The maximum hook length of d-distinct simultaneous core partitions

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Abstract

We exactly determine the maximum hook length of (s, t)-core partitions with d-distinct parts when there are finitely many such partitions. We also conjecture a recurrence for the number of $(k-1)$-distinct $(rk+1, (r+1)k+1)$-core partitions.

Contents

1 Introduction

A *partition* is a weakly decreasing tuple of positive integers $\lambda = (\lambda_1, \ldots, \lambda_n)$. The *size* of λ is $\lambda_1 + \cdots + \lambda_n$. Partitions have been studied not only for their number-theoretic and combinatorial properties, but also for their applications to group representation theory.

A partition can be visualized by its *Young diagram*, which is a left-justified array of cells where row i contains λ_i cells for $1 \le i \le n$. For each cell, we define its *hook* to be all the cells on its right, all the cells below it, and itself. The *hook length* of a cell is the number of cells in its hook. A notion of interest in representation theory is that of an *s-core partition*, a partition whose Young diagram contains no cells with hook length s [6, Chapter 2].

Anderson [1] generalized this notion to that of an *(s,t)-core partition*, which contain no cells with hook length s or t. In particular, she proved that there are $\binom{s+t}{s}/(s+t)$ such partitions when s and t are coprime; otherwise, there are infinitely many. Anderson's result has inspired several research directions related to (s,t)-core partitions (see [2, 8] and [5, Section 4] for three surveys on the subject).

One such direction has studied (s,t)-core partitions with distinct parts (see, e.g, [12, 11, 14, 9, 13, 3]). More generally, one can study *d-distinct* (s,t)-core partitions [10, 7, 4], in which $\lambda_i - \lambda_{i+1} \ge d$ for all $i \in [n-1]$. Kravitz [7, Lemma 2.4] proved that the number of d-distinct (s,t)-core partitions is finite if and only if $\gcd(s,t) \le d$, extending Anderson's result. Most work has focused on counting distinct or d-distinct (s,t)-core partitions, which has only been solved for a few choices of parameters. Similarly, closed-form expressions for the maximum size, maximum number of parts, and maximum hook length (also known as *perimeter*) of distinct or d-distinct (s,t)-core partitions were only known for a few choices of parameters.

In Section 2, we provide a closed-form expression for the maximum hook length of $(s,s+k)$-core partitions with distinct parts for all s and k. Then in Section 3, we generalize our formula for maximum hook length to d-distinct (s,t)-core partitions for all d when (s,t) are coprime. Finally, in Section 4, we extend our result to give a formula for the maximum hook length when $d \ge \gcd(s,s+k)$, which completely resolves the problem by Kravitz's result. Only loose upper bounds for the maximal hook length for general s and t were previously known.

Motivated by the form of our maximum hook length formula, we also study counting the number of d-distinct $(s,s+k)$-core partitions for $d = k-1$ and $(s,s+k) = (rk+1,(r+1)k+1)$, i.e. when $s = 1 \bmod k$. We conjecture a recurrence that allows us to calculate the number of such partitions for any given r and k.

2 Maximum hook length of $(s, s+k)$-core partitions with distinct parts

For any $(s,s+k)$-core partition λ with distinct parts, we denote its maximal hook length by H_λ. Let H be the maximum value of H_λ over all such partitions λ. We describe an algorithm for constructing such a partition with maximum hook length H. From this, we extract a closed-form formula for H.

Throughout, we use $a \bmod b$ to denote the modulo operation (remainder of Euclidean division of a by b) and $a \pmod b$ to denote a as an element of the ring $\mathbb{Z}/b\mathbb{Z}$.

Theorem 2.1. *Let s and k be positive, coprime integers with $s \geq 2$. Then, letting H denote the maximum hook length of an $(s, s+k)$-core partition with distinct parts, we have*

$$H(s,k) = \begin{cases} s-1 & \text{if } k = 1 \\ B-2 & \text{if } k > 1 \text{ and } m^{-1} \bmod k \leq \frac{k}{2} \\ B-1 & \text{if } k > 1 \text{ and } m^{-1} \bmod k > \frac{k}{2}, \end{cases}$$

where

$$\begin{aligned} &m = s \bmod k, \\ &\widetilde{m} = \min\{\pm m^{-1} \bmod k\}, \text{ and} \\ &B = \left\lfloor \frac{s-1}{k} \right\rfloor (k + s\widetilde{m}) + s\left(\left\lceil \frac{m-1}{k}\widetilde{m} \right\rceil + \widetilde{m} - 1\right) + m. \end{aligned}$$

The case of $k = 1$ proceeds quickly. First, observe that the β-set $\{s-1\}$ characterizes a valid $(s, s+1)$-core partition with distinct parts and maximum hook length $s-1$. Now, suppose S is another valid β-set and that some $n \in S$ for $n > s$ (recall that no β-set of an s-core partition may contain s). For S to be $(s, s+1)$-core, we must have $n-s, n-1-s \in B$. In this case, however, S contains consecutive non-negative integers, and the resulting partition does not have distinct parts.

In what follows, we focus on the case $k > 1$.

2.1 Correspondence with posets and order ideals

Following Baek, Nam, and Yu [3], we analyse (s,t)-core partitions by defining a poset $P_{(s,s+k)}$ and establishing a bijection between (s,t)-core partitions and order ideals on $P_{(s,s+k)}$.

Definition 2.2. Let $P_{(s,s+k)} := \{\mathbb{N}_{>0} \setminus \{n \in \mathbb{N}_{>0} \mid n = as + b(s+k) \text{ for some } a, b \in \mathbb{N}_{\geq 0}\}$ given by the covering relations $n \lessdot m$ iff $n - m = s$ or $n - m = s + k$.

Remark. It is well known that the unique maximal element of P is the Frobenius number $M = s(s+k) - s - (s+k)$.

In what follows, we refer to $P_{(s,s+k)}$ simply as P.

Note that P consists of all possible β values of an $(s, s+k)$-core partition. Furthermore, a core partition has distinct parts if and only if its β-set does not contain any consecutive integers, so core partitions with distinct parts correspond *precisely* to order ideals of P not containing consecutive integers. We call such order ideals 1-*distinct.* More generally, an ideal not containing elements whose difference is $\leq d$ is d-*distinct.*

Using this bijection, we can equivalently define H to be the largest element in any 1-distinct ideal of P. Note that if some 1-distinct ideal contains H, then it contains the principal ideal $\langle H \rangle$, which is also 1-distinct. Thus it suffices to construct the principal ideal $\langle H \rangle$, which is equivalent to maximizing the generator q over all 1-distinct principal ideals $\langle q \rangle$.

2.2 Structure of $P_{(s,s+k)}$

We now describe the structure of $P_{(s,s+k)} = P$ with the following lemmas and definitions. This will be useful in finding the maximum hook length H.

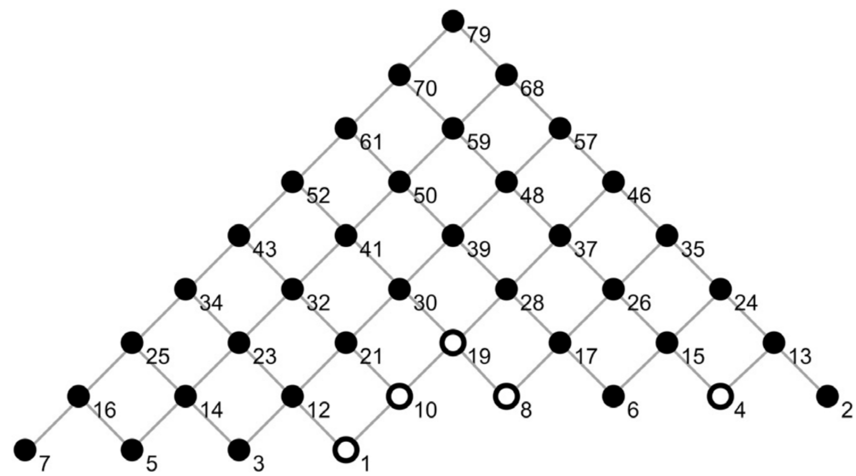

Figure 1: Hasse diagram of $P_{9,11}$ with its order ideal $I = \{19, 10, 8, 4, 1\}$

Lemma 2.3. *Any $n \in P$ has a unique representation of the form $M - as - bt$ for $a, b \in \mathbb{N}_{\geq 0}$*

Proof. It suffices to show that for any element $p \in P$, we cannot have both $p = M - a_1 s - b_1 t$ and $p = M - a_2 s - b_2 t$ where $a_1 \neq a_2$ and $b_1 \neq b_2$. Assume for the sake of contradiction that $p = M - a_1 s - b_1 t = M - a_2 s - b_2 t$. Then $(a_2 - a_1)s = (b_1 - b_2)t$, but $b_1 \neq b_2$ and $\gcd(s, t) = 1$, which implies that $s \mid b_1 - b_2$ since $s \mid (b_1 - b_2)t$. Then $b_1 - b_2 \geq s$, so that $b_1 \geq s$. However, then $p = M - a_1 s - b_1 t \leq M - s(s+k) = -2s - k < 0$, a contradiction. $\square$

Corollary 2.4. *P is a graded poset with rank function $f : P \to \mathbb{Z}$ given by $f(M - as - bt) = -(a + b)$.*

Corollary 2.5. *P is a join-semilattice.*

Definition 2.6. Define the set $E = P \cap [s + k - 1]$.

Note that E consists of all elements $p \in P$ such that $p - (s+k) < 0$, including all minimal elements $[s - 1]$.

Definition 2.7. We impose the following total ordering on E. For $a, b \in E$, we say $a \lessdot_E b$ if $a = b + k$ or $a = b - s$.

We may also refer to $a \lessdot_E b$ as *"a is before b"* or *"a is first"*.

Proposition 2.8. *The ordering on E from Definition 2.7 is total, with largest element $M - (s+k)\lfloor M/(s+k) \rfloor$ and smallest element $M - s\lfloor M/s \rfloor$.*

Proof. Notice that subtracting s decreases the rank by one. Thus $a \lessdot_E b$ whenever $f(a) < f(b)$. Since our ordering is clearly non-contradictory among elements of the same rank (i.e., the order does not imply that any element is greater than itself), then it is non-contradictory in general.

Now we show that each $a \in E$ other than $M-(s+k)\lfloor M/(s+k) \rfloor$ is covered by exactly one element. If $a > k$ and $a-k \in P$, then $a-k \in E$, and so $a \lessdot_E a-k$. Then $s+k-1 < a+s \notin E$, so a is covered by one element. Else if $a < k$, $a - k \notin E$ but $a + s \in E$ exactly unless $a = M - (s+k)\lfloor M/(s+k) \rfloor$. Then each $a \in E$ other than $M - (s+k)\lfloor M/(s+k) \rfloor$ is covered by exactly one element. Furthermore, the covering elements are all distinct because if distinct a, b are covered by the same element, then $a - s = b + k$, and then $b = a + s + k$, so $b \notin E$.

Therefore, each element but one also covers exactly one element. Since $b = M - s\lfloor M/s \rfloor$ is not covered by any element (as $b - s, b + k \notin E$), then all other elements are covered once. Since we know our ordering is non-contradictory, then we may deduce that it is total, with $M - (s+k)\lfloor M/(s+k) \rfloor$ the largest element and $M - s\lfloor M/s \rfloor$ the smallest. ☐

Definition 2.9. A *ledge* L_i is the set

$$\begin{aligned} L_i = &\{n \mid n \in [s-1] \text{ and } n \equiv i \pmod{k}\} \\ &\cup \ \{n \mid n \in E \setminus [s-1] \text{ and } n \equiv i + m \pmod{k}\}. \end{aligned}$$

Note that L_i partition elements in E.

Note that L_i partition elements in E. Throughout the paper, when we say "first" element or "last" element of a ledge, we refer to the ordering defined on E. We can also define an ordering on the ledges as follows.

Definition 2.10. Define a total order on the set $\mathcal{L}$ of ledges via

$$L_m \prec L_{2m} \prec \ldots \prec L_{-m} \prec L_0,$$

where subscripts are taken modulo k. We call a set of consecutive elements of the complete ordered sets of ledges an *adjacent* set of ledges.

Note that this ordering on ledges is compatible with the ordering on E, i.e. if L_i comes before L_j in the ledge ordering, we have that for any $x \in L_i$ and $y \in L_j$, we have $x \lessdot_E y$.

2.3 Optimizing q over principal ideals $\langle q \rangle$

Now that we have established the structure of P, we can now consider how to find the largest q such that the principal ideal $\langle q \rangle$ is 1-distinct. To do this, we first define the height of a principal ideal $\langle q \rangle \subseteq P$.

Definition 2.11. Given a principal ideal $\langle q \rangle \subseteq P$, let its *height* $h(\langle q \rangle)$ be

$$h(\langle q \rangle) = \left\lfloor \frac{q}{s} \right\rfloor + 1.$$

Lemma 2.12. *If $h(\langle q_1 \rangle) > h(\langle q_2 \rangle)$, then $q_1 > q_2$.*

Proof. The statement follows quickly from 2.11. □

Since a greater height corresponds to a greater maximal element of the principal order ideal, it suffices to consider principal ideals $\langle q \rangle$ with maximum height. We find that $\langle H \rangle$ is the ideal that also maximizes a_q, which we define below.

Definition 2.13. Let $a_{\langle q \rangle} = q - s \cdot (h(\langle q \rangle) - 1)$.

Proposition 2.14. *We have $\langle H \rangle$ is the principal ideal $\langle q \rangle$ with no consecutive elements maximizing $(h(\langle q \rangle), a_{\langle q \rangle})$ lexicographically.*

Proof. By Lemma 2.12, $\langle H \rangle$ is a principal ideal without consecutive element with maximum height. Let $\langle q_1 \rangle, \langle q_2 \rangle \subseteq P$ be principal ideals, and let $h(\langle q_1 \rangle) = h(\langle q_2 \rangle)$. Then notice that if $a_{\langle q_1 \rangle} > a_{\langle q_2 \rangle}$, we have $q_1 > q_2$. Also observe that if $\langle q_1 \rangle \neq \langle q_2 \rangle$, then $a_{\langle q_1 \rangle} \neq a_{\langle q_2 \rangle}$. Therefore, $h(\langle q \rangle)$ and $a_{\langle q \rangle}$ uniquely characterize $\langle q \rangle$. Then $\langle H \rangle$ is the principal ideal $\langle q \rangle \subseteq P$ with no consecutive integers of maximum height which has largest $a_{\langle q \rangle}$. □

2.4 Reduction to interval ideals of E

We now establish a bijection between d-distinct principal ideals $\langle q \rangle$ and d-distinct intervals in E that are ideals, which we call d-distinct interval ideals. This allows us to reduce our algorithm to finding the d-distinct interval ideal with the largest corresponding generator q. We first establish the correspondence between principal ideals and interval ideals.

Lemma 2.15. *For any principal ideal $\langle q \rangle \subseteq P$, we have $\langle q \rangle \cap E$ is a single interval ideal according to the total ordering on E.*

Proof. Suppose $a, b \in \langle q \rangle \cap E$ with $a <_E b$. Notice that $\langle q \rangle \cap E$ is an ideal in E and in P. We need to show that if $b + k \in E$, then $b + k \in \langle q \rangle$, and that if $b - s \in E$, then $b - s \in \langle q \rangle$. Then the lemma follows by induction. The second case follows from the covering definition for P.

So suppose $b + k \in E$. Consider $c = a \vee b \in \langle q \rangle$. Let $a = M - n_1 s - n_2(s + k)$. Since $a <_E b$, by the total ordering on E we know that there exists some integers x, y such that $b = a + x(s - (s + k)) + ys = M - (n_1 - x - y)s - (n_2 + x)(s + k)$. Thus, we have that $c = b + n(s + k)$ for some integer n , giving us that $b + (s + k) \in \langle q \rangle \Rightarrow b + k \in \langle q \rangle$, as desired. □

Corollary 2.16. *Any $\langle q\rangle$ is such that $\langle q\rangle \cap E$ is a union of an adjacent set of ledges $\{L_{im}, \ldots, L_{jm}\}$ together perhaps with a set of the largest elements of $\{L_{(i-1)m}\}$ and the smallest elements of $\{L_{(j+1)m}\}$.*

Proposition 2.17. *Let $\langle q\rangle$ be a principal ideal of P and L_i the first ledge such that $L_i \cap \langle q\rangle \neq \emptyset$. Then, $a_{\langle q\rangle}$ is the first element in $L_i \cap \langle q\rangle$ by the ordering on E. In other words, $a_{\langle q\rangle}$ is the smallest element of E in $\langle q\rangle$.*

Proof. Let $q \in P$. Since $1 \leq a_{\langle q\rangle} \leq s-1$, we know $a_{\langle q\rangle}$ lies in some ledge L_j. Suppose for the sake of contradiction that there is an element in $L_j \cap \langle q\rangle$ before $a_{\langle q\rangle}$ according to the ordering on E. Then by lemma 2.15 and since $a_{\langle q\rangle} \in \langle q\rangle$, either $a_{\langle q\rangle} - s \in L_j \cap \langle q\rangle$ or $a_{\langle q\rangle} + k \in L_j \cap \langle q\rangle$. The first case is impossible because $a_{\langle q\rangle} - s \leq 0$. Assume now that $a_{\langle q\rangle} + k \in L_j \cap \langle q\rangle$. We know that $a_{\langle q\rangle} + \ell s \in \langle q\rangle$ for $0 \leq \ell \leq \lfloor q/s \rfloor$. Furthermore, notice that the join of $x, x+k \in P$ is $x+s+k$, which covers both. Since $\langle q\rangle$ is principal, it is closed under joins. We proceed by induction. As a base case, we have that $a_{\langle q\rangle} + k \in \langle q\rangle$. Then assume that $a_{\langle q\rangle} + k + \ell s \in \langle q\rangle$. Since $a_{\langle q\rangle} + \ell s \in \langle q\rangle$, closure under join gives $a_{\langle q\rangle} + k + (\ell+1)s \in \langle q\rangle$. The induction step holds as long as $\ell \leq \lfloor q/s \rfloor$, and so in particular we find that $q + k = a_{\langle q\rangle} + k + \lfloor q/s \rfloor s \in \langle q\rangle$. This contradicts our assumption that $\langle q\rangle$ is generated by q since the generator of a principal ideal in P is always its greatest element.

It remains to show that $j = i$. Suppose not. Then there is an element $x \in L_i \cap \langle q\rangle$ such that $x <_E a_{\langle q\rangle}$. By Lemma 2.15, either $a_{\langle q\rangle} + k \in E \cap \langle q\rangle$ or $a_{\langle q\rangle} - s \in E \cap \langle q\rangle$, both of which are impossible by the previous paragraph. □

Lemma 2.18. *Given a principal ideal $\langle q\rangle$ such that $\langle q\rangle \cap E$ intersects ledges $L_{im}, L_{(i+1)m}, \ldots, L_{(i+j)m}$, we have that $h(\langle q\rangle) = |\langle q\rangle \cap E| = \sum_{n=i}^{i+j} |L_{nm} \cap \langle q\rangle|$.*

Proof. Consider the $h(\langle q\rangle)$ elements $A = \{a_{\langle q\rangle}, a_{\langle q\rangle} + s, \ldots, a_{\langle q\rangle} + (h(\langle q\rangle) - 1)s = q\} \subseteq \langle q\rangle$. We claim that

$$\begin{aligned} g : A &\to \langle q\rangle \cap E \\ a_{\langle q\rangle} + ds &\mapsto (a_{\langle q\rangle} + ds) - (s+k)\left\lfloor \frac{a_{\langle q\rangle} + ds}{s+k} \right\rfloor \end{aligned}$$

is a bijection.

First, we claim the image of g is in $\langle q\rangle \cap E$. It is clearly in $\langle q\rangle$. Since $s + k \nmid a_{\langle q\rangle} + ds$, we have

$$\begin{aligned} g(a_{\langle q\rangle} + ds) &> (a_{\langle q\rangle} + ds) - (s+k)\frac{a_{\langle q\rangle} + ds}{s+k} = 0 \\ g(a_{\langle q\rangle} + ds) &< (a_{\langle q\rangle} + ds) - (s+k)\left(\frac{a_{\langle q\rangle} + ds}{s+k} - 1\right) = s+k. \end{aligned}$$

Further, $g(a_{\langle q\rangle} + ds) \in P$, so $g(a_{\langle q\rangle} + ds) \in P \cap [s+k-1] = E$.

Recall that each element of P has a unique representation of the form $M - as - b(s+k)$. Note also that $a_{\langle q\rangle} + ds$ and $g(a_{\langle q\rangle} + ds)$ have the same s-term in their expansions.

Furthermore, all elements of A have pairwise distinct s-terms in their expansions. Therefore, all $g(a_{\langle q\rangle} + ds)$ have distinct s-terms in their expansions, so g is injective.

Let $q = M - a_1 s - b_1(s+k)$. Then for some $M - as - b(s+k) \in \langle q\rangle \cap E$, let

$$g^{-1}(M - as - b(s+k)) = M - as - b_1(s+k).$$

Observe that the image of g^{-1} is in A. So to prove that g is surjective, we need $g \circ g^{-1} = id$. This indeed the case because $g^{-1}(n) = n + \ell_1(s+k)$ for some $\ell_1 \geq 0$ and $n \in A$. Then $g(g^{-1}(n)) = n + \ell_1(s+k) - \ell_2(s+k)$ such that $n + \ell_1(s+k) - \ell_2(s+k) - (s+k) \leq 0$. Then since $n \in E$, we have that $n - (s+k) \leq 0$, which implies that $\ell_1 = \ell_2$, and $g(g^{-1}(n)) = n$. Then g is a bijection, and $h(\langle q\rangle) = |\langle q\rangle \cap E|$. □

To establish a correspondence from interval ideals to principal ideals, we now define the notion of the adjacent join closure of an interval ideal.

Definition 2.19. Let $S \subseteq E$ be an interval ideal. We add elements to S iteratively as long as possible in the following way. For any element p such that $p - s, p - (s+k)$ (the elements it covers) are in our set, add p. Let $\bigvee(S)$ denote resulting principal ideal, which we call the *adjacent join closure*.

Lemma 2.20. *If S is d-distinct, then $\bigvee(S)$ is d-distinct.*

Proof. We proceed by induction. Let S_i denote the closure of S after i pairwise join operations such that each join adds an element, so $S_0 = S$ and $S_m = \bigvee(S)$ for maximal m, i.e. we cannot add new elements under join. Initially, we have that $S_0 = S$ is d-distinct. Then it suffices to show that if S_i is d-distinct, then S_{i+1} is d-distinct for $i \in [m-1]$.

Suppose for the sake of contradiction that S_i is d-distinct S_{i+1} is not. Let $a + (s+k) = a \vee a + k$ be the new element of S_{i+1}, where $a, a+k \in S_i$ are (positive) elements. Then $\exists j \in S_i$ s.t. $a + (s+k) \sim j$. Note that $a, a+k \in S_i \Rightarrow d \leq k$, or else S_i would not be d-distinct.

If $j \notin [s-1]$, then we must have $j - s \in S_i$. Then $a + k \sim j - s$, but both $a+k, j-s \in S_i$, a contradiction to our assumption that S_i is d-distinct. Thus we must have $j \in [s-1] \Rightarrow a + (s+k) - j \geq a + k + 1$. But $a + (s+k) \sim j \Rightarrow a + (s+k) - j < d \Rightarrow a + k + 1 < d$. Thus we have $a + k + 1 < d \leq k$, a contradiction. Therefore, S_i is d-distinct gives us S_{i+1} is d-distinct, so inductively that $S_m = \bigvee(S)$ is d-distinct as desired. □

Now we can formally state and prove our bijection between d-distinct ideals $\langle q\rangle$ and d-distinct intervals on E.

Proposition 2.21. *Let I_d be the set of all d-distinct principal ideals on P, and let E_d be the set of all d-distinct interval ideals on E. The map*

$$\begin{aligned} f : I_d &\to E_d \\ \langle q\rangle &\mapsto \langle q\rangle \cap E \end{aligned}$$

is a bijection.

Proof. We first show that the map f is injective. Suppose $\langle q_1\rangle \cap E = \langle q_2\rangle \cap E = E'$ where $q_1 \neq q_2$. By Proposition 2.17, we have $a_{\langle q_1\rangle} = a_{\langle q_2\rangle}$ is the unique first element of E'. Then by Proposition 2.18, we have $q_1 = q_2 = h(\langle q\rangle) \cdot s \cdot (h(\langle q\rangle) - 1)$, where the height $h(\langle q\rangle) = h(\langle q_1\rangle = h(\langle q_2\rangle)$ is uniquely given by the length of E'. Thus $q_1 = q_2$, so f is injective.

It remains to show that f is surjective. By Lemma 2.20 we know that from any d-distinct interval ideal E' of E, we can generate a d-distinct principal ideal $\bigvee(E')$. Therefore, it suffices to show $\bigvee(E') \cap E = E'$. Clearly $E' \in \bigvee(E')$ by definition, and by construction $\bigvee(E')$ never adds any element in $E \setminus E'$ (i.e. outside the ideal E'), as no element of E covers two elements. Thus, we have established a bijection between d-distinct principal ideals $\langle q\rangle$ and d-distinct interval ideals on E. □

We have reduced the problem to finding the distinct interval ideal E' on E that corresponds to the $\langle q\rangle = \bigvee(E')$ with maximal q. Furthermore, we know we want to maximize $(h(\langle q\rangle), a_q)$ in lexicographic order by Lemma 2.14. We also have that $h(\langle q\rangle)$ is given by $|E'|$ by Lemma 2.18, and a_q is given by the first element of E' Lemma 2.17. Thus, we must maximize $|E'|$ and then the first element of E' in that order.

2.5 Structure of 1-distinct interval ideals

Now that we have established a bijection between d-distinct principal ideals $\langle q\rangle$ and d-distinct interval ideals of E, we can look at the structures of 1-distinct interval ideals of E that we use in our optimization algorithm.

Proposition 2.22. *We have*

$$|L_i| = \begin{cases} 0 & \text{if } s|i \text{ and } i \neq 0 \\ \left\lfloor \frac{s-1}{k} \right\rfloor & \text{if } i = 0 \\ \left\lfloor \frac{s-1}{k} \right\rfloor + 1 & \text{if } m \leq i \text{ and } s \nmid i \\ \left\lfloor \frac{s-1}{k} \right\rfloor + 2 & \text{if } 0 < i < m \end{cases}$$

Proof.

Case (1). $s|i$ and $i \neq 0$.

Suppose $n \in L_i$. Let $i = ds$ for some positive integer d, and thus there is no integer in $[s-1]$ equivalent to i. In particular $n \notin [s-1]$, so $n \equiv i + m \pmod k$ Since $s \leq i < k$, then $s < k$ and $m = s$. If $i + m = (d+1)s < k$, then n is a multiple of of s and hence does not appear in P. If $i + m = (d+1)s > k$, then $(d+1)s \bmod k = (d+1)s - k \in [s-1]$ because $ds < k$ and so $ds + s \bmod k < s$. Then, since $n \leq s + k - 1$, we have $n = ((d+1)s - k) + k = (d+1)s$, which does not appear in P as a multiple of s.

Case (2). $i = 0$.

If $n \in [s-1]$, then we may restrict n to one of the integers $0, k, 2k, \ldots, \lfloor (s-1)/k \rfloor$. Then specifically, $k, 2k, \ldots, \lfloor (s-1)/k \rfloor \in L_i$, a total of $\lfloor (s-1)/k \rfloor$ elements . Now if $n \in [s+k-1]$, we know that $n \equiv s \pmod k$. However, there are no such integers in $[s+1, s+k-1] \cap P$. Then there are no additional elements in this case, for a total of $\lfloor (s-1)/k \rfloor$.

Case (3). $m \leq i$ and $s \nmid i$.
Suppose that $n \in [s-1]$. Then $n \in \{i, i+k, \ldots, k(\lfloor (s-1)/k \rfloor - 1)\}$. Observe that $i + k(\lfloor (s-1)/k \rfloor - 1)$ is the largest such element in $[s-1]$ since $i + k(\lfloor (s-1)/k \rfloor) \geq s$ as $i \geq m$. If $n \in [s+1, s+k-1]$, then $n \equiv i + m \pmod{k}$. Since $|[s+1, s+k-1]| = k-1$, it has exactly one representative of every equivalence class modulo k except m. If $i + m \equiv m \pmod{k}$, then $i \equiv 0 \pmod{k}$, so $i = 0$, a contradiction. Therefore, there is exactly one $n \in [s+1, s+k-1]$ such that $n \equiv i + m \pmod{k}$, for a total of $\lfloor (s-1)/k \rfloor + 1$ elements in L_i.

Case (4). $0 < i < m$.
Suppose that $n \in [s-1]$. Then $n \in \{i, i+k, \ldots, k\lfloor (s-1)/k \rfloor\}$. Observe that $i + k\lfloor (s-1)/k \rfloor$ is the largest such element in $[s-1]$ since $i + k(\lfloor (s-1)/k \rfloor + 1) \geq s$ as $i > 0$. If $n \in [s+1, s+k-1]$, then $n \equiv i + m \pmod{k}$. Since $|[s+1, s+k-1]| = k-1$, it has exactly one representative of every equivalence class modulo k except m. If $i + m \equiv m \pmod{k}$, then $i \equiv 0 \pmod{k}$, so $i = 0$, a contradiction. Therefore, there is exactly one $n \in [s+1, s+k-1]$ such that $n \equiv i + m \pmod{k}$, for a total of $\lfloor (s-1)/k \rfloor + 2$ elements in L_i.

$\square$

2.6 Interval ideals to $\widetilde{m}$-intervals

By Corollary 2.16 and Lemma 2.18, we want to maximize the length of the ledges that comprise an interval ideal on E . We can represent the sequence of ledges as a sequence of their indices, which we define as an $\widetilde{m}$-interval below, and call this sequence I.

By Proposition 2.22, we see that either all ledges have the same length, or that some ledges are one element longer than others . Furthermore, the ledges that are longer are precisely the ledges indexed by i where $0 < i < m$. To maximize the length of the interval on E of adjacent ledges, we want to maximize the number of longer ledges. This is then equivalent to maximizing the size of $|I \cap [m-1]|$, where I is the corresponding $\widetilde{m}$-interval.

Definition 2.23. Recall that $\widetilde{m} := \min\{\pm m^{-1} \bmod k\}$. An $\widetilde{m}$*-interval* is an ordered subset of $\mathbb{Z}/k\mathbb{Z}$ of the form

$$\{d, d+m, \ldots, d + (\widetilde{m} - 1)m\}$$

for some integer d.

Since we know our ledges $L_m, L_{2m}, \ldots, L_{km}$ partition E according to the ordering on E, any $\widetilde{m}$-interval corresponds to a set of $\widetilde{m}$ adjacent ledges and thus an interval ideal of E. Furthermore, no two elements in this interval can be consecutive by definition of $\widetilde{m}$. Thus any $\widetilde{m}$-interval corresponds to a distinct interval ideal of E.

Lemma 2.24. *Let I and J be distinct $\widetilde{m}$-intervals not containing 0 or m. Then* $|I \cap [m-1]| = |J \cap [m-1]|$.

Proof. Since m and k are coprime, the sequence $0, m, 2m, \ldots, (k-1)m$ understood as elements of $\mathbb{Z}/k\mathbb{Z}$ contains each element exactly once. Without loss of generality, the first element of I, which we denote c, occurs earlier in this sequence than the first element of J, so that $J = I + zm \bmod k$ for some $z \in [k-1]$. Then, $I + ym$ does not contain any of $0, m, \ldots, \lfloor k/m \rfloor m$ for $0 \leq y \leq z$ (recall that $0 \notin I$).Therefore, it suffices to let $J = I + m$ and show that $|I \cap [m-1]| = |(I+m) \cap [m-1]|$. For this, it suffices to show that $c \in [m-1]$ if and only if $c + \widetilde{m}m \in [m-1]$.

Assume that $c \in [m-1]$. Suppose first that $m^{-1} \bmod k \leq \frac{k}{2}$, so that $\widetilde{m}m = 1$. Then $c + 1 \notin [m-1]$ if and only if $c = m-1$. However, then $c+1 = m$, and so $m \in I + m$, a contradiction as we assumed $m \notin I + m$. Suppose now that $m^{-1} \bmod k > \frac{k}{2}$, so that $\widetilde{m}m = -1$. Then $c - 1 \notin [m-1]$ if and only if $c = 1$. However, then $c - 1 = 0$, and so $0 \in I + m$, a contradiction. On the other hand, if $c \notin [m-1]$, a contradiction is similarly obtained. □

Lemma 2.25. *Let I be an $\widetilde{m}$-interval not containing 0 or m. Then*

$$|I \cap [m-1]| = \left\lceil \frac{m-1}{k} \widetilde{m} \right\rceil.$$

Proof. Let $\alpha = |I \cap [m-1]|$, and note that α does not depend on our choice of I by Lemma 2.24. Then it suffices to show

$$\frac{m-1}{k}\widetilde{m} \leq \alpha < \frac{m-1}{k}\widetilde{m} + 1.$$

Consider all distinct $\widetilde{m}$-intervals

$$I_j = \{jm, (j+1)m, \ldots, (j + \widetilde{m} - 1)m\}$$

for $j \in [k]$. Note that the $\widetilde{m}-$intervals not containing 0 or m are $I_2, I_3, \ldots, I_{k-\widetilde{m}}$, and the intervals containing 0 or m are $I_{k-\widetilde{m}+1}, \ldots, I_{k+1} = I_1$. Then the average of $|I_j \cap [m-1]|$ over all intervals $I_1, \ldots, I_k$ is $\frac{m-1}{k}\widetilde{m}$. To see this, every element of $[m-1]$ is counted in $\widetilde{m}$ intervals , and there are k distinct intervals I_c .

To show that $\frac{m-1}{k} \leq \alpha$, we show that any $\widetilde{m}-$interval containing 0 or m cannot contain more than α elements in $[m-1]$. Indeed, the only cases when $|I_{j+1} \cap [m-1]| > |I_j \cap [m-1]|$ is when the last element of I_{j+1} is in $[m-1]$ and the first element of I_j is not. This happens precisely in the following two cases.

Case (1). $(j + \widetilde{m})m = 1 \pmod{k}$ and $jm = 0 \pmod{k}$.

Then $j = 0$ and $I_{j+1} = m, 2m, \ldots, \widetilde{m}m$. But then $I_{j+2} = I_2 = 2m, \ldots, 1, m+1$, so $|I_{j+1} \cap [m-1]| = |I_{j+2} \cap [m-1]| = \alpha$.

Case (2). $(j + \widetilde{m})m = m - 1 \pmod{k}$ and $jm = m \pmod{k}$.

Then we have $j = 1$, but $I_{j+1} = I_2$ already has exactly α elements in $[m-1]$.

These are the only cases as if $(j+\widetilde{m})m = 2, 3, \ldots, m-2 \pmod{k}$, then $jm = (j+\widetilde{m})m \pm 1 \in [m-1]$.

Thus given that $|I_{k-\widetilde{m}} \cap [m-1]| = \alpha$, we have

$$|I_{k-\widetilde{m}+1} \cap [m-1]|, |I_{k-\widetilde{m}+2} \cap [m-1]| \ldots, |I_{k+1} \cap [m-1]| \leq |I_2 \cap [m-1]| = \alpha$$

Thus, α is at least the average of $|I \cap [m-1]| = \frac{m-1}{k}\widetilde{m}$.

To show that $\alpha < \frac{m-1}{k}$, we show that

$$|I_{k-\widetilde{m}+1} \cap [m-1]|, \ldots, |I_1 \cap [m-1]| \geq \alpha - 1.$$

We can apply a similar argument as above. The only cases when $|I_{j+1} \cap [m-1]| < |I_j \cap [m-1]|$, or equivalently when $|I_{j+1} \cap [m-1]| = |I_j \cap [m-1]| - 1$, is when the last element of $|I_{j+1} \cap [m-1]|$ is not in $[m-1]$ and the first element of I_j is. This happens when either $jm = 1$ and $(j+\widetilde{m})m = 0 \Rightarrow \widetilde{m}m = -1$ or $jm = m-1$ and $(j+\widetilde{m})m = m \Rightarrow \widetilde{m}m = 1$. However, note that these two cases are mutually exclusive, as $\widetilde{m}m$ is either -1 or 1. Therefore,

$$|I_{k-\widetilde{m}+1} \cap [m-1]|, \ldots, |I_1 \cap [m-1]| \geq |I_{k-\widetilde{m}} \cap [m-1]| - 1 = \alpha - 1.$$

Now we can write

$$\begin{aligned}
&(\widetilde{m}+1)(\alpha-1) + (k-\widetilde{m}-1)(\alpha) \leq k\left(\frac{m-1}{k}\widetilde{m}\right) = \sum_{c=1}^{k} |I_c \cap [m-1]| \\
&\Rightarrow k\alpha - (\widetilde{m}+1) \leq k\left(\frac{m-1}{k}\widetilde{m}\right) \\
&\Rightarrow \alpha \leq \frac{m-1}{k}\widetilde{m} + \frac{\widetilde{m}+1}{k} < \frac{m-1}{k}\widetilde{m} + 1
\end{aligned}$$

where the last inequality follows from $\widetilde{m} < \frac{k}{2}$ unless $(m,k) = (1,2)$, in which case we can simply check that $\alpha = \lceil \frac{m-1}{k}\widetilde{m} \rceil = 0$ still holds. $\square$

2.7 Optimization algorithm

Lemma 2.26. *Let $L_i, L_{i+m}, \ldots, L_{i+jm}$ be a sequence of adjacent intervals such that $|L_i| \geq 2$. Then $\mathcal{L} = L_i \cup L_{i+m} \cup \cdots \cup L_{i+jm}$ is an ideal, and it up-generates the principal ideal $\langle q \rangle$ such that $\langle q \rangle \cap E = \mathcal{L}$. If $|L_i| = 1$, then $\mathcal{L}$ is not an ideal.*

Proof. First, notice that no ledge in $L_i, L_{i+m}, \ldots, L_{i+jm}$ is empty because the empty ledges are the first ledges , if any exist, by Proposition 2.22. We show that $\mathcal{L}$ is an ideal. Notice that for any $x \in E$, $x-(s+k) < 0$ by definition, so $x-(s+k) \notin P$. Furthermore, $x - s \in P$ iff x is the last element in its ledge (the element which is not in $[s-1]$, of which there is only one as discussed int he proof of Proposition 2.22). Therefore, if the size of that ledge L_ℓ is at least 2, then $x - s \in L_\ell$, and otherwise $x - s \in L_{\ell-m}$ the previous ledge if $|L_\ell| = 1$. Therefore, we have that iff $|L_i| \geq 2$, then $\mathcal{L} = L_i, L_{i+m}, \ldots, L_{i+jm}$ is an ideal.

Next, we show that $\mathcal{L}$ up-generates a principal ideal. Notice that when we create the up-generation $\hat{\mathcal{L}}$ by beginning with some set $T = \mathcal{L}$ and adding elements to T via the up-generation rule, the order in which we add does not matter, and if we ever reach a position where we cannot add any elements, we have created the up-generation. Notice further that we may add elements one rank at a time in ascending order of rank because adding elements on an upper rank cannot newly allow us to add elements on a lower rank. Therefore, we begin at the lowest rank of $P \cap L$, which we call the zeroth rank. Clearly there are no elements to add because in order to add an element all the elements it covers must already be in T, but there are no such elements of lower rank. On the first rank (immediately above the zeroth), we may add exactly those elements of the form $x - s$ where x is an element of L_i other than the last element, as these elements x are precisely the elements for which $x - s \notin T$, and for all such elements, $x - s - k$ is in T by construction (in fact, $x - s - k \in L_i$). Then in general on the nth rank, we may add $x - s$ where x is either an element we have already added, the last element of the $L_{i+(n-2)m}$, or any element but the last of $L_{i+(n-1)m}$ by a similar reasoning. Finally, on the $(j+1)$st rank, we add all elements $x - s$ where x is either an element we have already added, the last element of the $L_{i+(j-1)m}$, or any element but the last of L_{i+jm}. Notice inductively that any rank of T consists of an arithmetic progression of common difference k. The specifically the $(j+1)$st rank of T forms an arithmetic progression of length d for some $d > 0$ with common difference k, and furthermore T does not yet contain any elements in any higher ranks. Then on each subsequent rank we may add again all integers $x - s$ where x is not the least integer in the previous rank (or equivalently $x - (s_k)$ where x is not the greatest integer of the previous rank), creating an arithmetic sequence of one fewer integer on the subsequent rank. Then inductively, we conclude after $d - 1$ ranks by adding a single element, and no element may be added to the next higher rank.

Thus, we have constructed the up-generation $\hat{\mathcal{L}}$. Notice that for each element $x \in \hat{\mathcal{L}}$ other than the last one we added, either $x - s \in \hat{\mathcal{L}}$ or $x - (s + k) \in \hat{\mathcal{L}}$. Therefore, $\hat{\mathcal{L}}$ is a principal ideal, generated by that final element. By construction, its intersection with E is exactly $\mathcal{L}$, since at no point did we add any $x \in E$ when up-generating $\hat{\mathcal{L}}$ $\square$

Given both Lemma 2.18 and Lemma 2.14, we know that $\langle H \rangle \cap E$ must have maximal length, and given the length $|\langle H \rangle \cap E|$ that the first element of $\langle H \rangle \cap E$ must be maximal. We can now present the precise interval on E that gives us the largest hook length H in all possible cases. Recall that $\widetilde{m} := \min\{\pm m^{-1} \bmod k\}$.

Proposition 2.27. *We have that $\langle H \rangle \cap E$ is precisely the union of $\widetilde{m}$ adjacent ledges beginning with L_i, where*

$$i = \begin{cases} k-1 & \text{if } m = 1 \\ m-2 & \text{if } m \geq 3 \text{ and } m^{-1} \bmod k < \frac{k}{2} \\ m-1 & \text{if } m^{-1} \bmod k > \frac{k}{2}. \end{cases}$$

Proof. **Case I:** $m = 1$. In this case, by Proposition 2.22, all ledges L_i for $i > 0$ have the same size, namely $\lfloor (s-1)/k \rfloor + 1 \geq 2$, while L_0 has one fewer element. Consider the first and last points in an interval of E of length $\lfloor (s-1)/k \rfloor + 2$, which we denote x and y respectively. Then,

$$y = x - \left\lfloor \frac{s-1}{k} \right\rfloor k + s = x - 1.$$

Thus, any interval of E of length at least $\lfloor (s-1)/k \rfloor + 2$ contains consecutive elements. This proves that the principal ideal $\langle q \rangle$ up-generated by L_{k-1} maximizes the size of $\langle q \rangle \cap E$ among those principal ideals with no consecutive elements.

We have $a_{\langle q \rangle} = s-2$. The longest interval of E starting at $s-1$ is L_0, which has length $\lfloor (s-1)/k \rfloor < |L_{k-1}|$. This proves that $\langle q \rangle = \langle H \rangle$.

Case II: $m \geq 3$ and $m^{-1} \bmod k < k/2$. Consider the ledges

$$\mathcal{L} = L_{m-2}, L_{2m-2}, \ldots, L_{\widetilde{m}m-2} = L_{m-2}, L_{2m-2}, \ldots, L_{k-1}.$$

We usually think of $\mathcal{L}$ as the union of the above ledges, although occasionally it may become convenient to consider $\mathcal{L}$ as a set of ledges. We have that $\mathcal{L}$ is an interval not containing L_0, because $L_{m-1}, L_{2m-1}, \ldots, L_0$ is an interval of the same length containing L_0, and L_{m-2} is not in the latter interval. Given an interval $I \subseteq E$ of length greater than the length of $\mathcal{L}$ in E, it must intersect more than $\widetilde{m}$ ledges, because any interval of $\widetilde{m}$ ledges has length in E at most that of $\mathcal{L}$ by Lemma 2.24. Thus, for some ℓ, I intersects L_ℓ and $L_{\ell+1}$. We can translate I so that its first element is the first element of L_ℓ, and the translated interval has consecutive integers if and only if the original one does. So assume I is such a translated interval. If I does not contain consecutive elements, $|L_{\ell+1}| = |L_\ell| + 1$, because for each element $x \in L_\ell$, we have $x+1 \in L_{\ell+1}$. This is impossible by Proposition 2.22, so $\mathcal{L}$ is the longest interval in E without consecutive elements.

Since $a_{\langle H \rangle}$ is the first element of $\langle H \rangle \cap E$, it suffices to show that $\mathcal{L}$ has the greatest first element among all intervals of E beginning in $[s-1]$ of the same length. The first element of $\mathcal{L}$ is $s-2$, so the only greater element in $[s-1]$ is $s-1$, the first element of L_{m-1}. But the longest interval in E starting with L_{m-1} is $L_{m-1}, L_{2m-1}, \ldots, L_0$, which is shorter by Proposition 2.22.

Case III: $m^{-1} \bmod k > k/2$. Consider the ledges

$$\mathcal{L} = L_{m-1}, L_{2m-1}, \ldots, L_{\widetilde{m}m-1} = L_{k-2}.$$

We have that $\mathcal{L}$ is an interval not containing L_0 because $\mathcal{L}$ is immediately preceded by the interval $L_m, L_{2m}, \ldots, L_{\widetilde{m}m} = L_{k-1}$; Then since $2\widetilde{m} < k$, we know that $L_{2\widetilde{m}m} = L_{k-2}$ precedes L_0. By similar reasoning as the previous case, $\mathcal{L}$ is the longest interval in E without consecutive elements.

Since $\mathcal{L}$ begins with $s-1$, it up-generates the ideal $\langle H \rangle$ with maximal $a_{\langle H \rangle}$. □

2.8 Obtaining a closed form

Proof of Theorem 2.1. We have $H = a_{\langle H \rangle} + s \cdot (h(\langle H \rangle) - 1)$.

Case I: $m = 1$. We have $a_{\langle H \rangle} = s-2$, and $h(\langle H \rangle) = \lfloor (s-1)/k \rfloor + 1$. Thus,

$$H = s - 2 + s \left\lfloor \frac{s-1}{k} \right\rfloor = B - 2.$$

Case II: $m \neq 1$. We have

$$a_{\langle H \rangle} = i + \left\lfloor \frac{s-1}{k} \right\rfloor k$$

where $i = m-2$ if $m^{-1} \bmod k < k/2$ and $i = m-1$ if $m^{-1} \bmod k > k/2$. Using Lemma 2.25, we have

$$s(h(\langle H \rangle) - 1) = s\left(\left\lceil \frac{m-1}{k}\widetilde{m}\right\rceil + \widetilde{m}\left(\left\lfloor \frac{s-1}{k}\right\rfloor + 1\right) - 1\right).$$

Thus,

$$H = i + \left\lfloor \frac{s-1}{k}\right\rfloor k + s\left(\left\lceil \frac{m-1}{k}\widetilde{m}\right\rceil + \widetilde{m}\left(\left\lfloor \frac{s-1}{k}\right\rfloor + 1\right) - 1\right) = B - j,$$

where $j = 2$ if $m^{-1} \bmod k < k/2$ and $j = 1$ if $m^{-1} \bmod k > k/2$. □

3 Generalization to maximum hook length of $(s, s+k)$-core partitions with d-distinct parts for gcd$(s, k) = 1$.

Now we turn to the question of finding the maximum hook length over all (s, t)-core partitions whose parts differ by at least d in size, which we denote by H_d. We again provide an algorithm for how to find H_d given any s and t. We can formalize this as follows.

Definition 3.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a partition where $\lambda_i \geq \lambda_i$. We call λ *d-distinct* if $\lambda_i \geq \lambda_{i+1} + d$ for all $i \in [m-1]$.

Definition 3.2. Let S be a set of distinct integers. We call S *d-distinct* if $|s_i - s_j| \geq d$ for all $s_i, s_j \in S$ where $s_i \neq s_j$.

Definition 3.3. Let $a, b \in \mathbb{Z}$. Then $a \sim b$ denotes that $|a - b| \leq d$.

As in Section 2, we have a bijection between d-distinct principal ideals $\langle q \rangle$ and d-distinct intervals on E. This allow us to reduce our algorithm to finding the optimal d-distinct interval on E that gives the largest corresponding hook length q.

For now, assume $d < k/2$.

Definition 3.4. A *ledge* L_i is the set

$$L_i = \{n \in E \mid n \equiv i \pmod{k}\}.$$

Definition 3.5. Let $\widetilde{m}_d = \min\{\ell m^{-1} \bmod k \mid -d \leq \ell \leq d, \ell \neq 0\}$. In the future we let ℓ denote the minimizing value of ℓ.

Definition 3.6. An $\widetilde{m}_d$-interval is an ordered subset of $\mathbb{Z}/k\mathbb{Z}$ of the form

$$\{d, d+m, \ldots, d + (\widetilde{m}_d - 1)m\}$$

for some integer d.

Definition 3.7. For any $\widetilde{m}_d$-interval I, define $I + a := \{i + a \bmod k \mid i \in I\}$.

Lemma 3.8. *Assume $d < k/2$. We have*

$$\alpha := \max_I |I \cap [m-1]| = \left\lceil \frac{m\widetilde{m}_d - 1}{k} \right\rceil,$$

where the maximum is taken over all $\widetilde{m}_d$-intervals I that do not include both 0 and m. Further, if $\ell \neq 1$, the $\widetilde{m}_d$-interval $\{m-1, \dots, \ell-1\}$ attains this maximum. Finally, for intervals I and I' not containing 0, we have

$$||I \cap [m-1]| - |I' \cap [m-1]|| \leq 1.$$

Proof. First we show that $\left\lceil \frac{m-1}{k}\widetilde{m}_d \right\rceil \leq \alpha$. Note that over all $\widetilde{m}_d$-intervals I, the average size of $|I \cap [m-1]| = \frac{m-1}{k}\widetilde{m}_d$. Furthermore, for any $\widetilde{m}_d$-interval I' that contains both 0 and m and any $\widetilde{m}_d$-interval I that does not contain both 0 and m, we have

$$|I' \cap [m-1]| \leq |I \cap [m-1]|$$

for any interval I that does not contain both 0 and m. To see this, I' cannot contain either 1 or $m-1$ by definition of $\widetilde{m}_d$. Then $I'+1$ cannot contain both 0 and m, and for every element in $i \in I' \cap [m-1]$, we have $i+1 \in I \cap [m-1]$. Furthermore, for any interval I' containing only 0, we have

$$1 \in I'+1 \cap [m-1] \Rightarrow |I'+1 \cap [m-1]| > |I' \cap [m-1]|.$$

Thus, the average of $|I \cap [m-1]| = \frac{m-1}{k}\widetilde{m}_d$ over all $\widetilde{m}_d$-intervals I that do not contain both 0 and m is strictly greater than $\frac{m-1}{k}\widetilde{m}_d$, and by definition of α as a maximum, we have

$$\alpha > \frac{m-1}{k}\widetilde{m}_d \Rightarrow \alpha \geq \left\lceil \frac{m-1}{k}\widetilde{m}_d \right\rceil$$

as desired.

Now we show that $\alpha \leq \left\lceil \frac{m\widetilde{m}_d}{k} \right\rceil$. Define $\alpha' = \max_I |I \cap [0, m-1]|$ over all $\widetilde{m}_d$-intervals I. Then clearly $\alpha \leq \alpha'$, so it suffices to show that $\alpha' \leq \left\lceil \frac{\widetilde{m}_d m}{k} \right\rceil$. Further note that the average of $|I \cap [0, m-1]|$ over all $\widetilde{m}_d$-intervals I is $\frac{m\widetilde{m}_d}{k}$.

Consider the sequence $I, I+1, \dots, I+(k-1)$ of all k $\widetilde{m}_d$-intervals I (starting with any I). Then it suffices to show that for any j, $|I \cap [0, m-1]|$ and $|I+j \cap [0, m-1]$ differ by at most 1. This would show that over all intervals I, $|I \cap [m-1]|$ varies by at most one, and thus $\alpha' < \frac{m\widetilde{m}_d}{k} + 1 \Rightarrow \alpha' \leq \left\lceil \frac{m\widetilde{m}_d}{k} \right\rceil$.

Let $[m-j, m-1]$ denote the interval $[m-j \bmod k, k-1] \cup [0, m-1]$. We claim that

$$|I+j \cap [0, m-1]| = |I \cap [0, m-1]| + |I \cap [k-j, k-1]| - |I \cap [m-j, m-1]|.$$

To see this, note that $|I \cap [k-j, k-1]|$ counts the number of times there is an element $i \notin |I+a \cap [0, m-1]|$ such that $i+1 \in I+(a+1) \cap [m-1]|$, i.e. the number of times we gain an element in $[0, m-1]$ from $I+a$ to $I+(a+1)$. Similarly, $|I \cap [m-j, m-1]|$

counts the number of times such that there is an element $i \in |I + a \cap [0, m-1]|$ such that $i + 1 \notin |I + (a+1) \cap [0, m-1]|$, i.e. the number of times we lose an element in $[0, m-1]$ from $I + a$ to $I + (a+1)$.

So it remains to show that $|I \cap [k-j, k-1]|$ and $|I \cap [m-j, m-1]|$ differ by at most one. However, note that given that I is an $\widetilde{m}_d$-interval, we have

$$i \in I \cap [k-j, k-1] \Rightarrow i + m \in |I \cap [m-j, m-1]|$$

unless i is the last element of I, and similarly that

$$j \in |I \cap [m-j, m-1]| \Rightarrow j - m \in |I \cap [k-j, k-1]|$$

unless j is the first element of I. Let A denote the event that the last element of I lies in $[k-j, k-1]$, and let B denote the event that the first element of I lies in $[m-j, m-1]$. Both when neither A or B occur and when both A and B occur, we have that $|I \cap [k-j, k-1]| = |I \cap [m-j, m-1]|$. When A occurs and B does not, then $|I \cap [k-j, k-1]| = |I \cap [m-j, m-1]| + 1$. When B occurs and A does not, then $|I \cap [m-j, m-1]| = |I \cap [k-j, k-1]| + 1$. In all of these cases we have $|I \cap [k-j, k-1]|$ and $|I \cap [m-j, m-1]|$ differ by at most one, as desired.

Now notice that $\lceil (\widetilde{m}m - 1)/k \rceil = \lceil \widetilde{m}m/k \rceil$ precisely when $\ell \neq 1$. Now let $\ell \leq 1$. Notice that in this case $-k < \ell - 1 \leq 0$ as $\ell \leq d \leq k/2$. So it now suffices show that $-k < \ell - \widetilde{m}_d \leq 0$. If $\ell > -k/2$, then this holds as $1 \leq \widetilde{m}_d \leq k/2$. If $\ell = -k/2$, then $\widetilde{m}_d = 1$, and so the inequality again follows. In particular we see that for $\ell < 0$, the bounds $\lceil (m-1)\widetilde{m}_d/k \rceil \leq \alpha \leq \lceil m\widetilde{m}_d/k \rceil$ imply that

$$\alpha = \lceil (m\widetilde{m}_d - 1)/k \rceil, \ \ell < 0.$$

When $\ell = 1$, we may appeal to Lemma 2.25. The the only case that remains is $\ell > 1$.

Suppose $\ell > 1$. Over all $\widetilde{m}_d$-intervals I, the average of $|I \cap [0, m-1]|$ is

$$\frac{m\widetilde{m}_d}{k}.$$

Hence, there is at least one $\widetilde{m}_d$-interval with

$$|I \cap [0, m-1]| \geq \left\lceil \frac{m\widetilde{m}_d}{k} \right\rceil.$$

It remains to show that there is such an $\widetilde{m}_d$-interval not containing 0. Observe that the $\widetilde{m}_d$-interval $I = \{m-1, \ldots, \ell - 1\}$ does not include 0 (or else it would include m, which is impossible). Thus, it suffices to show that I is an $\widetilde{m}_d$-interval maximizing $|I \cap [0, m-1]|$. (In fact, the proof of this claim only relies on $\ell \neq 1$ and hence proves the second part of the lemma statement.) Given any $\widetilde{m}_d$-interval J not containing $m-1$, we have

$$|(J+1) \cap [0, m-1]| \geq |J \cap [0, m-1]|.$$

In fact, we still have this inequality if J contains $m-1$ and $k-1$, which happens if and only if J contains $m-1$ not as its first element. Therefore, by repeatedly adding 1 to J until we obtain I, we conclude that

$$|I \cap [0, m-1]| \geq |J \cap [0, m-1]|,$$

as desired. $\square$

Proposition 3.9. *We have*

$$|L_i| = \begin{cases} 0 & \text{if } s|i \text{ and } i \neq 0 \\ \left\lfloor \frac{s-1}{k} \right\rfloor & \text{if } i = m \\ 1 & \text{if } i = \lceil k/s \rceil \, s \bmod k \text{ and } k > s \\ \left\lfloor \frac{s-1}{k} \right\rfloor + 1 & \text{if } i = 0 \\ \left\lfloor \frac{s-1}{k} \right\rfloor + 1 & \text{if } m < i \text{ and } s \nmid i \\ \left\lfloor \frac{s-1}{k} \right\rfloor + 2 & \text{if } 0 < i < m \text{ and } i \neq \lceil k/s \rceil \, s \bmod k. \end{cases}$$

Proof.
Case (1). $s|i$ and $i \neq 0$
It suffices to notice that $i \notin P$, since i can in this case be written as a linear combination of $s, s+k$. Then also $i+k$ is a linear combination of $s, s+k$. Recall that $E = P \cap [s+k-1]$. As $s < k$ in this case, then $i + 2k \geq s + k$, and so $i + 2k \notin E$. Then no integer congruent to $i \pmod k$ is in E, and thus L_i is empty.
Case (2). $i = m$
We claim that $m + bk \notin E$ for $b \geq \left\lfloor \frac{s-1}{k} \right\rfloor$. Note that when $b = \left\lfloor \frac{s-1}{k} \right\rfloor$, we have

$$\begin{aligned} m + bk = m + \left\lfloor \frac{s-1}{k} \right\rfloor \cdot k &= (m-1) + \left\lfloor \frac{s-1}{k} \right\rfloor \cdot k + 1 \\ &= (s - 1 \bmod k) + \left\lfloor \frac{s-1}{k} \right\rfloor \cdot k + 1 \\ &= (s-1) + 1 = s, \end{aligned}$$

since $(s,k) = 1 \Rightarrow m > 0$. But $s \notin P \Rightarrow s \notin E$. Also, for any $b > \left\lfloor \frac{s-1}{k} \right\rfloor$, we have $m + bk \geq s + k$, and thus again $m + bk \notin E$.

Now we claim that for any $b < \left\lfloor \frac{s-1}{k} \right\rfloor$, $m + bk \in E$. Indeed, for $b < \left\lfloor \frac{s-1}{k} \right\rfloor$ we have $m + bk < s \Rightarrow m + bk \in E$.

Thus we have that

$$L_m = \{m + bk \mid 0 \leq b \leq \left\lfloor \frac{s-1}{k} \right\rfloor - 1\},$$

and so $|L_m| = \left\lfloor \frac{s-1}{k} \right\rfloor$.

Case (3). $i = \lceil k/s \rceil s \bmod k$ and $k > s$
We have

$$k < \left\lceil \frac{k}{s} \right\rceil s < k + s.$$

Since $k > s$,

$$0 < \left\lceil \frac{k}{s} \right\rceil s \bmod k < s,$$

so $i \in E$.

Next we show that $i + bk \geq s + k$ for any $b \geq 2$, and hence $i + bk \notin E$. It suffices to show for $b = 2$. We have $i + 2k > 2k > s + k$.

Finally, we notice that $i + k = \lceil k/s \rceil s$. Then $s \mid i + k$, and so $i + k \notin P$. Then i is the only element of its ledge.

Case (4). $i = 0$
First we show that $bk \in E$ for $1 \leq b \leq \left\lfloor \frac{s-1}{k} \right\rfloor + 1$. Note that $bk \leq \left(\left\lfloor \frac{s-1}{k} \right\rfloor + 1\right) k \leq \left(\frac{s-1}{k} + 1\right) k = s + k - 1$. Then to show that each of these elements is in P, we must show that they are not linear combinations of $s, s+k$. If they are, then they need to be multiples of s, as they are each smaller than $s + k$. Since s, k are coprime, $s \mid bk \Rightarrow s \mid b \Rightarrow s \leq b$. Thus, it suffices to show that $b < s$. This follows from $k > 1$ and $s > 1$. Thus $bk \in E$.

Now we show that $bk \notin E$ for $b > \left\lfloor \frac{s-1}{k} \right\rfloor + 1$. Note that when $b = \left\lfloor \frac{s-1}{k} \right\rfloor + 2$, we have

$$bk = \left(\left\lfloor \frac{s-1}{k} \right\rfloor + 2\right) k > \left(\frac{s-1}{k} + 1\right) k = s + k - 1,$$

so $bk \notin E$.

Thus we have that

$$L_0 = \{bk \mid 1 \leq b \leq \left\lfloor \frac{s-1}{k} \right\rfloor + 1\},$$

and so $|L_0| = \left\lfloor \frac{s-1}{k} \right\rfloor + 1$.

The reader is advised to remove his socks, lest they be blown off by the proof of the following case.

Case (5). $m < i$ and $s \nmid i$

We claim that $i + bk \notin E$ for $b \geq \left\lfloor \frac{s-1}{k} \right\rfloor + 1$. Note that when $b = \left\lfloor \frac{s-1}{k} \right\rfloor + 1$, we have

$$i + bk = i + \left(\left\lfloor \frac{s-1}{k} \right\rfloor + 1\right) \cdot k = i + \left\lfloor \frac{s+k-1}{k} \right\rfloor \cdot k > s + k - 1,$$

where the last inequality follows since

$$k\left(\frac{s+k-1}{k} - \left\lfloor \frac{s+k-1}{k} \right\rfloor\right) = s - 1 \bmod k = m - 1 < i.$$

Now it suffices to show that $i + bk \in E$ when $b = \left\lfloor \frac{s-1}{k} \right\rfloor$. We have that

$$i + \left\lfloor \frac{s-1}{k} \right\rfloor k \leq i + s - 1 < s + k - 1,$$

so it remains to show that $i+\left\lfloor\frac{s-1}{k}\right\rfloor k \in P$, which is equivalent to showing that $s \nmid i+\left\lfloor\frac{s-1}{k}\right\rfloor k$. Note that from Case (2), $s \mid m+\left\lfloor\frac{s-1}{k}\right\rfloor k$, so it suffices to show that $i \not\equiv m \pmod{s}$. If $s > k$, then this follows from $i > m$, for suppose $i \equiv m \pmod{s}$. Then $i \geq m+s > k$, a contradiction. If $s < k$, then $m = s$, and so $s \nmid i \Rightarrow i \not\equiv m \pmod{s}$.

Thus, we have that

$$L_i = \{i + bk \mid 0 \leq b \leq \left\lfloor\frac{s-1}{k}\right\rfloor\}$$

and so $|L_i| = \left\lfloor\frac{s-1}{k}\right\rfloor + 1$.

Case (6). $0 < i < m$ and $i \neq \lceil k/s \rceil\, s \bmod k$.

We claim that $i + bk \notin E$ for $b \geq \left\lfloor\frac{s-1}{k}\right\rfloor + 2$. Notice that for $b = \left\lfloor\frac{s-1}{k}\right\rfloor + 2 > \frac{s-1}{k} + 1$, we have $bk > s + k - 1$, and so $i + bk \notin E$.

Now let $b = \left\lfloor\frac{s-1}{k}\right\rfloor + 1$. Then

$$i + bk = i + \left(\left\lfloor\frac{s-1}{k}\right\rfloor + 1\right) k \leq s + k - 1,$$

where the last inequality follows since $k(\frac{s-1}{k} - \left\lfloor\frac{s-1}{k}\right\rfloor) = m - 1 \geq i$.

We claim $s \nmid i + bk$. From Case (2) we know that $s \mid (b-1)k + m$. Thus, it suffices to show $s \nmid k + i - m$. To see this, first suppose $s < k$. Then, we have $s = m$, so it suffices to show that $i + k \not\equiv 0 \pmod{s}$. Note that

$$\left(\left\lceil\frac{k}{s}\right\rceil s \bmod k\right) + k = \left\lceil\frac{k}{s}\right\rceil s \equiv 0 \pmod{s}.$$

We also have $\left(\left\lceil\frac{k}{s}\right\rceil s \bmod k\right) < s$, and $i < m \Rightarrow i < s$. Then $i \neq \left\lceil\frac{k}{s}\right\rceil s \bmod k$ gives us that $i \not\equiv \left\lceil\frac{k}{s}\right\rceil s \bmod k \pmod{s}$, and so $i + k \not\equiv 0 \pmod{s}$.

Now suppose $k < s$. Suppose for contradiction that $i + k \equiv m \pmod{s}$. Note that $i + k < m + s$, so we must have $i + k = m$. But $m = s \bmod k < k$, a contradiction.

Thus, $s \nmid i + bk$, so $i + bk \in E$. Hence, $|L_i| = \left\lfloor\frac{s-1}{k}\right\rfloor + 2$.

□

Proposition 3.10. *Suppose that $d < k/2$. We have that $\langle H\rangle \cap E$ is the interval $\mathcal{I}$ of E, where $\mathcal{I}$ contains the union of $\widetilde{m}_d$ adjacent ledges beginning at L_i—excluding the non-minimal element in L_i, if any—where*

$$i = \begin{cases} k-1 & \text{if } m = 1 \\ m-2 & \text{if } m \geq 3 \text{ and } \ell = 1 \\ m-1 & \text{if } \ell = -1 \text{ or } \ell \geq 2 \\ m-1-\ell & \text{if } \ell \leq -2. \end{cases}$$

If $\ell \leq -2$, $\mathcal{I}$ additionally contains the last element of L_{i-m} and the first element of L_i. If $\ell \leq 1$, $\mathcal{I}$ additionally contains the first element in $L_{i+\ell}$. These are all the elements in $\mathcal{I}$.

Proof. If $\ell = \pm 1$, then the proof proceeds exactly as in the case of $d = 1$. From now on in the proof, suppose that $\ell \neq \pm 1$.

Case I: $\ell \geq 2$. In this case, we have

$$\mathcal{I} = L'_{m-1}, L_{2m-1}, \ldots, L_{m\widetilde{m}_d - 1},$$

where L'_{m-1} excludes the non-minimal element in L_{m-1}, if any.

Suppose for the sake of contradiction there is an interval ideal $\mathcal{I}' \subseteq E$ longer than $\mathcal{I}$. Then, $\mathcal{I}'$ must intersect more than $\widetilde{m}_d$ ledges by Lemma 3.8 and Proposition 3.9, as the longest length of a union of $\widetilde{m}_d$ adjacent ledges excluding the first non-minimal element (if any) is determined by the number of those ledges L_j where $j \in [m-1]$. Thus, for some j, $\mathcal{I}'$ intersects L_j and $L_{j+\ell}$.

We claim one can translate $\mathcal{I}'$ within E so that its first element is the first minimal element in L_j, all while preserving d-distinctness. Suppose for contradiction that such a translated interval is not d-distinct, so it contains $a \in L_j$ and $a + \ell \in L_{j+\ell}$. We have that if $a - k \in L_j$, then $a + \ell - k \in L_{j+\ell}$, because $a + \ell - k > a - k$. Thus, for some b, $a - bk, a + \ell - bk \in \mathcal{I}'$, so $\mathcal{I}'$ is not d-distinct. Therefore, we now suppose that the first element in $\mathcal{I}'$ is the first minimal element in L_j.

Since $\mathcal{I}'$ contains every element in $L_j \cap [s-1]$, $\mathcal{I}'$ cannot contain any elements in $L_{j+\ell} \cap [\ell+1, s+\ell-1]$. Therefore, if $a \in L_{j+\ell} \cap \mathcal{I}'$, then $a \geq s + \ell > s$, and so in particular a is non-minimal in P. Then $|L_{j+\ell} \cap \mathcal{I}'| = 1$. Further, since $s + \ell \leq a \leq s + k - 1$, we have $s \leq a - \ell \leq s + k - 1 - \ell$, where $a - \ell \equiv j \pmod{k}$. Thus, $j \notin [m-\ell, m-1]$.

First, suppose $j \leq m - \ell - 1$. We claim that $\mathcal{I}'$ cannot be longer than $\mathcal{I}$, contradicting our choice of $\mathcal{I}'$. Let $J = \{m-1, 2m-1, \ldots, m\widetilde{m}_d - 1\}$ and $J' = \{j, j+m, \ldots, j + (\widetilde{m}_d - 1)m\}$ be $\widetilde{m}_d$-intervals. It suffices to show that $|J' \cap [m-1]| < |J \cap [m-1]|$. Since $0 \notin J \cup J'$, this is equivalent to $|J' \cap [0, m-1]| < |J \cap [0, m-1]|$. Since $J = J' + m - j - 1$, this is in turn equivalent to $|J' \cap [j+1, m-1]| < |J' \cap [k+j+1-m, k-1]|$. Observe that $j \notin [j+1, m-1]$, $j + (\widetilde{m}_d - 1)m = j + \ell - m \in [k+j+1-m, k-1]$, and every other element of J' in $[k+j+1-m, k-1]$ is immediately followed by an element in $[j+1, m-1]$. It follows that $|J' \cap [j+1, m-1]| + 1 = |J' \cap [k+j+1-m, k-1]|$, completing the proof of the claim that $\mathcal{I}'$ cannot be longer than $\mathcal{I}$.

Now suppose for contradiction that $j \geq m$. Let I' and I be the corresponding $\widetilde{m}_d$-intervals for $\mathcal{I}'$ and $\mathcal{I}$, respectively, so

$$\begin{aligned} I' &= \{j, j+m, \ldots, j + (\widetilde{m}_d - 1)m\} \\ I &= \{m-1, 2m-1, \ldots, \widetilde{m}_d m - 1\}. \end{aligned}$$

Since $|L_{j+\ell} \cap \mathcal{I}'| = 1$, in order for $\mathcal{I}'$ to be strictly longer than $\mathcal{I}$, we must have $|I' \cap [m-1]| \geq |I \cap [m-1]|$. Note that by considering $I', I'-1, I'-2, \ldots, I$, i.e. repeatedly subtracting 1 from all elements in I', we have

$$0 \geq |I \cap [m-1]| - |I' \cap [m-1]| = |I' \cap [m, j]| - |I' \cap [0, j-m]|.$$

But note that for any element $e \in I' \cap [0, j-m]$, we have that $e + m \in I' \cap [m, j]$ unless e is the last element of I', which corresponds to the last full ledge in interval $\mathcal{I}'$. The last element of I' is exactly $j + (\widetilde{m}_d - 1)m = j - m + \ell$. However, note that $\ell \leq m$. To see this, suppose that $\ell > m$. We have $\ell \leq d \Rightarrow m < d$, so $\widetilde{m} = 1 \Rightarrow \ell = m$. Thus $j - m < j - m + \ell \leq j$, so $j - m + \ell \bmod k \notin [0, j - m]$. Thus, we have

$$e \in I' \cap [0, j - m] \Rightarrow e + m \in I' \cap [m, j]$$

for all e, which implies that $|I' \cap [0, j - m]| \leq |I' \cap [m, j]|$. In addition, we similarly have that for any element $e \in [m, j]$ that $e - m \in [0, j - m]$ unless e is the first element of I'. Indeed, $j \in I' \cap [m, j]$, and j is our first element of I' with no corresponding element $j - m \in I' \cap [0, j - m]$. Thus, we actually have

$$|I' \cap [m, j]| = |I' \cap [0, j - m]| + 1 > |I' \cap [0, j - m]|,$$

a contradiction.

Thus we have shown that $\mathcal{I}$ is the longest interval of E we can choose. Since ledge L_{m-1} has the maximum first minimal element, by Proposition 2.14, we have $\mathcal{I}$ yields the largest H, as desired.

Case II: $\ell \leq -2$. In this case, we have

$$\mathcal{I} = a, L_{m-1-\ell}, \ldots, L_{k-1}, b,$$

where a is the last element of $L_{-1-\ell}$, and b the first element of L_{m-1}. First, we show that $\mathcal{I}$ is d-distinct. In particular, since the union of $\widetilde{m}_d$ ledges $L_{m-1-\ell}, \ldots, L_{k-1}$ is d-distinct, then it suffices to show that adding elements a, b to those ledges preserves d-distinctness. Notice that a may only be within d of elements of L_{k-1}. Since $d < k/2$, then a can only be within d of with at most one element on that ledge (and more strongly, only one integer in residue class $k - 1$). Furthermore, since $a + \ell = -1$, which lies in the residue class of $k - 1$, then a is not within d of any element of L_{k-1}. Similarly, $b = s + k - 1$, so $b - \ell \notin E$ is the only element x in the residue class of $m - 1 - \ell$ for which $|b - x| \leq d$. Thus, b is not within d of any elements in $\mathcal{I}$.

Suppose for the sake of contradiction there is an interval ideal $\mathcal{I}' \subseteq E$ longer than $\mathcal{I}$. Then by the final statement in Lemma 3.8, $\mathcal{I}'$ must intersect more than $\widetilde{m}_d$ ledges. Suppose that it doesn't, so it only intersects ledges L_j to $L_{j+\ell-m}$. Then since it cannot intersect the first element of L_j, the length of the interval is at most 2 elements longer than the length of $L_{m-1-\ell}, \ldots, L_{k-1}$ without the first element of $L_{m-1-\ell}$, whereas the I is 3 elements longer than the length of $L_{m-1-\ell}, \ldots, L_{k-1}$ without the first element of $L_{m-1-\ell}$. Thus, for some j, $\mathcal{I}'$ intersects L_j and $L_{j+\ell}$.

One can translate $\mathcal{I}'$ until it contains only the last element of L_j or it contains all of L_0, all while preserving d-distinctness and the length of $\mathcal{I}'$. To see this, suppose the translated $\mathcal{I}'$ contains all of L_0 but more than one element in L_j. Then $j = -\ell$, so $k - \ell \in L_j \cap \mathcal{I}'$, which is within d of $k \in L_0$. Thus, we may assume $\mathcal{I}'$ is a translated interval containing only the last element of L_j.

By Lemma 3.8,

$$\mathcal{I}' \supseteq j, L_{j+m}, \ldots, L_{j+\ell}, b'.$$

We claim that we actually have equality above. Suppose not, so $b'-k \in \mathcal{I}'$. Then, $b'-k-\ell \in L_{j+m}$, since $b'-k-\ell < b' \leq s+k-1 \Rightarrow b'-k-\ell \in E$, which contradicts d-distinctness.

Then, since $\mathcal{I}'$ is longer than $\mathcal{I}$, we have

$$|\{j+m, j+2m, \ldots, j+\ell\} \cap [m-1]| = \alpha.$$

Note that we have $j+m \leq m-\ell$. We claim that $m < m-\ell \leq k \Rightarrow m-\ell \notin [m-1]$. Suppose for contradiction that $m-\ell > k \Rightarrow m > k+\ell$, contradicting the definition of ℓ.

Finally we claim that $j+\ell+m < m$. Given that we have $j \in \mathcal{I}'$, we must have $j+\ell \leq 0$, otherwise j would be within d of $j+\ell \in L_{j+\ell}$. If $j+\ell = 0$, then $L_{j+m+\ell} = L_m$, which is impossible since L_m is the first ledge in P. Thus, $j+\ell+m \in [m-1]$.

But then

$$|\{j+2m, \ldots, j+\ell, j+\ell+m\} \cap [m-1]| = \alpha+1,$$

which contradicts the maximality of α.

Thus we have shown that $\mathcal{I}$ is the longest interval of E we can choose. Given a d-distinct interval ideal $\mathcal{I}'$ of the same length as $\mathcal{I}$, we claim it can only contain one element in L_j, the first ledge $\mathcal{I}'$ intersects. If not, the largest element in $L_j \cap \mathcal{I}'$ is within d of an element in $L_{j+\ell} \cap \mathcal{I}'$, contradicting the d-distinctness of $\mathcal{I}'$. Then, by the final statement of Lemma 3.8, $L_{j+\ell} \subseteq \mathcal{I}'$, so $j \leq -\ell$. If $j = -\ell$, then the longest interval in E starting at j is shorter than $\mathcal{I}$, contradicting our choice of $\mathcal{I}'$. Then, $j = -\ell-1$ is the greatest possible first element of $\mathcal{I}'$, which is achieved by $\mathcal{I}$. By Proposition 2.14, $\mathcal{I}$ yields the largest H, as desired. □

Proposition 3.11. *Suppose that $k/2 \leq d < k$. Then we have that $\langle H \rangle \cap E$ is an interval $\mathcal{I}$ in E, where $\mathcal{I}$ contains L_i—excluding the non-minimal element in L_i, if any—where*

$$i = \begin{cases} k-1 & \text{if } m = 1 \\ m-1 & \text{if } 1 < m \leq d \\ k-1 & \text{if } d < m < k-1 \\ m-1 & \text{if } d < m = k-1. \end{cases}$$

If $d < m < k-1$, $\mathcal{I}$ additionally contains the last element of L_{i-m} and the first element of L_i. If $m \notin (1, d]$, then $\mathcal{I}$ additionally includes the first element of L_{i+m}.

Proof. **Case I:** $m = 1$. In this case, we have

$$\mathcal{I} = L'_{k-1}, b,$$

where b is the first element of L_0.

Suppose for the sake of contradiction there is an interval ideal $\mathcal{I}' \subseteq E$ longer than $\mathcal{I}$. Then, for some j, either

$$\mathcal{I}' \supseteq a, L_j,$$

where a is the last element of L_{j-1}, or

$$|\mathcal{I}' \cap L_j| \geq 2$$
$$|\mathcal{I}' \cap L_{j+1}| \geq 2.$$

In the first case, $a+1$ is the last element of L_j, and hence $\mathcal{I}'$ is not d-distinct. In the second case, $c+1 \in \mathcal{I}'$, where c is the first element of $\mathcal{I}' \cap L_j$. So in general, $\mathcal{I}'$ is not d-distinct.

Over all intervals of length $\mathcal{I}$, $\mathcal{I}$ has the largest first element, and so $\mathcal{I}$ is the optimal interval in E.

Case II: $1 < m \leq d$. In this case, we have $\mathcal{I} = L'_{m-1}$. Suppose for the sake of contradiction there is an interval ideal $\mathcal{I}' \subseteq E$ longer than $\mathcal{I}$. Then, for some j, $\mathcal{I}$ intersects L_j and L_{j+m}.

If $j \geq m$, then $\mathcal{I}'$ is longer than L_j by Proposition 3.9, so if a is the first element in $L_j \cap \mathcal{I}'$, then $a+m \in L_{j+m} \cap \mathcal{I}'$. Then $\mathcal{I}'$ is not d-distinct as $|a-(a+m)| \leq d$.

If $j < m$, then let a be the first element of L'_j. Since $j < m$, the last element of L_{j-m} is $j-m+k$, which gives $a = j-m+s$. The first element of L_{j+m} is $j+s$, which is within m of a, and thus within d of a. Then since $\mathcal{I}'$ is at least as long as L_j by Proposition 3.9, we have that $\mathcal{I}'$ contains both $a-bk = (j-m+s)-bk$ and $(j+s)-bk$ for some non-negative integer b, and thus is not d-distinct.

Thus we have shown that $\mathcal{I}$ is the longest interval of E we can choose. Since ledge L_{m-1} has the maximum first minimal element, by Proposition 2.14, we have $\mathcal{I}$ yields the largest H, as desired.

Case III: $d < m < k-1$. In this case, we have

$$\mathcal{I} = a, L_{k-1}, b,$$

where a is the last element of L_{k-1-m} and b is the first element of L_{m-1}. First, we show that $\mathcal{I}$ is d-distinct. It suffices to show that adding a and b to L_{k-1} preserves d-distinctness. Since $m < k-1$, we have $a = k-1-m$. Then $k-1$ and a differ by more than d, and $k-1$ is the smallest element on L_{k-1}, so a, L_{k-1} is d-distinct. Now note that $b = s+k-1$ is the largest element of E, and the largest element of L_{k-1} is $a+s = k-1-m+s$. Then b differs by at least m from any element in L_{k-1}, and thus b, L_{k-1} is d-distinct. Clearly a, b differ by at least d, as their difference is $m+s$, so a, L_{k-1}, b is d-distinct.

Suppose for the sake of contradiction there is an interval ideal $\mathcal{I}' \subseteq E$ strictly longer than $\mathcal{I}$. Then by Proposition 3.9, $\mathcal{I}$ intersects L_j and L_{j+m} for some j. As in the proof of Proposition 3.10, we may translate $\mathcal{I}'$ until its first element is the last element in L_j or it contains all of L_0, all while preserving d-distinctness. In the latter case, $\mathcal{I}$ contains two elements of $L_j = L_{k-m}$, so is not d-distinct. Thus, we may assume $\mathcal{I}'$ is a translated interval containing only the last element of L_j.

If $j+m \bmod k \geq m$, then $\mathcal{I}'$ contains all of L_{j+m} and the first two elements of L_{j+2m} by Proposition 3.9. The second element of L_{j+2m} is within d of an element in L_{j+m}, and hence $\mathcal{I}'$ is not d-distinct.

If $j+m \bmod k < m$, then $\mathcal{I}'$ contains all of L_{j+m} and the first element of L_{j+2m} by Proposition 3.9. Since $j+m \bmod k < m$, we have $j+m \bmod k = j+m-k$. Note that the last element of L_j is j. Then the difference between elements $j+m \bmod k$ and j is $k-m$. Recall that $k/2 \leq d < m < k-1$, so $k-m < k/2 \leq d$, and hence $\mathcal{I}'$ is not d-distinct.

Thus we have shown $\mathcal{I}$ is an interval of E of maximum length. Given a d-distinct interval $\mathcal{I}'$ of the same length as $\mathcal{I}$, we claim that it can only contain one element in L_j, the first ledge

$\mathcal{I}'$ intersects. If not, the largest element in $L_j \cap \mathcal{I}'$ is within d of an element in $L_{j+m} \cap \mathcal{I}'$, contradicting the d-distinctness of $\mathcal{I}'$. Then, by the final statement of Lemma 3.8, $L_{j+m} \subseteq \mathcal{I}'$, so $j \leq k-m$. If $j = k-m$, then the longest interval in E starting at j is shorter than $\mathcal{I}$, contradicting our choice of $\mathcal{I}'$. Then, $j = k-m-1$ is the greatest possible first element of $\mathcal{I}'$, which is achieved by $\mathcal{I}$. By Proposition 2.14, $\mathcal{I}$ yields the largest H, as desired.

Case IV: $d < m = k-1$. In this case, we have

$$\mathcal{I} = L'_{k-2}, b,$$

where b is the first element of $L_{m-2} = L_{k-3}$. Suppose for the sake of contradiction there is an interval ideal $\mathcal{I}' \subseteq E$ longer than $\mathcal{I}$. Then, for some j, either

$$\mathcal{I}' \supseteq a, L_j,$$

where a is the last element of L_{j+1}, or

$$\begin{aligned} |\mathcal{I}' \cap L_j| &\geq 2 \\ |\mathcal{I}' \cap L_{j-1}| &\geq 2. \end{aligned}$$

In the first case, $a-1$ is the last element of L_j when $a > 1$, and hence $\mathcal{I}'$ is not d-distinct. When $a = 1$, there is no interval of E beginning at a which is longer than $\mathcal{I}$ since $L_j = L_0$, the last ledge.

In the second case, $c - 1 \in \mathcal{I}'$, where c is the first element of $\mathcal{I}' \cap L_j$. Note that $c \neq 1$, as if $c = 1$ we are in the first case of the previous paragraph. So in general, $\mathcal{I}'$ is not d-distinct.

Thus we have shown that $\mathcal{I}$ is the longest interval of E we can choose. Since ledge L_{m-1} has the maximum first minimal element, by Proposition 2.14, we have $\mathcal{I}$ yields the largest H, as desired. □

3.1 A closed form

Theorem 3.12. *Let s and k be positive, coprime integers with $s \geq 2$. The maximum hook length H_d of an $(s, s+k)$-core partition with d-distinct parts can be computed as*

$$H_d(s,k) = \begin{cases} s-1 & \text{if } k = 1 \text{ or } k, s \leq d \\ s+k-1 & \text{if } k \neq 1 \text{ and } k \leq d < s \\ B-2 & \text{if } d < k \text{ and } m\widetilde{m}_d \bmod k = 1 \\ B-s-1 & \text{if } d < k \text{ and } 1 < m\widetilde{m}_d \bmod k \leq d \\ B+k-m\widetilde{m}_d-1 & \text{if } d < k \text{ and } d < m\widetilde{m}_d \bmod k < k-1 \\ B-1 & \text{if } d < k \text{ and } 1 < m\widetilde{m}_d \bmod k = k-1, \end{cases}$$

where

$$\begin{aligned} B &= \left\lfloor \frac{s-1}{k} \right\rfloor (k + s\widetilde{m}_d) + s\left(\left\lceil \frac{m\widetilde{m}_d - 1}{k} \right\rceil + \widetilde{m}_d - 1\right) + m, \\ m &= s \bmod k, \ \text{ and} \\ \widetilde{m}_d &= \min\{\ell m^{-1} \bmod k \mid -d \leq \ell \leq d, \ell \neq 0\}. \end{aligned}$$

Proof. **Case I:** $k = 1$ or $k, s \leq d$. The case $k = 1$ has already been proven above. If $k, s \leq d$, any two minimal elements are within d of each other, and any element is within d of its children. Hence, a d-distinct β-set has only one element. Since $s - 1$ is the maximum minimal element in P, $H_d = s - 1$.

Case II: $k \neq 1$ and $k \leq d < s$. In this case, consecutive elements in E are within d, so $\langle H_d \rangle$ can only have 1 minimal element. Since $s + k - 1$ is the maximum element with this property (given that it is the maximum element in E), $H_d = s + k - 1$.

Cases III–VI: $d < k$. In each of these cases, we use Proposition 3.10 and Proposition 3.11. We use the following formula to calculate H_d, which follows immediately from the definition $a_{\langle q \rangle}$ in Section 2.3:

$$H_d = a_{\langle H_d \rangle} + s \cdot (h(\langle H_d \rangle) - 1),$$

where we can calculate $h(\langle H_d \rangle)$ using Proposition 2.18 and Proposition 3.9. □

4 Maximum hook length of $(s, s + k)$-core partitions with d-distinct parts.

In the 1-distinct case, recall that the number of $(s, s+k)$-core partitions is finite iff $\gcd(s, s+k) = 1$. However, in general, we have the following condition.

Theorem 4.1 ([7])**.** *The number of $(s, s+k)$-core partitions with d-distinct parts is finite if and only if $d \geq gcd(s, s+k)$.*

Consequently, there exists a maximum hook length exactly when $d \geq \gcd(s, s+k)$. For a d-distinct (bs, bk)-core partition, we can reduce the formula for maximum hook length to the previous cases in Section 3, where s, k are coprime. We present the closed form in the following theorem.

Theorem 4.2. *Let $gcd(s, k) = 1$. Then for any integer $b > 1$ and $0 \leq c < b$, we have*

$$H_{bd+c}(bs, bk) = \begin{cases} b\left(H_d\left(s, k\right) + 2\right) - 1 & \text{if } k = 1 \text{ and } d < s \\ b\left(H_d\left(s, k\right) + 1\right) - 1 & \text{if } k = 1 \text{ and } d \geq s \\ b\left(H_d\left(s, k\right) + 2\right) - 1 & \text{if } d < k \text{ and } (m\widetilde{m}_d \bmod k = 1 \\ & \quad \text{or } d < m\widetilde{m}_d \bmod k = k - 1) \\ b\left(H_d\left(s, k\right) + 1\right) - 1 & \text{if } k \neq 1 \text{ and } (m\widetilde{m}_d \bmod k = k - 1 = d \\ & \quad \text{or } (m\widetilde{m}_d \bmod k \neq 1, k - 1) \text{ or } d \geq k). \end{cases}$$

By $\langle n \rangle_b$, we mean $\{n - a_1 bs - a_2 b(s + k) \geq 0 \mid a_1, a_2 \in \mathbb{N}_{\geq 0}\}$. Notice that $\langle n \rangle_1 = \langle n \rangle$ for $n \in P$.

Lemma 4.3. *We have $H_{bd+c}(bs, bk) = H_{bd}(bs, bk)$.*

Proof. Note first that since $bd + c \geq bd$, we have $H_{bd+c}(bs, bk) \leq H_{bd}(bs, bk)$. Now observe that all elements in $\langle H_{bd}(bs, bk)\rangle_b$ are equivalent modulo b. Therefore, any two elements within $bd + c$ of each other are also within bd, since $c < b$. Hence, $H_{bd}(bs, bk)$ is $(bd + c)$-distinct, and so $H_{bd+c}(bs, bk) \geq H_{bd}(bs, bk)$. □

Proof of Theorem 4.2. By Lemma 4.3, we may assume that $c = 0$. We repeatedly use implicitly the fact that if $\langle n\rangle_b$ is not bd-distinct, then neither is $\langle n\rangle_b + m$ for any $m \in \mathbb{Z}^+$. Additionally, it follows from Propositions 3.10 and 3.11 that $s-1 \in \langle H_d(s,k)\rangle_1$ or $s+k-1 \in \langle H_d(s,k)\rangle_1$. In particular, $-1 \in \{H_d(s, k) - a_1 s - a_2(s + k) \mid a_1, a_2 \in \mathbb{N}_{\geq 0}\}$, and we rely on this observation in cases **III** and **IV**.

Case I: $k = 1$ and $d < s$.

In this case, $H_d(s, 1) = s - 1$. First, we prove $H_{bd}(bs, b) \geq b(s + 1) - 1$. We have $\langle b(s + 1) - 1\rangle_b = \{bs + b - 1, b - 1\}$, which is bd-distinct.

Now, we prove $H_{bd}(bs, b) \leq b(s + 1) - 1$. If $H_{bd}(bs, b) = b(s + 1)$, then $\langle H_{bd}(bs, b)\rangle_b/b = \langle s + 1\rangle_1 = \{s + 1, 1, 0\}$ is d-distinct, a contradiction. It follows that $H_{bd}(bs, b) < b(s + 1)$.

Case II: $k = 1$ and $d \geq s$.

In this case, $H_d(s, 1) = s - 1$. First, we prove $H_{bd}(bs, b) \geq bs - 1$. This follows from the fact that $\langle bs - 1\rangle_b = \{bs - 1\}$ is clearly bd-distinct.

Now, we prove $H_{bd}(bs, b) \leq bs - 1$. If $H_{bd}(bs, b) = bs$, then $\langle H_{bd}(bs, b)\rangle_b/b = \langle s\rangle_1 = s, 0$ is d-distinct, a contradiction. It follows that $H_{bd}(bs, b) < bs$.

Case III: $d < k$ and ($m\widetilde{m}_d \bmod k = 1$ or $d < m\widetilde{m}_d \bmod k = k - 1$).

First, we prove $H_{bd}(bs, bk) \geq b(H_d(s, k) + 2) - 1$. Since $b(x + 2) - 1 \geq 0$ if and only if $x \geq -1$, we have

$$\langle b(H_d(s, k) + 2) - 1\rangle_b = b((\langle H_d(s, k)\rangle_1 \cup \{-1\}) + 2) - 1.$$

Thus, it suffices to prove that $\langle H_d(s, k)\rangle_1 \cup \{-1\}$ is d-distinct. In particular, since $\langle H_d(s, k)\rangle_1$ is d-distinct, it suffices to show that $[d-1] \cap \langle H_d(s, k)\rangle_1 = \emptyset$. First, suppose $m\widetilde{m}_d \bmod k = 1$. If $|-1-x| \leq d$ for some $x \in \langle H_d(s, k)\rangle_1$, then $|(-1+k)-(x+k)| \leq d$, so $x+k \notin \langle H_d(s, k)\rangle_1$. Since $x - (s + k) < x - d \leq -1$, $x \in E$. Since $x + k \notin \langle H_d(s, k)\rangle_1$, x is the first minimal element in its ledge L_i. In fact, since $|-1-x| \leq d < k$, x is the only minimal element in L_i. By Propositions 3.10 and 3.11, $i \in \{k-1, m-2\}$. Since $|-1-(k-1)| = k > d$, $x = m-2$. By Proposition 3.9, $s < k$, so $x = s - 2$ and hence $d \geq s - 1$. However, in this case, we have that $m\widetilde{m}_d \neq 1$ because $|k - (1 - m \bmod k)| = s - 1 \leq d$. This contradiction proves $[d - 1] \cap \langle H_d(s, k)\rangle_1 = \emptyset$ when $m\widetilde{m}_d \bmod k = 1$. Now suppose $d < m\widetilde{m}_d \bmod k = k - 1$. If $|-1-x| \leq d$ for some $x \in \langle H_d(s, k)\rangle_1$, then $|(-1+s)-(x+s)| \leq d$, so $x+s \notin \langle H_d(s, k)\rangle_1$. Since $\langle H_d(s, k)\rangle_1$, we may assume $x \in [s - 1]$. But, using Propositions 3.10 and 3.11, we see that if $x \in [s - 1]$, then $x + s \in \langle H_d(s, k)\rangle_1$, a contradiction.

Now, we prove $H_{bd}(bs, bk) \leq b(H_d(s, k) + 2) - 1$. If $H_{bd}(bs, bk) = b(H_d(s, k) + 2)$, then $\langle H_{bd}(bs, bk)\rangle_b/b = \langle H_d(s, k) + 2\rangle_1$ is d-distinct. By Propositions 3.10 and 3.11, $-1, -2 \in \{H_d(s, k) - a_1 s - a_2(s + k) \mid a_1, a_2 \in \mathbb{N}_{\geq 0}\}$, so $0, 1 \in \langle H_d(s, k) + 2\rangle_1$, contradicting d-distinctness. It follows that $H_{bd}(bs, bk) < b(H_d(s, k) + 2)$.

Case IV: $k \neq 1$ and ($m\widetilde{m}_d \bmod k = k - 1 = d$ or ($m\widetilde{m}_d \bmod k \neq 1, k - 1$) or $d \geq k$)

First, we prove that $H_{bd}(bs, bk) \geq b(H_d(s,k)+1)-1$ by showing that $\langle b(H_d(s,k)+1)-1\rangle_b$ is bd-distinct. Note that this is sufficient, since $b(H_d(s,k)+1)-1 \equiv -1 \pmod{b}$, and thus cannot be a linear combination of bs and bk. Since $b(x+1)-1 \geq 0$ if and only if $x \geq 0$, we have

$$\langle b(H_d(s,k)+1)-1\rangle_b = b(\langle H_d(s,k)\rangle_1 + 1) - 1.$$

Since $\langle H_d(s,k)\rangle_1$ is d-distinct, then we have $\langle b(H_d(s,k)+1)-1\rangle_b$ is bd-distinct.

Now, we prove $H_{bd}(bs, bk) \leq b(H_d(s,k)+1)-1$. If $H_{bd}(bs, bk) = b(H_d(s,k)+1)$, then $\langle H_{bd}(bs,bk)\rangle_b/b = \langle H_d(s,k)+1\rangle_1$ is d-distinct. This is equivalent to $[d-1] \cap \langle H_d(s,k)\rangle_1 = \emptyset$, which we now show is impossible. If $m\widetilde{m}_d \bmod k = k-1 = d$, by Proposition 3.11, $m-1 \in [d-1] \cap \langle H_d(s,k)\rangle_1$. Now suppose $m\widetilde{m}_d \bmod k \neq 1, k-1$. If $d < k/2$ and $\ell \geq 2$, then by Proposition 3.10, $m\widetilde{m}_d - 1 \in [d-1] \cap \langle H_d(s,k)\rangle_1$. If $d < k/2$ and $\ell \leq -2$, then $-1-\ell \in [d-1] \cap \langle H_d(s,k)\rangle_1$. If $k/2 \leq d < k$ and $m \leq d$, then by Proposition 3.11, $m-1 \in [d-1] \cap \langle H_d(s,k)\rangle_1$. Finally, if $k/2 \leq d < k$ and $d < m$, then $k-1-m \in [d-1] \cap \langle H_d(s,k)\rangle_1$. Suppose finally that $d \geq k$. If $s \leq d$, $\langle H_d(s,k)\rangle_1 = \{s-1\}$, and $s-1 \in [d-1] \cap \langle H_d(s,k)\rangle_1$. If $s > d$, $\langle H_d(s,k)\rangle_1 = \{s+k-1, k-1\}$, and $k-1 \in [d-1] \cap \langle H_d(s,k)\rangle_1$. It follows that $H_{bd}(bs, bk) < b(H_d(s,k)+1)$. □

5 Counting $(k-1)$-distinct cores

We also investigate the number of $(s, s+k)$-core partitions for specific values for s, k, and d. Based on our formula for the maximum hook length in Section 2, we see that the largest hook length depends heavily on $m = s \bmod k$. Hence, we are motivated to look at infinite families that parameterize s in terms of k. In particular, we look at the number of $(k-1)$-distinct partitions for both $(rk-1, (r+1)k-1)$-core partitions and $(rk+1, (r+1)k+1)$-core partitions, which give $m = k-1$ and $m = 1$ respectively.

5.1 $(k-1)$-distinct $(rk+1, (r+1)k+1)$-core partitions

We have the following recurrence that allows us to compute the number of $(k-1)$-distinct $(rk+1, (r+1)k+1)$-core partitions for any given values of r, k.

Let

$$C_n = \frac{\binom{2n}{n}}{n+1} \qquad \text{(Catalan numbers)}$$

$$E(\ell, r) = C_\ell + \left(\sum_{i=2}^{r-1}(C_i - C_{i-1})C_{\ell-i}\right) - (C_{r+1} - 2C_r + C_{r-1})C_{\ell-r} \qquad \text{(conjectured)}$$

$$D(\ell, k, a, b) = \sum_{i=1}^{\ell-1}\left((C_{i+1} - C_i)\left(\sum_{\alpha=1}^{a-1} D(\ell-i, k, \alpha, b)\right) + (C_i - C_{i-1})\left(\sum_{\alpha=a+1}^{k} D(\ell-i, k, \alpha, b)\right)\right)$$
$$+ \delta_{ab}(C_{\ell+1} - C_\ell) \qquad (\delta \text{ is Kronecker delta})$$

$$T_0(r, k) = \left(\sum_{i=1}^{r}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k} D(i, k, \alpha, \beta)\right) + 1$$

$$T_1(r, k) = \sum_{i=1}^{r}(k-1) \cdot E(r+1, i) + \sum_{\ell=i+1}^{r}\sum_{j=2}^{k} E(\ell, i) \cdot \left(\sum_{\alpha=1}^{j-1}(*)\right) + E(\ell-1, i) \cdot \left(\sum_{\alpha=j+1}^{k}(*)\right),$$

where

$$(*) = \left(\sum_{\beta=1}^{j-2} D(r-\ell, k, \alpha, \beta)\right) + \left(\sum_{\beta=j}^{k} D(r-\ell+1, k, \alpha, \beta)\right).$$

Then, the number of $(k-1)$-distinct $(rk+1, (r+1)k+1)$-core partitions is $T_0(r, k) + T_1(r, k)$. The value of E is conjectured, but appears true empirically.

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