominating a mutation infinite divide in this sense is also mutation infinite.

In yet more partial progress towards a complete classification of algebraic divides, we develop a novel combinatorial axiomatization of divides, from which corresponding quivers may be easily retrieved. We anticipate not only that such an axiomatization may facilitate the complete classification of divides by mutation finiteness of their quivers, but also that such a combinatorial tool might allow future researchers to analyse divides in a systematic and combinatorial way which has not yet been pursued.

2. Quivers

In this and the following section, we review quivers and quiver mutations and the mechanics by which we extract quivers from real morsifications of plane curves.

Definition 2.1. *A quiver is a tuple $Q = (V, E, s, t)$ where V and E are sets and $s : E \to V$ and $t : E \to V$ are maps. We call elements of V vertices and the elements of E are oriented edges or arrows. For an arrow $\alpha \in E$, $s(\alpha) \in V$ returns the starting vertex and $t(\alpha) \in V$ the terminal vertex.*

In this paper, we require a quiver Q to have no loops or oriented 2-cycles, namely the maps s and t satisfy:

(1) for any edge $\alpha \in E$, we have $s(\alpha) \neq t(\alpha)$;
(2) there do not exist edges $\alpha, \beta \in E$ such that $s(\alpha) = t(\beta)$ and $s(\beta) = t(\alpha)$.

Equivalently, we can simply describe a quiver to be a directed graph with no loops or 2-cycles.

Definition 2.2. *Let Q be a quiver with a vertex k. The quiver mutation μ_k transforms Q into a new quiver $Q' = \mu_k(Q)$ via a sequence of three steps:*

(1) for each oriented two-arrow path $i \to k \to j$, introduce a new arrow $i \to j$ (unless both i and j are frozen, in which case do nothing);
(2) reverse the direction of all arrows incident to the vertex k;
(3) remove all oriented 2-cycles.

For any quiver Q and any vertex k in Q, μ_k is an involution on Q.

Example 2.3. *Example of a quiver mutation. Mutating at the circled vertex.*

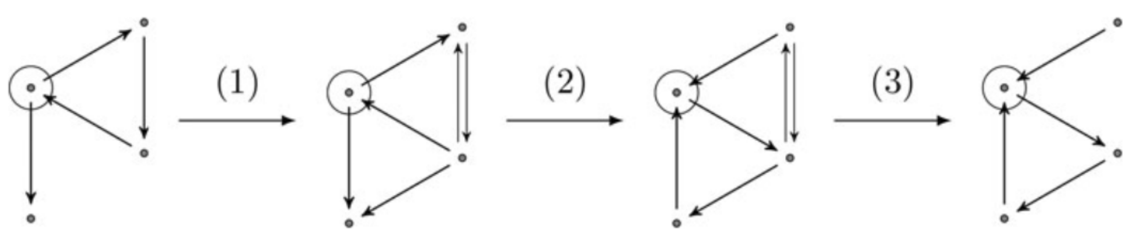
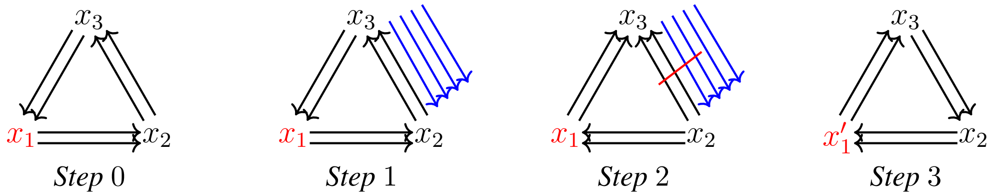


FIGURE 3. Example of quiver mutation from [Law16]

Example 2.4. *Another example of a quiver mutation. Mutating at x_1.*



Definition 2.5. *Let Q and Q' be quivers such that Q' is obtained by mutating some finite sequence of vertices in Q. Then, Q and Q' are said to be mutationally equivalent. Furthermore, the set of all quivers mutationally equivalent to Q is called the mutation class of Q.*

The mutation class of a quiver Q may be infinite, in which case we call Q a *mutation-infinite* quiver. If the mutation class of Q is finite, then it is *mutation-finite*. We shall also denote these notions by calling Q *infinite mutation type* or *finite mutation type* respectively.

Quivers have been completely classified based on their mutation type.

Definition 2.6. *A block is a quiver isomorphic to a single vertex or one of the the following quivers shown in Figure 4. Vertices marked in white are called outlets and vertices in black are named dead-ends.*

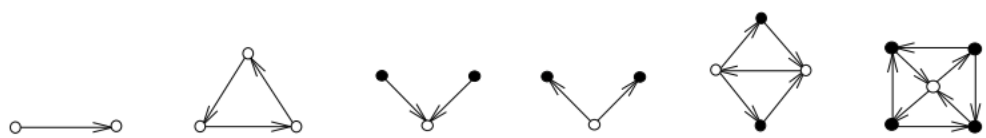


FIGURE 4. Blocks. Figures taken from [FST12]

Definition 2.7. *A connected quiver Q is called block-decomposable if it can be obtained from a collection of blocks by identifying outlets of different blocks along some partial matching. We shall have two edges with the same endpoints and opposite directions cancel out, whereas two edges with the same endpoints and direction form an edge of weight 2.*

If Q is a quiver with more than one components, then it is called block-decomposable if Q satisfies the above definition, or if Q is a disjoint union of several mutually orthogonal quivers satisfying the above definition. If Q is not block-decomposable, then we say that it is *non-decomposable*.

From [FST06] [sic]: "It is shown in [FST] that block-decomposable quivers have a nice geometrical interpretation: they are in one-to-one correspondence with adjacency matrices of arcs of ideal (tagged) triangulations of bordered two-dimensional surfaces with marked points (see [FST, Section 13] for the detailed explanations). Mutations of block-decomposable quivers correspond to flips of triangulations. In particular, this description implies that mutation class of any block-decomposable quiver is finite (indeed, the absolute value of an entry of adjacency matrix can not exceed 2). Another immediate corollary is that any subquiver of a block-decomposable is block-decomposable too."

Example 2.8. *A block decomposition of quivers, if it exists, is not necessarily unique. See, for example, Figure Figure 5*

FIGURE 5. Two distinct decompositions of a quiver into blocks.

Theorem 2.9. [FST06] *A quiver arises from an ideal triangulation of a bordered surface with marked points if and only if it is block-decomposable.*

Theorem 2.10. [FST12] *Any quiver Q, $|Q| \geq 3$, with finite mutation lass is either a quiver arising from the triangulation of a bordered two-dimensional surface or a quiver mutationally equivalent to one of the eleven exceptional types: $E_6, E_7, E_8, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, X_6, X_7$.*

Representatives of the exceptional quivers are illustrated in 6.

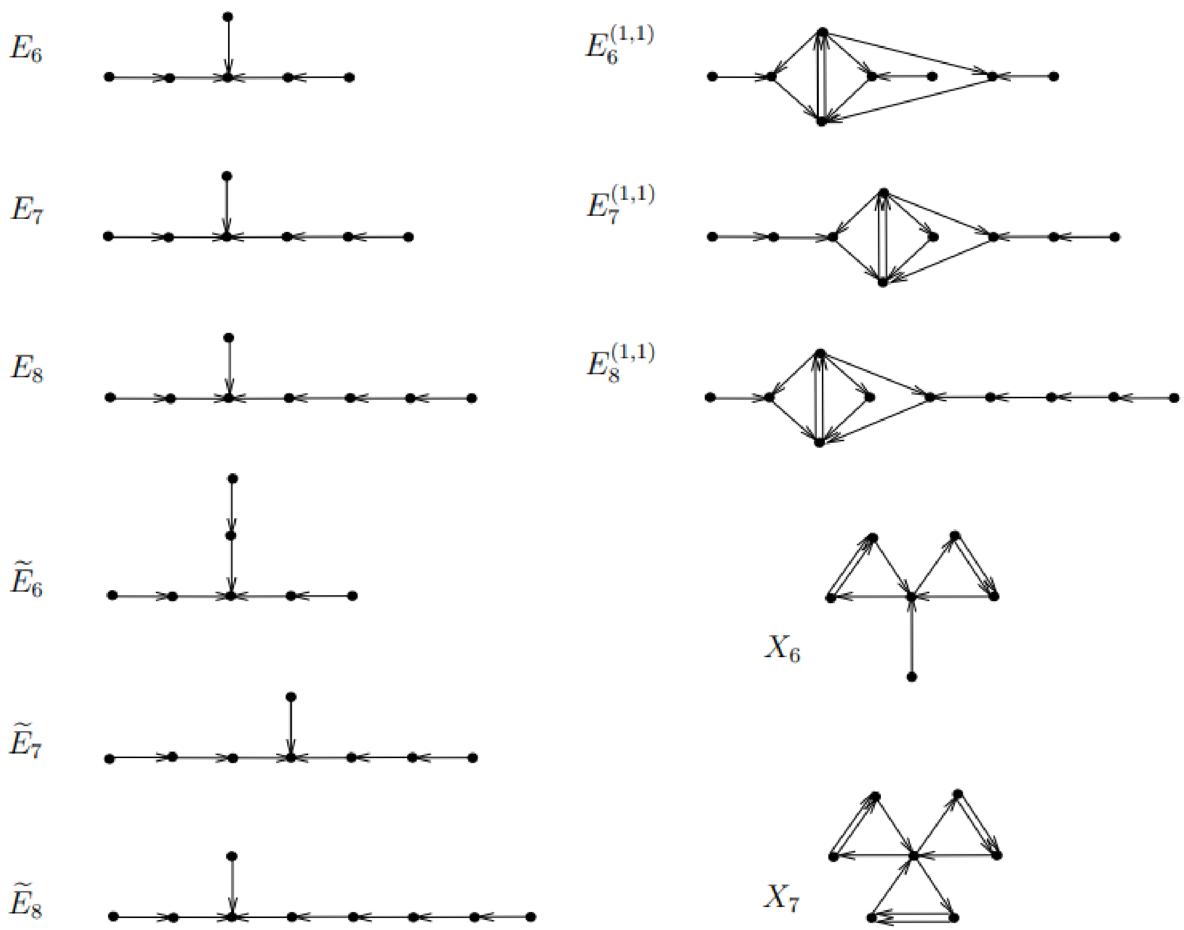


FIGURE 6. The eleven exceptional types.

The corollary below follows as an immediate result of the last two theorems.

Corollary 2.11. *Any mutation finite quiver with at least three vertices is either block decomposable or isomorphic to one of the eleven exceptional types.*

3. Quivers of Finite Mutation Type

Proposition 3.1. *A block type V quiver comes from exactly one connected divide, the intersection of two circles.*

Proof. Assume for contradiction that there is a divide D whose quiver is Q, a single block V. Let v_4 be the vertex of degree 4, and let v_0, v_1, v_2, v_3 be the other four vertices, where v_i is adjacent to $v_{\pm i}$, where i is taken mod 4. First, assume that v_4 corresponds to an intersection point p in D. Then v_4 is connected to exactly the vertices corresponding to the bounded faces of D with p on their boundary. Thus, there must be exactly 4 such faces, and then the vertices associated to these faces are connected to each other in cyclic order. The central intersection along with these four faces thus correspond to the complete set of vertices in Q. However, two adjacent bounded regions share at least two intersections, so Q must contain an additional vertex, a contradiction.

Now suppose that v_4 corresponds to a bounded region $A \subset D$. That is, let A be an n-gon for $n \geq 1$. If $n \geq 3$, this is impossible because the vertices of the quiver corresponding to the vertices of the n-gon A are pairwise unconnected. If A_4 is a 2-gon, then in order to have $v_4 = 4$, both adjacent regions (across each side of A) must be distinct and bounded, giving five total vertices in the patter of a block type V. Thus, our divide cannot have any additional intersections or bounded regions. Then D is topologically equivalent to the union of two circles. □

Proposition 3.2. *No block decomposable quiver Q strictly containing a block type V ever arises from a connected divide D.*

Proof. Assume for contradiction that there is such a divide D. The outlet v of the block type V in Q has degree at least 5. Then since each vertex in D is adjacent to at most four bounded regions, v must correspond to a bounded region in D, specifically an n-gon A for $3 \leq n \leq 5$. If $n = 5$, then none of the regions adjacent to A in D may be bounded (lest v have degree at least 6). But then all vertices adjacent to v are pairwise unconnected, meaning v cannot be the outlet of a type V. If $v = 4$, then similarly only one of the four neighboring regions is bounded, and the resulting quiver can have undirected 4-cycle whose vertices are all adjacent (but not equal to) v. Hence again we have a contradiction. Finally, suppose $n = 3$, and then two of the regions adjacent to A in D may be bounded. Again, no four vertices adjacent to v form an undirected 4-cycle. Then we have a contradiction in all cases, and the statement follows. □

Conjecture 3.3. *For every block decomposable quiver Q arising from an algebraic divide D, there is a block decomposition of Q not using blocks of type II.*

Lemma 3.4. *Let Q be a block decomposable quiver, and let $v \in Q$. Then the induced subquiver $Q \setminus v$ is also block decomposable.*

Proof. Fix a block decomposition of Q. We proceed by casework and explain how to reach a block decomposition of $Q \setminus v$ from that of Q. Suppose that v is contained in a certain block P, and suppose we can block decompose $P \setminus v$ st all outlets of P (except perhaps v) are outlets of the new block decomposition. In particular, we allow the decomposition of $P \setminus v$ to be empty. Then we may block decompose $Q \setminus v$ by replacing all Pv. with such decompositions. Then in particular it suffices to show that for any block P and $v \in P$, that $P \setminus v$ is block decomposable st the outlets of P remain outlets of $P \setminus v$. We call such a block decomposition of $P \setminus v$ happy.

Suppose P is type I. Then the empty decomposition is happy. If v is part of a block type II along with two other vertices a, b, then a single type I connecting a and b is happy. If v is the central vertex of a type III along with with a, b, then the empty decomposition is happy. If v is not the source of its type II, then a single type I connecting a and b is happy. If v is the outlet of a type IV, with the other outlet being a and the remaining vertices c, d, then a type II with outlet a is happy. If v is not an outlet of its type IV, then a type II connecting the other three vertices is happy. If v is the outlet of a type V with other vertices a, b, c, d, then four type Is arranged in a cycle $abcd$ form a happy decomposition. If a rather than v is the outlet of a type V, then a type IV with outlet at a is a happy decomposition. These being all the block types, we conclude that $Q \setminus v$ is block decomposable. □

Corollary 3.5. *Any induced subquiver of a block decomposable quiver is also block decomposable.*

Definition 3.6. *Let Q be a quiver containing vertices a, b. We define an operation on Q called $M_{a,b}(Q)$ as follows. First, add a new vertex a' to Q such that for all $v \in Q$, the (signed) edge weight from a' to v is the sum f the edge weights from a to v and from b to v. We then remove vertices a and b.*

Lemma 3.7. *Let Q be a block decomposable quiver, and let v, w be adjacent vertices of Q. Then $M_{v,w}(Q)$ is also block decomposable.*

Proof. Fix a block decomposition of Q. Since Q is block decomposable, then the edge vw must be contained in either one block P_1 or two blocks P_1, P_2. (No more than two, since each outlet may only be shared by two blocks.) We mean by $P_1 \cup P_2$ the quiver formed by these two blocks. If P_1, P_2 may be replaced by another block or pair of blocks P_1', P_2' (or an empty set of blocks) which form a block decomposition of $M_{v,w}(P_1 \cup P_2)$ such that the outlets of $P_1 \cup P_2$ are also outlets of $P_1' \cup P_2'$ (except v, w), then $M_{v,w}(Q)$ is also block decomposable. We call such a block decomposition $P_1' \cup P_2'$ cheerful. Then it suffices to consider what these blocks P_1, P_2 might be, and we proceed by casework.

First, we consider the case where there is only one block P_1 joining v, w. If P_1 is type I, then the empty block decomposition is cheerful. If P_1 is type II, then the empty decomposition is also cheerful. If P_1 is a type III containing v, w, a, then a single type I connecting v' to a is cheerful. If v, w are the two outlets of a type IV, then the empty decomposition is cheerful. If v, a are the outlets of a type IV also containing w, b, then two type Is from v' to b and b to a form a cheerful decomposition. If v, w, a, b, c constitute a type V with outlet a, then a type III connecting a, b, c and type Is connecting b, v' and c, v' form a cheerful decomposition. If v is the outlet of a type V also containing a, b, c st a is the vertex not adjacent to w, then a type I from v' to a and a type III with outlet at a form a cheerful decomposition.

Now we consider cases where vw has weight two. Notice then that each P_1, P_2 has at least two outlets and thus are either types I, II, or IV, and hence there are only six cases to consider. In every such case, the empty decomposition is cheerful. □

Conjecture 3.8. *If Q is mutation equivalent to an exceptional case quiver, then $M_{v,w}(Q)$ is of finite mutation type for any $v, w \in Q$.*

Remark. We tested this conjecture by iterating a nondeterministic method that works as follows: For each such Q and each edge (u, v), perform a random mutation sequence of length $500 \cdot |Q|$ on $M_{u,v}(Q)$. If there are triple edges present, then $M_{u,v}(Q)$ is of infinite mutation type. If there are no triple edges, then there is a high probability that $M_{u,v}(Q)$ is of finite mutation type. After repeatedly applying this method, no mutation infinite quivers were found.

Proposition 3.9. *Let D be a divide whose quiver is mutation finite. Let v be an intersection of D such that of the four regions a, b, c, d adjacent to v (labeled in cyclic order), all those which are bounded are pairwise distinct. Then uncrossing the crossing at v preserves mutation finiteness.*

Proof. WLOG, it suffices to consider an uncrossing which connects regions a and c by symmetry. If either a, c is unbounded, then such an uncrossing results in the deletion of the vertices at v, a, c (if they existed before). If both a, c are bounded regions, then the uncrossing results in the quiver obtained by first merging v with a and then v' with c. Thus, uncrossing at v corresponds on the quiver level to a combination of mergers and deletions. Note that deletion always preserves mutation finiteness because an induced subquiver of a mutation finite quiver is mutation finite and deletion corresponds to taking induced subquivers. Hence if D is block decomposable, the result follows by Lemma 3.7, and if D is an exceptional case, the result follows by Conjecture 3.8. $\square$

4. Divides

We begin by recalling definitions. By singularity, we refer to the *germ* $(C, z) \in \mathbb{C}^2$ of a reduced analytic curve C in the complex plane $\mathbb{C}^2$ at some singular point $z \in C$. Without loss of generality, we assume $z = 0$.

Definition 4.1. *A nodal deformation of a singularity (C, z) inside the Milnor ball $\boldsymbol{B}$ is an analytic family of curves $C_t \cup \mathrm{B}$ such that*

- *the complex parameter t varies in a (small) disk centered at $0 \in \mathbb{C}$;*
- *for $t = 0$, we recover the original curve $C_0 = C$;*
- *each curve C_t is smooth along $\partial\mathrm{B}$, and intersects $\partial\boldsymbol{B}$ transversally;*
- *for any $t \neq 0$, the curve C_t has only ordinary nodes inside $\boldsymbol{B}$;*
- *the number of these nodes does not depend on t.*

Definition 4.2. *a real nodal deformation of a real singularity (C, z) is obtained by taking a nodal deformation $(C_t \cap \boldsymbol{B})$ which is equivariant with respect to complex conjugation, and restricting t to a (small) interval $[0, \tau) \in \mathbb{R}$.*

Definition 4.3. *A real morsification of a real singularity (C, z) is a real nodal deformation $C_t = \{f_t(x, y) = 0\}$ such that*

- *all critical points of f_t are Morse (i.e., with non-degenerate Hessian);*
- *all saddle points of f_t are at the zero level (i.e., lie on C_t).*

It is conjectured by [FPST22] that every real plane curve singularity possesses a real morsification. We now recall how to reach a quiver from such a real morsification.

Definition 4.4. *Loosely speaking, a divide D in a closed disk $\boldsymbol{D} \subset \mathbb{R}^2$ is the image of a generic relative immersion of a finite set of intervals and circles into $\boldsymbol{D}$. More precisely, the images of immersed intervals and circles, collectively called the branches of D, must satisfy the conditions (D1)–(D6) below. In particular:*

- *(D1) the immersed circles do not intersect the boundary $\partial\boldsymbol{D}$;*
- *(D2) the immersed intervals have pairwise distinct endpoints which lie on $\partial\boldsymbol{D}$; moreover these immersed intervals intersect $\partial\boldsymbol{D}$ transversally;*
- *(D3) all intersections and self-intersections of the branches are transversal;*
- *(D4) no triple (self-)intersections are allowed.*

We are only interested in the topology of a divide. That is, we do not distinguish between divides related by a diffeomorphism between their respective ambient disks.

The connected components of the complement $\boldsymbol{D} \setminus D$ which are disjoint from $\partial\boldsymbol{D}$ are the regions of D. We say two regions are adjacent if the intersection of their closures contains a 1-cell. The closure of the union of all regions and all singular points of D (its nodes) is called the body of the divide, denoted $I(D)$. We require that

- *(D5) the body of the divide is connected, as is the union of its branches;*
- *(D6) each region is homeomorphic to an open disk.*

In what follows, we don't always draw the boundary of the ambient disk $\boldsymbol{D}$.

Definition 4.5. *Any real morsification $(C_t)_{t\in[0,\tau)}$ of a real plane curve singularity (C, z) defines a divide in the following natural way. The sets $\mathbb{R}C_t$ of real points of the deformed curves C_t, for $0 < t < \tau$, are all isotopic to each other in the "Milnor disk" $\boldsymbol{D} = \mathbb{R}\boldsymbol{B} \subset \mathbb{R}^2$ consisting of the real points of the Milnor ball $\boldsymbol{B}$. Each real curve $\mathbb{R}C_t \cap \boldsymbol{D}$, viewed up to isotopy, defines the divide associated with the morsification. Conditions (D1)–(D4) and (D6) above are readily checked. Condition (D5) follows from the connectedness of the Dynkin diagram of a singularity and from Guseın-Zade's algorithm* [FPST22] *that constructs this diagram from a divide. A divide arising in this way from a real morsification is called algebraic.*

As an intermediate step on the journey from morsification to quiver, we recall the notion of an AΓ-diagram.

Definition 4.6. *Given a divide D, its A'Campo-Guseın-Zade diagram $A\Gamma(D)$ ($A\Gamma$-diagram for short) is a vertex-colored graph constructed as follows:*

- *place a vertex at each node of D, and color it black;*
- *place one vertex into each region of D; color these vertices $\oplus$ or $\ominus$ so that adjacent regions receive different colors (signs), and non-adjacent regions sharing a node receive the same color;*
- *for each 1-cell separating two regions, draw an edge connecting the vertices located inside these regions;*
- *for each region R, say bounded by k one-dimensional cells, draw k edges connecting the nodes on the boundary of R to the vertex located inside R; these edges correspond to the k distinct (up to isotopy) ways to draw a simple curve contained in R (except for one of the endpoints) connecting the interior vertex to a boundary node.*

Finally, [FPST22] provides the following way to reach a quiver from a an AΓ-diagram (and hence from a real morsification).

Definition 4.7. *Given a divide D, its associated quiver Q(D) is constructed from the $A\Gamma$-diagram $A\Gamma(D)$ as follows:*

- *first, orient the edges of $A\Gamma(D)$ using the rule $\cdot \to \oplus \to \ominus \to \cdot$;*
- *then remove the marking of the vertices.*

Since we consider quivers up to global reversal of arrows, the choice of signs in the $A\Gamma$-diagram does not matter.

Furthermore, we have the following theorem.

Theorem 4.8. [FPST22] *Let $Q(D)$ be the quiver obtained from an algebraic divide D which is associated with a real morsification of some singularity. Then $Q(D)$ uniquely determines the complex topological type of the singularity.*

Finally, we present the underlying conjecture which arises from this line of inquiry, on which [FPST22] makes partial progress.

Conjecture 4.9. *Given two real morsifications of real isolated plane curve singularities, the following are equivalent:*

- *the two singularities have the same complex topological type;*
- *the quivers associated with the two morsifications are mutation equivalent.*

5. Scannable Divides of Finite Mutation Type

Definition 5.1. [FPST22] *A scannable divide is a divide D drawn inside a rectangle of the form $[a_0, a] \times [b_0, b] \subset \mathbb{R}^2$ so that the following conditions hold, for some $a_0 < a_1 < a_2 < a$. For every point (x_0, y_0) on D such that the tangent line to a local branch of D at (x_0, y_0) is vertical (i.e., given by the equation $x = x_0$), we require that*

- *(x_0, y_0) is a smooth point of D (i.e., not a node);*
- *either $x_0 = a_1$ or $x_0 = a_2$;*
- *if $x_0 = a_1$, then the local branch of D lies to the right of the tangent;*
- *if $x_0 = a_2$, then the local branch of D lies to the left of the tangent.*

Notably, all scannable divides are algebraic (i.e., they arise from real morsifications).

Definition 5.2. *We say a scannable divide D is uncapped if there are no points (x_0, y_0) on D such that the tangent line to a local branch of D at (x_0, y_0) is vertical. Else, we say D is capped.*

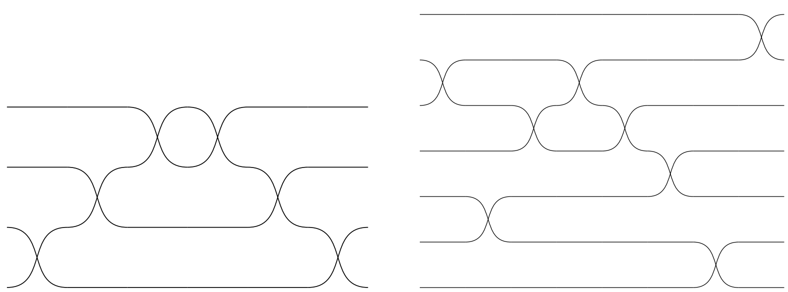


FIGURE 7. Uncapped scannable divides.

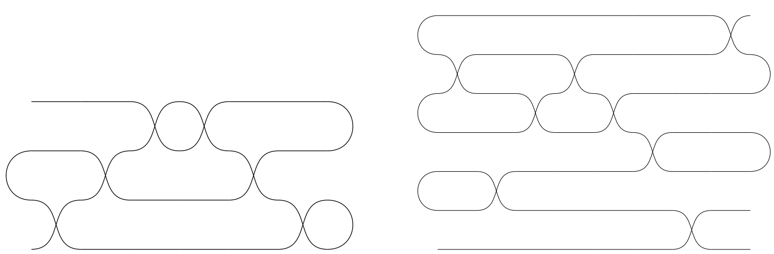


FIGURE 8. Capped scannable divides.

To each uncapped scannable divide D with $n + 1$ strands we may associate a word $w(D)$ in the letters $\{1, 2, \ldots, n\}$. Recall that in any such uncapped scannable divide D, each vertical line intersecting with D must intersect D in n places. We may draw D so that if two such vertical lines ℓ_1, ℓ_2 intersect D and such that for some $\epsilon > 0$, the closest distance from either ℓ_1 or ℓ_2 to an intersection of strands in D is greater than ϵ, then the y-coordinate of the kth intersection of ℓ_1 with D is equal to the y-coordinate of the kth intersection of ℓ_2 with D. Then away from intersections of D, all points in D lie at one of n distinct y-coordinates. We may then label the horizontal strips or gaps between these y coordinates by the numbers $1, 2, 3 \ldots, n$, where 1 represents the highest such gap. Then observe that each intersection of D may be drawn to lie in exactly one such gap (and for convenience at exactly the vertical midpoint). Then from left to write, the order of intersections in D as labeled by gap numbers constitutes $w(D)$. Going forwards, we may refer to $w(D)$ simply as D, since the sets of words and uncapped scannable divides are in bijection.

By $[a, b]$ for $a, b \in \mathbb{Z}$, we mean the set of all integers at least a and no greater than b. By $[a]$ for $a \in \mathbb{Z}^+$ we mean $[1, a]$.

Definition 5.3. *We say two words w, v on letters $[n]$ are equivalent if w may be transformed into v via one or more of the following actions.*

(1) Horizontal reversal: reversing the order of letters in a word
(2) Vertical reversal: swapping the letters $1 \longleftrightarrow n, 2 \longleftrightarrow (n-1), \ldots$.
(3) Vertical shift: adding ± 1 to each letter, allowed iff the result word is still on letters $[n]$.
(4) Braid move: replacing a sequence $k, k \pm 1, k$ with $k \pm 1, k, k \pm 1$.

Example 5.4. *Consider the following words.*

(1) 121543;
(2) 345121;
(3) 432656;
(4) 543767;
(5) 543676.

The second word is obtained by the first via horizontal reversal, the third from the second by vertical reversal (with $n = 6$), the fourth from the third by a vertical shift, and the fifth from the fourth y a braid move. Thus, all five above words are equivalent.

In the future, we often omit the statement 'up to equivalence' when context is clear.

Lemma 5.5. *Any two equivalent words w, v in $[n]$ represent divides whose quivers are mutation equivalent.*

Proof. It suffices to show that if v, w differ by any of the four actions in 5.3, they represent divides with mutation equivalent quivers. It is known that braid moves one a divide preserve mutation equivalence of the associated quivers. The horizontal and vertical reversal correspond to rigid transformations of D in $\mathbb{R}^2$ and hence also to rigid transformations of the associated quivers. A positive (negative, respectively) vertical shift corresponds to moving the highest (lowest) strand in D, a horizontal line segment disjoint from the rest of D to the bottom (top) of D and then shifting all of D upwards (downwards). Since a removing or adding a disjoint line segment does not alter the quiver (by not introducing any new intersections or bounded regions to the divide), and as the vertical shift is a rigid transformation, vertical shift preserves quiver mutation class. □

Now we introduce a key relationship between words.

Definition 5.6. *Let w, v be words on $[n]$. We say w is a subword of v if deleting some subset of the letters in v (and preserving the order of the rest) yields a word equivalent to w.*

Lemma 5.7. *If an uncapped scannable divide D contains some mutation-infinite divide D' as a subword, then D is also mutation-infinite.*

Proof. Notice that for any intersection v of D, the surrounding bounded regions are all distinct. Notice that the regions above and below v lie in different gaps then both each other and than the regions to the right and left. Hence they cannot lie in the same bounded regions as each other or as the right and left regions. Furthermore, the right an left regions are either distinct from each other or unbounded. Hence, we meet the conditions of Proposition 3.9.

Observe that deleting the k-th letter of $w(D)$ corresponds to uncrossing the k-th intersection of D, rectifying horizontally. Therefore, any subword $w(D')$ of $w(D)$ corresponds to the divide which is obtained from D by removing all crossings in $w(D) \setminus w(D')$. By 3.9, mutation finiteness of D then implies mutation finiteness of D'. □

Definition 5.8. *We say a word w on $[n]$ is reducible if it equivalent to the concatenation of two words w_1, w_2 where w_1 is on $[k]$ and w_2 is on $[k+1, n]$. We allow w_1 or w_2 to be the empty word.*

Notice that then the concatenation of two reducible words on disjoint sets of letters is also reducible.

Lemma 5.9. *If a word is reducible, its associated quiver is the disjoint union of the quivers associated to w_1, w_2.*

Proof. If one of w_1, w_2 is empty, the statement holds trivially. Then let w_1, w_2 both be nonempty. Let $D(w)$ be the divide corresponding to w. If w either does not contain the letter k or does not contain the letter $k+1$, then $D(w)$ is not connected. Specifically, D has empty intersection with the k-th or $(k+1)$-st strip.

Else, notice that $D(w_1)$ has empty intersection with the $(k+1)$st strip and $D(w_2)$ has empty intersection with the kth strip. Then there are no bounded regions which intersect the kth or $(k+1)$st strip. Notice that for any scannable divide D, an edge in the corresponding quiver $Q(D)$ either connects an intersection in the mth strip to a bounded region in the mth or $(m \pm 1)$st strip, or else it connect a bounded region in the mth strip to one in the $(m \pm 1)$st strip. Therefore, any path in $Q(D)$ between points corresponding to points in the m_1th and m_2th strip of D for $m_1 < m_2$ must contain points in each strip m for $m_1 < m < m_2$. In our case, there are no bounded regions in the kth or $(k+1)$st strip, so there cannot be a path in $Q(D(w))$ which connects points in the pth strip to points in the qth strip for $p \leq k, q \geq k+1$. In other words,$D(w_1)$ and $D(w_2)$ are disjoint as subsets of $D(w)$. □

Then when classifying uncapped scannable divides, it suffices to classify the irreducible words because to check whether a quiver is mutation finite, it suffices to check the its connected components. Specifically, we have the following theorem.

Theorem 5.10. *If D is an uncapped irreducible scannable divide whose quiver is mutation finite, the D is of one of the following forms.*

(1) 1,1,1,2,1,3,2,...,n,n-1
(2) 1,2,...,n-1,n,n,n-1,...,2,1
(3) 1,2,...,n-1,n,n-1,...,2,1
(4) 1,1,2,1,1
(5) 1,1,2,1,1,2
(6) 1,2,1,3,2,1.

Furthermore, all mutation finite divides are given by capping the above six types according to one of finitely many ways.

Remark. Although the three infinite families of 5.10 are distinct as scannable divides, they all can be seen to share the same structure. Clockwise rotations of the second and third families yield divides isomorphic to the first family, up to capping. As we show below, the quivers for all three infinite families are block-decomposable and arise from the m-punctured disk.

Lemma 5.11. *If a word w on $[n]$ does not contain each letter at least once, it is reducible.*

Proof. If w does not contain either n or 0, then it is reducible by defn. Else, suppose $k \notin w$ for $1 < k < n$. Then we may commute any letter greater than k with any letter less than k, so that we may write w as $w_1 w_2$ with w_1 in $[0, k-1]$ and w_2 in $[k+1, n]$. ☐

Definition 5.12. *We say a word w is minimally mutation infinite if its corresponding quiver is mutation infinite, but the quiver corresponding to every subword of w is mutation finite.*

Lemma 5.13. *The following words are minimally mutation infinite. We will in the future refer to these words by their number, and we consider them up to equivalence.*

(1) $2, 1, 1, 1, 2$
(2) $1, 1, 2, 2, 1$
(3) $1, 1, 2, 1, 1, 1$
(4) $2, 1, 1, 3, 2$
(5) $2, 1, 3, 2, 2$
(6) $1, 2, 2, 3, 1$
(7) $1, 1, 3, 2, 1, 3$
(8) $2, 1, 4, 3, 2, 4$

Proof. All eight of the above strings are mutation infinite. However, the words with all instances of each letter placed consecutively and $2, 1, 1, 2$ and $1, 1, 2, 1$ and $1, 2, 1, 1, 1$ and $1, 1, 2, 1, 1$ and $2, 1, 3, 2$ and $1, 2, 3, 1$ and $1, 3, 2, 1, 3$ are mutation finite, and they constitute all the subwords of the above list of words. ☐

6. PROOF OF THEOREM 5.10 FOR UNCAPPED SCANNABLE DIVIDES.

Theorem 6.1. *Let D be an irreducible scannable divide such that $Q(D)$ is block-decomposable. Then $w(D)$, the word form of D is one of the following:*

(1) $1, 1, \ldots, 1, 2, 1, 3, 2, 4, 3, 5, 4, \ldots, n, n-1$
(2) $1, 2, \ldots, n-1, n, n, n-1, \ldots, 2, 1$
(3) $1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1.$

Lemma 6.2. *Quivers arising from the above three words are block decomposable, and they arise from a punctured disk in the standard sense.*

Lemma 6.3. *A type 1, 2, or 3 word for $n \geq 2$ is maximal block decomposable irreducible word with respect to adding letters. That is, inserting any letter at any point in the word either creates another such word or a non block decomposable word.*

Proof. Let w be a type 1 word. First, assume that $n \geq 3$. First, we consider inserting a 1. Inserting a 1 at the beginning creates another type 1, but inserting a 1 between the 2s produces the infinite subword 21132. Placing a 1 after the second 2 creates a subword 214324 if $n \geq 4$, or 113213 if there is at least 1 initial 1, or a type 1 if $n = 3$ and there is no leading 1. Inserting a 2 before the first 2 yields either a subword 21322. Inserting a 2 after the last 2 yields the word 21322. There is only one other place to insert a 2, between the two existing 2's but not adjacent to either (adjacent placing would yield a result identical to placing the 2 before or after both existing 2's. This placement is equivalent to a type 1 word via braid moves. Let $2 < k < n$. Inserting a k before the first existing k or after the second creates a subword 5. The only remaining placement yields a 1 via braid moves. Finally, consider inserting and n. Inserting one before the existing n yields either a subword 21132 or 214324. Placing an n at the end of the word creates a type 1 word via braid moves.

Now, let $n = 2$. Inserting a 2 at the end creates a word equivalent to $1, \ldots, 1, 2, 1$, a type 1 word. Suppose there are no leading 1s. Then inserting a 2 at the beginning creates a reducible word. Otherwise, the only mutation finite possibilities after inserting a 1 are the words $1, 2, 2, 1$ and $1, 2, 1, 2, 1$ and $2, 1, 1, 2, 1$ and $1, 2, 1, 1, 2, 1$. However, none of them are block decomposable.

Let w be a type 2 word. Inserting a letter from $[2, n]$ between the two n's creates an mutation infinite subword 10001, and inserting a 1 there creates a subword 12121, which is not block decomposable. Inserting a letter in $[1, n-1]$ anywhere else creates an infinite subword 11221, and placing an n yields a subword 12212, which is $\tilde{E}_7$ and hence not block decomposable. Having one occurence of the new letter 0 leads to a reducible quiver, and the only irreducible quivers attained by placing $n + 1$ are of type 3 or 2 on $[n + 1]$.

Now suppose w is a word of type 3. Inserting a 1 at the beginning or end of the word creates an mutation infinite subword of 11221, and inserting it anywhere else leads to a subword 12121. Inserting any letter in $[2, n-1]$ creates a mutation infinite subword 10001. Inserting n annywhere but adjacent to the existing n creates a mutation infinite subword 12231. Inserting any letter not in $[n]$ yields a reducible word.

Thus, having handled all cases, the statement follows. □

Corollary 6.4. *Any irreducible word which properly contains a type 1, 3, or 2 subword on at least 2 letters, and contains three of some letter is mutation infinite. The only exception is a type 1 word with many leading 1s.*

Lemma 6.5. *Let w be a word on $[n]$ with subword 1221. Then, w is a type 3 or 2 word.*

Proof. Assume the subword 1221 is $k, (k + 1), (k + 1), k$, without loss of generality. By Corollary 6.4, we may assume that w contains at most two of each letter. Suppose w only has exactly one instance of $k+2$ (or equivalently $k-1$ by vertical reflection). Then to maintain irreducibility of w, w must have subword $k, k+1, k+2, k+1, k$. If $n > k+2$, then in order to maintain irreducibility, w must have subword $k, k + 1, k + 3, k + 2, k + 3, k + 1, k$. If $n > k + 3$, then there must be an instance of $k+4$. Placing it between the two instances of $k+3$ creates a mutation infinite subword, and placing all instances of $k + 4$ before the first instance of $k + 3$ or after the last causes w to be reducible. Therefore, we must have subword $k, k + 1, k + 4k + 3, k + 2, k + 3, k + 4k + 1, k$. By the same logic, we must have a subword of the form

$$k, k + 1, n, n - 1, \ldots, k + 4, k + 3, k + 2, k + 3, k + 4, \ldots, n - 1, n, k + 1, k.$$

If there is only one instance of $k - 1$, then it must occurs between the two ks let w be reducible. But then we have an infinite subword of type 21132, and so there must be two instances of $k - 1$, positioned on the outsides of the two instances of k. Then inductively, the word w is

$$1, \ldots, k, k + 1, n, n - 1, \ldots, k + 4, k + 3, k + 2, k + 3, k + 4, \ldots, n - 1, n, k + 1, k, \ldots, 1.$$

This subword is equivalent by braid moves to a word of type 3. Note that our reasoning also implies that if there is any ℓ which appears only once in w then ℓ is the only such letter, and w is of type 3.

Now assume that every letter in w appears exactly twice. Given the subword $k, k + 1, k + 1, k$, both instances of $k + 2$ must lie between the instances of $k + 1$ in order to maintain block decomposability because neither of $k, k+2, k+1, k+1, k+2, k$ or $k, k+1, k+2, k+1, k+2, k$ is block decomposable. Similarly, both instances of $k-1$ must lie one before the first k and the other after the second k. Then inductively, our word is $1, 2, \ldots, n-1, n, n, n-1, \ldots, 1$, a type 2. □

Put another way, Lemma 6.5 shows that having a subword 1221 is a sufficient condition to show that a block-decomposable word is of type 3 or 2. In fact it is necessary also. We show below that if a block decomposable word does not contain a subword 1221, it is of type 1.

Proof of Theorem 6.1. By Lemma 6.5, it suffices to prove that block decomposable words not containing subwords equivalent to 1221 are of type 1. By Corollary 6.4, it suffices to consider words with at most two of each letter. Since w is irreducible, it contains a subword equivalent to $k, k+1, k$ (or else w reduces as words in $[1, k]$ and $[k, n]$). Suppose that there is only one instance of $k+1$. Then in order for w to be irreducible, it must contain subword equivalent to $k, k+2, k+1, k+2, k$, which falls under the conditions of Lemma 6.5 (by containing a subword 1221). Then either $k+1 = n$, or there are two instances of $k+1$. In the latter case, we must have a subword $k, k+1, k, k+1$ so as not to fall under Lemma 6.5. We next consider placing $k+2$ into the word. If there is only one such instance, we must have a subword $k, k+1, k, k+2, k+1$, as if the $k+2$ lies at the extremes of this subword, the word w is reducible. If w contains $k+3$, then in order for w not to be reducible, we must have that either w falls under Lemma 6.5, or else there are two instances of $k+2$ arranged in a subword $k, k+1, k+2, k, k+1, k+2$ (as $k+2, k, k+1, k+2, k, k+1$ is not block decomposable). The same logic holds for inserting letters smaller than k in descending order. Then inductively, w has a subword of the form $213243 \ldots nn-1$, a type 1. Then by lemma 6.3, w is a word of type 1. □

Theorem 6.6. *If $Q(D)$ is finite mutation type but not block decomposable, then it is equivalent to one of the following.*

(1) $1, 1, 2, 1, 1$
(2) $1, 1, 2, 1, 1, 2$
(3) $1, 2, 1, 3, 2, 1$.

The proof of Theorem 6.6 relies ultimately on a finite computer search, but beforehand we prove some useful lemmas. Going forward, we let $|D|$ denote the number of intersections in the divide D.

Lemma 6.7. *If D is an uncapped irreducible scannable divide on n strands, then $|Q| = 2|D| - (n-1)$.*

Proof. Recall that by Lemma 5.11, D contains a crossing at each gap, and further notice that each bounded region may naturally be assigned the gap number of the gap with which it has largest intersection. Then since each bounded region in the kth gap is bounded by two intersections, we have that the number of bounded regions in the kth gap is $|D|_k - 1$, where $|D|_k$ is the number of intersections in D in the kth gap. Then summing over all $n-1$ possible gaps, we find that

$$|Q| = |D| + \sum_{i=1}^{n-1} |D|_i - 1 = 2|D| - (n-1),$$

where $|Q|$ is the size of the quiver corresponding to D. □

Lemma 6.8. *If all instances of the largest (smallest) letter of a word appear before (or after) all instances of the second largest (smallest), then the word is reducible.*

Proof. The statement follows immediately from Definition 5.3. □

Lemma 6.9. *Let D be a scannable divide on at least 5 strands such that at most one letter n in its word appears more than once. Then D is reducible.*

Proof. By Lemma 5.11, we may assume that each letter appears at least once. If each letter appears exactly once, the result follows by Lemma 6.8. Then the largest letter may be commuted to the end or beginning (respectively) of the word in order to render it into a reducible form. Now assume there is sme n appearing at least twice. Since the number of strands is at least 5, then the word of D either contains $n \pm 2$. Without loss of generality, assume $n + 2$. Since $n + 1$ appears only once, then assume again with no loss of generality that $n + 2$ appears before $n + 1$ in the word. Then all letters at leas $n + 2$ may be commuted to the end of the word, rendering it reducible. □

Corollary 6.10. *If the number of gaps n is at least $|D|$ for a scannable uncapped divide D, then D is reducible.*

Proof of Theorem 6.6. All block decomposable quivers coming from scannable divides are found in Theorem 6.1, and so by Theorem 2.10, we need only identify divides which produce one of the exceptional case quivers. Each of these quivers has size at most 10, so we need only analyse divides such that $2\,|D| - (n - 1) \leq 10$. Then by Lemma 6 and 6.10, the only pairs $(n, |D|)$ to check are $(7, 8), (6, 7), (5, 7), (5, 6), (4, 6), (4, 5), (3, 5), (3, 4)$. For each such pair, we enumerate all words on $n - 1$ letters of length $|D|$, up to word equivalence. We then use Lemmas 5.11,6.8,6,6.9, and Corollary 6.10 to eliminate many reducible words. The remaining set contains 84 words. A manual search quickly yields that only 33 of these are irreducible. After eliminating members of block-decomposable families and mutation-infinite words (as checked by random mutations until an edge of weight three appears), we are left with only three words: 11211, 112112, and 121321. □

Corollary 6.11. *The scannable divides without capping are precisely the words whose irreducible parts are enumerated in the two theorems Theorem 6.1 and Theorem 6.6.*

ACKNOWLEDGEMENTS

This project was partially supported by RTG grant NSF/DMS-1745638. It was supervised as part of the University of Minnesota School of Mathematics Summer 2022 REU program. The authors would like to thank Pavlo Pylyavskyy, Gregg Musiker, and the University of Minnesota, Twin Cities.

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