(s, t)-core partitions into d-distinct parts

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Outline

1. Partition recap
2. Core partitions and the poset P
3. Maximum hook length
4. The d-distinct generalization
5. (s, t)-core enumeration

Partitions

Definition

A *partition* is a weakly decreasing tuple of positive integers $\lambda = (\lambda_1, \ldots, \lambda_n)$.

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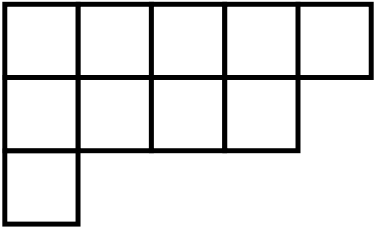

Figure: Young diagram for partition $\lambda = (5, 4, 1)$

Hook Lengths and Cores

Definition

The *hook length* of a cell C in the Young Diagram of a partition is the number of cells directly below or to the right of C, including C itself.

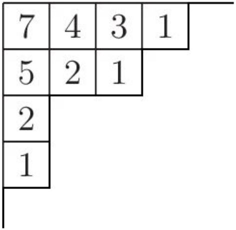

Figure: Young diagram for partition $\lambda = (4, 3, 1, 1)$.

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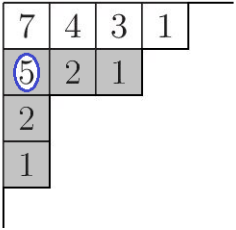


Figure: Young diagram for partition $\lambda = (4, 3, 1, 1)$.

Hook lengths and cores

Definition

An *s-core partition* is one without any hook lengths of size s. An (s,t)*-core partition* is one without hook lengths of size s or t.

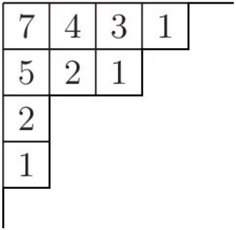


Figure: We have $\lambda = (4,3,1,1)$ is a $(6,8)$-core partition.

Hook lengths and cores

Definition

The β-*set* of a partition λ is the set of hook lengths in the first column.

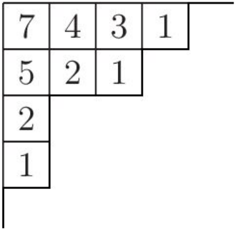
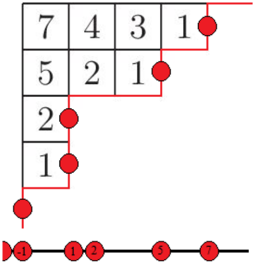


Figure: Young diagram for partition $\lambda = (4, 3, 1, 1)$, with β-set $(1, 2, 5, 7)$.

Frame Title

Lemma

The β-set of λ is obtained by the positions of vertical segments of the profile of the Young diagram, indexing the first horizontal segment as 0.



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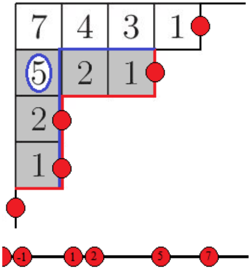
Hook lengths and cores

Lemma

A hook length of n corresponds to a 'bead' n spaces in front of a 'spacer'.



Corollary

The β-set of an s-core partition is closed under subtracting s.

Hook lengths and cores

Theorem (Anderson, 2002 [And02])

There are a finite number of (s,t)-core partitions iff $\gcd(s,t) = 1$. In particular, there are $\frac{\binom{s+t}{s}}{s+t}$ (s,t)-core partitions when $\gcd(s,t) = 1$.

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- For any (s, t) core partition λ, we have $a \in \beta \implies a - s, a - t \in \beta$.
- Since $0 \notin \beta$, then $c_1 s + c_2 t \notin \beta$.
- Iff $gcd(s, t) > 1$, there are finitely many integers not of the form $c_1 s + c_2 t \notin \beta \implies$ there are finitely many (s, t)-core partitions.

$(s, s+k)$-core partitions with distinct parts

Definition

We say a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ has distinct parts if all λ_i are distinct.

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Questions

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Questions

1. How many $(s, s+k)$-core partitions with distinct parts are there?
2. What is the maximum hook length over $(s, s+k)$-core partitions with distinct parts?

Maximum hook length of distinct $(s, s+k)$-core partitions

Theorem (PPX 2022)

Let s and k be positive, coprime integers with $s \geq 2$. Then, letting H denote the maximum hook length of an $(s, s+k)$-core partition with distinct parts, we have

$$H = \begin{cases} s-1 & \text{if } k = 1 \\ B-2 & \text{if } k > 1 \text{ and } m^{-1} \bmod k \leq \frac{k}{2} \\ B-1 & \text{if } k > 1 \text{ and } m^{-1} \bmod k > \frac{k}{2}, \end{cases}$$

where

$$m = s \bmod k,$$
$$\widetilde{m} = \min\{\pm m^{-1} \bmod k\}, \text{ and}$$
$$B = \left\lfloor \frac{s-1}{k} \right\rfloor (k + s\widetilde{m}) + s\left(\left\lceil \frac{m-1}{k} \widetilde{m} \right\rceil + \widetilde{m} - 1 \right) + m.$$

The poset $P_{(s,s+k)}$

Definition

Let $P_{(s,s+k)} = \mathbb{N} \setminus \{n \mid n = as + b(s+k) \text{ for some } a, b \in \mathbb{N}_0\}$, where $n \gtrdot m$ if and only if $n - m = s$ or $n - m = s + k$.

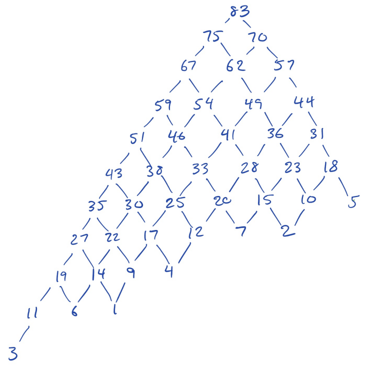


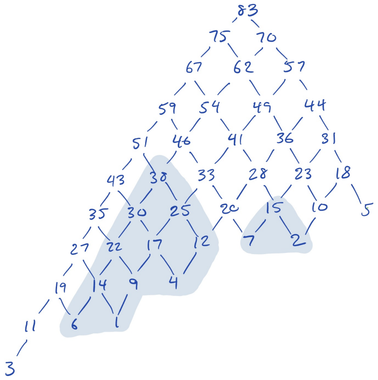
Figure: The Hasse diagram of $P_{8,13}$

Correspondence between β-sets and down sets of $P_{(s,s+k)}$

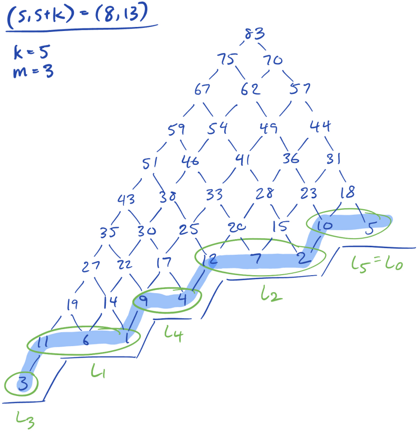
Corollary

The β-set of an $(s, s+k)$-core partition is closed under subtracting s and $s+k$.

The β-sets of $(s, s+k)$-core partitions are exactly the down sets of $P_{(s,s+k)}$.



Structure of $P_{(s,s+k)}$



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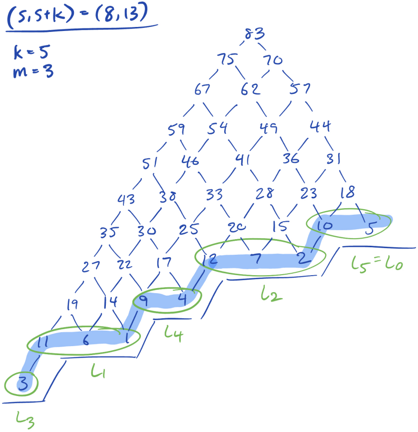

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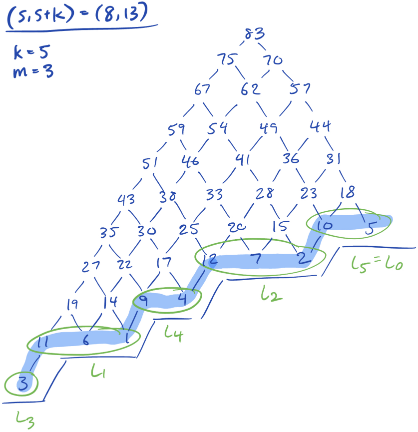

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- Ledges partition shaded region

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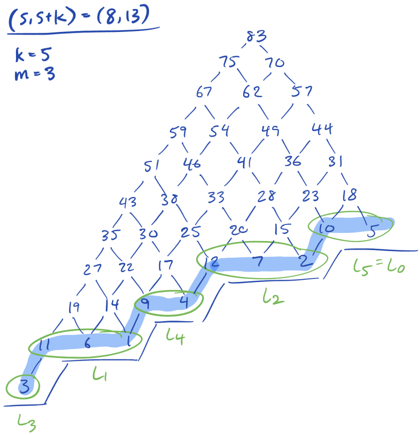
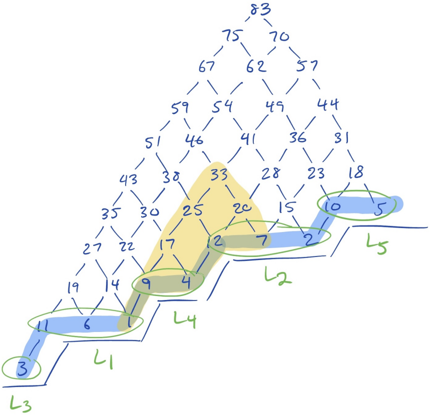


- Ledges L_i correspond to elements $i \pmod{k}$
- Ledges partition shaded region
- Adjacent ledges differ by $s \equiv m \pmod{k}$

Down sets and ledges of $P_{(s,s+k)}$



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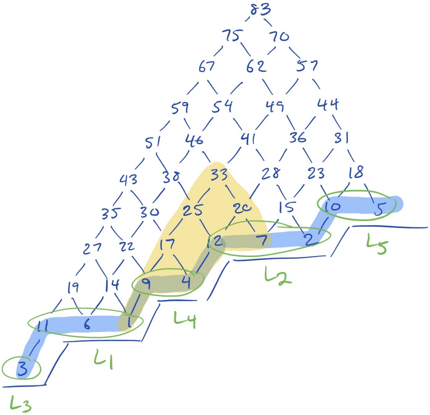

- Down set intersects shaded region at an interval

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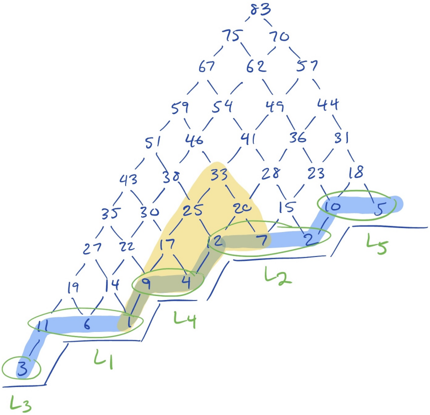
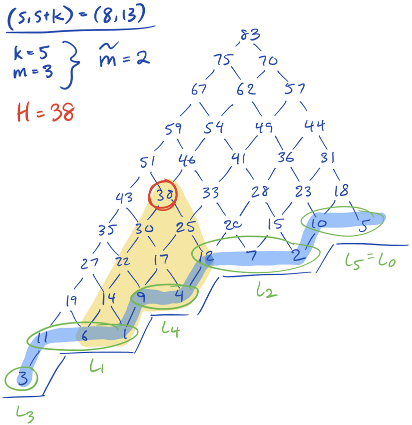


- Down set intersects shaded region at an interval
- Maximize hook length ⇒ maximize the length of intersection with shaded region

Example for distinct $(s, s+k)$-core partitions



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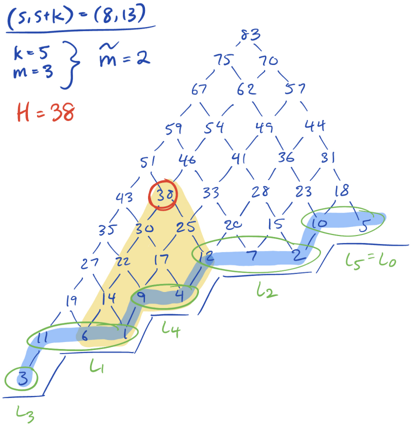

- Ledges L_i correspond to elements $i \pmod{k}$
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- $\widetilde{m} := \min\{\pm m^{-1} \bmod k\} = 2$

Generalization to $d-$distinct Core Partitions

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We say a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is $d-$distinct if $\lambda_i - \lambda_{i+1} \geq d$ for all $1 \leq i < n$.

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Question

What is the maximum hook length over d-distinct $(s, s+k)$-core partitions?

Maximum hook length of d-distinct $(s, s+k)$-cores

Theorem (PPX 2022)

Let s and k be positive coprime integers with $s \geq 2$. The maximum hook length H_d of an $(s, s+k)$-core with d-distinct parts can be computed as

$$H_d = \begin{cases} s-1 & \text{if } k=1 \text{ or } k,s \leq d \\ s+k-1 & \text{if } k \neq 1 \text{ and } k \leq d < s \\ B-2 & \text{if } d<k \text{ and } m\widetilde{m}_d \bmod k = 1 \\ B-s-1 & \text{if } d<k \text{ and } 1 < m\widetilde{m}_d \bmod k \leq d \\ B+k-m\widetilde{m}_d-1 & \text{if } d<k \text{ and } d < m\widetilde{m}_d \bmod k < k-1 \\ B-1 & \text{if } d<k \text{ and } 1 < m\widetilde{m}_d \bmod k = k-1, \end{cases}$$

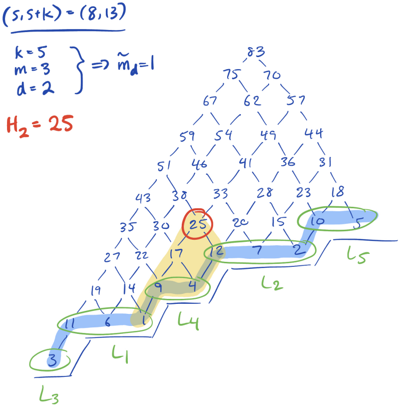
where

$$B = \left\lfloor \frac{s-1}{k} \right\rfloor (k + s\widetilde{m}_d) + s\left(\left\lceil \frac{m\widetilde{m}_d - 1}{k} \right\rceil + \widetilde{m}_d - 1 \right) + m,$$

$$m = s \bmod k, \text{ and}$$

$$\widetilde{m}_d = \min\{\ell m^{-1} \bmod k \mid -d \leq \ell \leq d, \ell \neq 0\}.$$

d-distinct $(s, s+k)$-core example



- $s = 8, k = 5 \Rightarrow m = 3$
- $\widetilde{m}_d = \min\{\ell m^{-1} \mid -d \leq \ell \leq d, \ell \neq 0\} = 1$
- $H_d = 25$

$(k-1)$-distinct $(rk+1,(r+1)k+1)$-core partitions

The number of d-distinct $(s, s+k)$-core partitions is well-understood when $d \geq k$ [Kra19].

But, "the case of general $d < k$ is fundamentally more complicated" [Kra19].

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But, "the case of general $d < k$ is fundamentally more complicated" [Kra19].

It's natural to start with $d = k-1$. It turns out to be convenient to have $s \equiv 1 \pmod{k}$.

Hence, we study $(k-1)$-distinct $(rk+1,(r+1)k+1)$-core partitions.

$(k-1)$-distinct $(rk+1,(r+1)k+1)$-core partitions

Conjecture

For a fixed r, the number of $(k-1)$*-distinct* $(rk\pm1,(r+1)k\pm1)$*-core partitions is a polynomial in* k*.*

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$$\#(2k-1,3k-1)=\frac{1}{2}k^2+\frac{5}{2}k-3$$

$$\dagger\#(3k-1,4k-1)=\frac{1}{6}k^3+\frac{7}{2}k^2+\frac{13}{3}k-8$$

$$\#(2k+1,3k+1)=\frac{1}{2}k^2+\frac{19}{2}k-5$$

$$\dagger\#(3k+1,4k+1)=\frac{1}{6}k^3+\frac{9}{2}k^2+\frac{106}{3}k-26$$

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We have a recurrence for the plus case (conditional on one conjecture)!

$(k-1)$-distinct $(rk+1,(r+1)k+1))$-core partitions

Let

$$C_n = \frac{\binom{2n}{n}}{n+1} \qquad \text{(Catalan numbers)}$$

$$E(\ell, r) = C_\ell + \left(\sum_{i=2}^{r-1}(C_i - C_{i-1})C_{\ell-i}\right) - (C_{r+1} - 2C_r + C_{r-1})C_{\ell-r} \qquad \text{(conjectured)}$$

$$D(\ell, k, a, b) = \sum_{i=1}^{\ell-1}\left((C_{i+1} - C_i)\left(\sum_{\alpha=1}^{a-1} D(\ell-i, k, \alpha, b)\right) + (C_i - C_{i-1})\left(\sum_{\alpha=a+1}^{k} D(\ell-i, k, \alpha, b)\right)\right)$$
$$+ \delta_{ab}(C_{\ell+1} - C_\ell) \qquad (\delta \text{ is Kronecker delta})$$

$$T_0(r, k) = \left(\sum_{i=1}^{r}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k} D(i, k, \alpha, \beta)\right) + 1$$

$$T_1(r, k) = \sum_{i=1}^{r}(k-1)\cdot E(r+1, i) + \sum_{\ell=i+1}^{r}\sum_{j=2}^{k} E(\ell, i)\cdot\left(\sum_{\alpha=1}^{j-1}(*)\right) + E(\ell-1, i)\cdot\left(\sum_{\alpha=j+1}^{k}(*)\right),$$

where

$$(*) = \left(\sum_{\beta=1}^{j-2} D(r-\ell, k, \alpha, \beta)\right) + \left(\sum_{\beta=j}^{k} D(r-\ell+1, k, \alpha, \beta)\right).$$

Then, the number of $(k-1)$-distinct $(rk+1,(r+1)k+1)$-core partitions is $T_0(r,k) + T_1(r,k)$.

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