BENDER–KNUTH INVOLUTIONS ON TABLEAUX AND LINEAR EXTENSIONS OF POSETS

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Abstract. In this report, we study the Bender–Knuth involutions on tableaux and linear extensions of posets. We introduce the linear extension group of a poset, the permutation group generated by Bender–Knuth involutions on the set of its linear extensions, and study posets according to properties of their linear extension groups. We also study relations satisfied by the Bender–Knuth involutions on linear extensions, with special attention to the cactus relations. Finally, we prove sufficient conditions the Berenstein–Kirillov group of a tableau to be transitive.

Contents

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1. Introduction

First introduced by Bender and Knuth in their study of enumerations of plane partitions and Schur polynomials [BK72], the *Bender–Knuth (BK) involutions* on the set of *column-strict (semi-standard) Young tableaux* have seen a wide range of applications across different areas of combinatorics. They were studied in the context of Gelfand-Tsetlin patterns by Berenstein and Kirillov [KB95] and by Halacheva [Hal20]. They have also been studied in the context of Grothendieck polynomials (by, e.g., Ikeda-Shimazaki [IS14] and Galashin-Grinberg-Liu [GGL16]), and shifted tableaux (by, e.g., Stembridge [Ste90] and Rodrigues [Rod21]). More recently in [CGP20], Chmutov, Glick, and Pylyavskyy showed that the action of the Bender–Knuth involutions on column-strict tableaux satisfies the *cactus relations*, a group of relations satisfied by interval reversals in a coboundary category studied by Henriques and Kamnitzer [HK06]; see also [Dev99, DJS03].

Recall that given a *partition* λ and a tuple $\alpha = (\alpha_1, \dots, \alpha_n)$, a column-strict (semi-standard) tableau T of *shape* λ and *content* α is a filling of the squares of the *Ferrers diagram* of λ, such that the labels are increasing weakly along rows and strictly along columns, and there are α_i occurrences of i in T for $i = 1, \dots, n$. Informally, for $i = 1, \dots, n-1$, the BK involution t_i acts on a column-strict tableau of content α of length n by swapping the contents of i and $i+1$ in each row, fixing an i (resp. $i+1$) when there is an $i+1$ below (resp. i above). Alternatively, BK involutions can be treated as permutations of order 2 in the set of all column-strict tableaux of shape λ. For example, Figure 1 shows the action of the BK involutions t_1 and t_2 on the set of column-strict tableaux of shape $(2,1)$ and content in $\{1,2,3\}$. Observe that the orbit on the right contains all *standard Young tableaux* of shape $(2,1)$, i.e. column-strict tableaux with content $(1,\dots,1)$, and that on the left contains all column-strict tableaux of shape $(2,1)$ and content being a permutation of $(2,1)$. This transitive behavior is not always exhibited in general; the second part of this report will be dedicated to study the orbits of the BK involutions on the set of column-strict tableaux of any given shape λ and content drawing from all permutations of a given tuple μ.

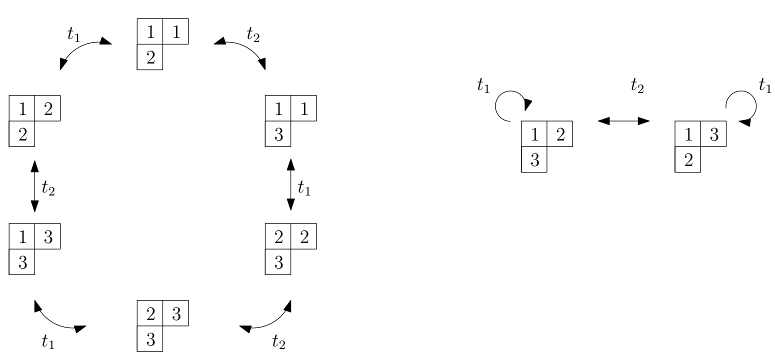


Figure 1. The action of BK involutions on column-strict tableaux.

We can also generalize the BK action on standard Young tableaux to an action on the set of linear extensions of a poset. Recall that the set of standard Young tableaux of shape λ is in bijection with the set of linear extensions of a Ferrers poset of shape λ. In this case, the BK involution t_i swaps the labels i and $i+1$ if they label incomparable elements of the poset, and fixes them otherwise. This action on linear extensions of a Ferrers poset can be naturally extended to define the BK involutions on the set of linear extensions of an arbitrary poset. This generalization was first introduced by Stanley [Sta09] to study promotion and evacuation, which are operators on linear extensions going back to Schützenberger [Sch72, Sch76]. Figure 2 shows an example of the action of BK involutions on the set of linear extensions of the ordinal sum of two antichains of size 2.

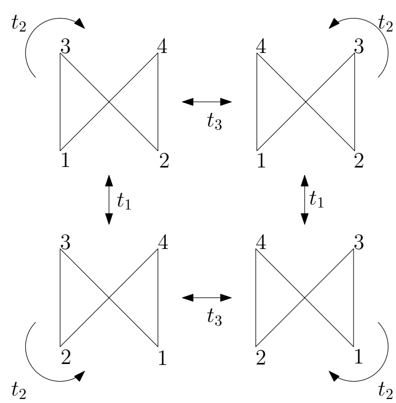


Figure 2. BK action on linear extensions of a poset

Given a poset P, the BK involutions t_i on the set of linear extensions of P satisfy the relations for the infinite Coxeter group

$$W_n = \langle t_1, \ldots, t_{n-1} : t_i^2 = 1 \text{ and } t_i t_j = t_j t_i \text{ if } |i-j| \geq 2 \rangle,$$

which induces a group homomorphism $\varphi_P : W_n \to \mathfrak{S}_{\mathrm{LinExt}(P)}$. We call the image of this map the *linear extension group of* P and denote it by H_P. For a poset P, the *linear extension graph* or *adjacent transposition graph* of P is defined to be the graph with vertices given by linear extensions of P and edges given by Bender–Knuth involutions; historically, it has been used in the study of linear extension generation since the 1990s [PR91, Rus92, Sta92, Wes93, Naa00, BM13]. In this report, we will study the properties of H_P as well as the kernel of φ_P, i.e. identifying the relations that are satisfied by the BK involutions.

1.1. **Outline of the paper.** In section 2, we define the Bender–Knuth involutions on column strict tableaux and on linear extensions of a poset. We also briefly discuss the cactus group and the cactus relations.

In section 3, we study the properties of the linear extension group. In subsection 3.1, we develop some general properties of H_P with respect to basic constructions of posets, such as the ordinal sum, disjoint union, and dual. In subsection 3.3, we show that the Bender–Knuth involutions satisfy the braid relations if and only if the underlying poset is a disjoint union of chains, and use the to prove that an n-element disjoint union of chains has linear extension group S_n. In subsection 3.4, we study which posets P have H_P isomorphic to the symmetric group on all linear extensions, which we call *LE-symmetric* posets. We show that the only disconnected LE-symmetric posets are the disjoint union of a singleton and a chain. Then in subsection 3.5, we study posets P for which H_P is a primitive permutation group, which we call *LE-primitive*. We show that a disconnected poset is LE-primitive if and only if it is a disjoint union of two chains of different lengths or an antichain of size 2. In subsection 3.6, we study the order $k(P)$ of the stabilizer subgroup of H_P. We show that $k(P)$ is either 1 or even for all posets P, and further classify the posets P for which $24 \nmid k(P)$. We also establish a lower bound for $k(P)$ in terms the height of the poset P.

In section 4, we study the properties of *LE-cactus posets* whose linear extension group satisfies a family of relations called the *cactus relations*. In subsection 4.1, we give several constructions of posets that preserves this property, e.g, disjoint union and ordinal sum with an antichain of size 1 or 2. In subsection 4.2, we consider non-LE-cactus posets and prove that no LE-cactus poset has three bottom elements. Lastly, in subsection 4.3, we detailed our attempts to generate LE-cactus preserving constructions for posets, featuring a plethora of examples, counterexamples, and conjectures.

In section 5, we return to the Bender–Knuth action on column-strict tableaux, where we are particularly interested in finding column-strict tableaux for which the Bender–Knuth action is transitive. In subsection 5.1, we give a few criterion to identify posets for which this transitivity holds. We provide data on the number of the Bender–Knuth action in subsection 5.2.

Finally in section 6, we investigate the order of promotion acting on column-strict tableaux of *staircase shapes* with various contents. The motivating example is a result due to Haiman's that that the order of promotion on a standard Young tableau of staircase shape has order $2N$, where $N = \binom{n}{2}$ is the number of blocks in the staircase. We give a column-strict tableau with staircase shape and entries in $\{1, \dots, N\}$ which does not have order $2N$. In subsection 6.1, we provide data for the order of promotion for staircases of length at most five.

2. Basic definitions and constructions

In this section, we provide the backgrounds on several algebraic and combinatorial objects that are fundamental to this report, including column-strict (semi-standard) and standard Young tableaux, poset operations, and linear extensions.

2.1. Column-strict tableaux and Bender–Knuth involutions. Given a *partition* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, a *column-strict (semi-standard) tableau* T is a filling of the squares of the *Ferrers diagram* of λ, increasing weakly along rows and strictly along columns. We say T has *content* $\alpha = (\alpha_1, \dots, \alpha_n)$ if there are α_i occurences of i in T for $i = 1, \dots, n/$. The set of all column-strict tableaux of shape λ and content α is denoted by $\mathrm{CST}(\lambda, \alpha)$. For example,

$$\mathrm{CST}((3,2),(1,2,1,1)) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 4 & \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 4 & \\ \cline{1-2}\end{array}, \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 & \\ \cline{1-2}\end{array} \right\}.$$

Denote the set of all column-strict tableaux T of shape λ and entries in $[n] = \{1, 2, \dots, n\}$ by $\mathrm{CST}(\lambda, [n])$, and the set of standard Young tableaux of shape λ by $\mathrm{SYT}(\lambda) := \mathrm{CST}(\lambda, (1, 1, \dots, 1))$. Historically, the study of column-strict tableaux arose from investigating Schur polynomials [BK72].

Following [CGP20], we define the Bender–Knuth involutions t_i for $1 \leq i \leq n-1$ to be are operators on the set of column-strict Young tableaux. Let T be such a column-strict Young tableau and let S be the skew tableau obtained by taking only the boxes of T with entry equal to i and $i+1$. Each row of S, read from left to right, consists of

(1) a entries equal to i that lie directly above an $i+1$,
(2) b entries equal to i that are alone in their columns,
(3) c entries equal to $i+1$ that are alone in their columns, and
(4) d entries equal to $i+1$ that lie directly below an i

for some $a, b, c, d \geq 0$. Define a tableau S' by switching the b and c values in each row of S. Note that the new tableaux is still column-strict. We define $t_i(T)$ to be the tableaux obtained by replacing S with S' in T.

Consequently, the t_i also act on each of these sets where the shape λ is fixed, and the content varies through the permutations $w(\alpha)$ of α:

$$\bigsqcup_{w \in \mathfrak{S}_n} \mathrm{CST}(\lambda, w(\alpha)).$$

2.2. **Poset operations.** First we introduce some notation. Let P be poset with n elements. Let $\mathrm{LinExt}(P)$ denote set of linear extensions of P. For any linear extension $\ell \in \mathrm{LinExt}(P)$ and any subset $S \subseteq P$, let $\ell(S)$ denote the image of S under ℓ. Let $P \oplus Q$ denote the ordinal sum of posets P and Q, where all elements of P are less than every element of Q. Note that $P \oplus Q \neq Q \oplus P$. Let $P+Q$ denote the disjoint union of posets P and Q. Note that disjoint union is commutative. Let P^* be the dual of the poset P: the relation $x \leq y$ holds in P^* if and only if $y \leq x$ holds in P. Let $\mathfrak{S}_X$ denote the symmetric group of all permutations of a set X.

2.3. **The cactus relations.** Henriques and Kamnitzer [HK06] gave the name *cactus group* $\mathcal{C}_n$ to a group that naturally acts on a coboundary category, a special monoidal category. The cactus group $\mathcal{C}_n$ is generated by q_{ij} for $1 \leq i < j \leq n$ with the relations

(1) $q_{ij}^2 = 1$,
(2) $q_{ij}q_{kl} = q_{kl}q_{ij}$ if $j < k$,
(3) $q_{ij}q_{kl}q_{ij} = q_{i+j-l,i+j-k}$ if $i \leq k < l \leq j$.

The cactus group has topological significance, which first appeared in the work of Devadoss [Dev99] and Davis-Januszkiewicz-Scott [DJS03]. The cactus group surjects onto the symmetric group S_n and its kernel is the fundamental group of the Deligne-Mumford compactification of the moduli space of real genus 0 curves with $n+1$ marked points. A useful presentation presentation of the cactus group $\mathcal{C}_n$ was given in Berenstein-Kirillov [KB95] and in Chmutov-Glick-Pylyavskyy [CGP20, Theorem 1.8]. Namely, $\mathcal{C}_n$ is isomorphic to another presentation generated by $t_1, \ldots, t_{n-1}$, where the isomorphism sends

$$q_{ij} \mapsto q_{j-1}q_{j-i}q_{j-1}, \text{ where } q_i = t_1(t_2t_1)(t_3t_2t_1)\cdots(t_it_{i-1}\cdots t_1),$$

and the t_i are subject to the relations

(C1) $t_i^2 = 1$,
(C2) $t_it_j = t_jt_i$ if $|i-j| > 1$,
(C3) $(t_iq_{jk})^2 = 1$, where $i+1 < j < k$.

Of particular importance to us is the result in Chmutov-Glick-Pylyavskyy [CGP20, Theorem 1.4], stating that $\mathcal{C}_n$ acts on the set of column-strict tableaux $\mathrm{CST}(\lambda, [n])$ of a given shape λ. Put another way, the Bender–Knuth involutions t_i acting on column-strict tableaux satisfy the relations (C1), (C2), and (C3). With the relations (C1), (C2), and (C3), we see that $\mathcal{C}_n$ is a quotient of the infinite Coxeter group W_n defined in (1).

Observe that the action the Bender–Knuth action on $\mathrm{SYT}(\lambda)$ can be extended to a Bender–Knuth action on the set of linear extensions of a given poset. Given an n-element poset P, let $\mathrm{LinExt}(P)$ be the set of its linear extensions, where $\ell \in \mathrm{LinExt}(P)$ is an order-preserving bijection from P to $[n]$. For $i \in [n-1]$, define the operator $t_i : \mathrm{LinExt}(P) \to \mathrm{LinExt}(P)$ by

$$t_i\ell = \begin{cases} \ell & \text{if } \ell^{-1}(i) < \ell^{-1}(i+1) \\ (i, i+1) \circ \ell & \text{otherwise,} \end{cases}$$

where $(i, i+1)$ transposes i and $i+1$. In other words, t_i transposes adjacent labels in the linear extension if and only if it is possible.

As mentioned above, the t_i satisfy the relations for the infinite Coxeter group

$$W_n = \langle t_1, \dots, t_{n-1} : t_i^2 = 1 \text{ and } t_it_j = t_jt_i \text{ if } |i-j| \geq 2 \rangle, \tag{1}$$

so the BK involutions give a group homomorphism $\varphi : W_n \to \mathfrak{S}_X$ for the various sets X on which the t_i act. This motivated us to ask the following questions about $\ker(\varphi)$ and $\mathrm{im}(\varphi)$.

Question 2.1. What further relations hold among the t_i acting on $\mathrm{LinExt}(P)$ depending on the structure of P?

Question 2.2. For which posets P is the the image of the Bender–Knuth involutions t_i on $\mathrm{LinExt}(P)$ the symmetric group $\mathfrak{S}_{\mathrm{LinExt}(P)}$? In other words, which posets P are LE-symmetric?

We also call $\mathrm{im}(\varphi)$ the *linear extension group* of P, and we denote it H_P.

3. Properties of the linear extension group

We introduce the linear extension group of a poset, the permutation group generated by Bender–Knuth involutions on the set of its linear extensions, and study posets according to properties of their linear extension groups. In particular, we discuss properties and examaples of *LE-symmetric* posets, which are posets for which the linear extension group is $\mathfrak{S}_{\mathrm{LinExt}(P)}$, *LE-braided* posets, which are posets for which the linear extension group satisfies the braid relations, and study the size of the stabilizer subgroup of a linear extension.

3.1. **Linear extension group properties.** In this section, we prove two basic properties of H_P: it is a transitive subgroup of $\mathfrak{S}_{\mathrm{LinExt}(P)}$ (Proposition 3.2) and that the linear extension group of an ordinal sum of posets is the direct product of the linear extension groups of the posets (Proposition 3.3).

We consider linear extensions on poset P of size n as order-preserving bijections $\ell : P \to \{1, 2, \dots, n\}$. For any subset $S \subseteq P$, let $\ell(S)$ denote the image of S under this bijection.

Definition 3.1. Call the poset P *LE-symmetric* if $H_P = \mathfrak{S}_{\mathrm{LinExt}(P)}$.

Proposition 3.2. *H_P is a transitive subgroup of $\mathfrak{S}_{\mathrm{LinExt}(P)}$.*

Proof. To show that H_P is a transitive subgroup of $\mathfrak{S}_{\mathrm{LinExt}(P)}$, it suffices to show that for any two linear extensions $\ell_i, \ell_j \in \mathrm{LinExt}(P)$, there exists some $w \in H_P$ such that $w\ell_i = \ell_j$.

We proceed by induction on the size of P. Our base case is when P is a singleton element, so H_P is clearly transitive.

Now suppose H_P is transitive for any poset P where $|P| \le n$. It suffices to show that H_P is transitive for any poset of size $|P| = n$. Let m be a maximal element of P, and define $M = \{\ell \in \mathrm{LinExt}(P) \mid \ell(m) = n\}$ to be the set of linear extensions of P in which m is the maximum element. By induction, we know that for any two linear extensions $\ell_i, \ell_j \in M$, there exists some $w \in H_P$ such that $w\ell_i = \ell_j$. Thus to show transitivity, it suffices to show that for any linear extension $\ell \notin M$, there exists some $\ell' \in M$ and some $w \in H_P$ such that $w\ell = \ell'$. Suppose that $\ell(m) = i$, where $i < n$ by definition of M. Then note that since m was a maximal element of P, we have that all the elements $\ell^{-1}(i+1), \ell^{-1}(i+1), \ldots, \ell^{-1}(n)$ are incomparable with m, as otherwise ℓ would not be a linear extension. Thus, we have that $t_{n-1}t_{n-2}\ldots t_i\ell \in M$, as desired. □

Proposition 3.3. $H_{P\oplus Q} = H_P \times H_Q$.

Proof. We have that any linear extension of $P \oplus Q$ is composed of a linear extension of P and a linear extension of Q. Formally, let $|P| = m$ and $|Q| = n$. Then there is a bijection

$$\begin{aligned} L : \mathrm{LinExt}(P) \times \mathrm{LinExt}(Q) &\to \mathrm{LinExt}(P \oplus Q) \\ L(\ell_i, \ell_j) &\mapsto \ell_{ij}(P \oplus Q) \end{aligned}$$

where

$$\begin{aligned} \ell_{ij}(p) &= \ell_i(p) \quad \forall p \in P \\ \ell_{ij}(q) &= \ell_j(q) + m \quad \forall q \in Q. \end{aligned}$$

Note that ℓ_{ij} must be a linear extension of $P \oplus Q$ since it respects the ordering of P and Q by construction, and $\ell(p) \le m < m+1 \le \ell(q)$ for any $p \in P, q \in Q$. Clearly this map is injective, so it suffices to show that it is surjective. Indeed, for any linear extension $\ell \in \mathrm{LinExt}(P \oplus Q)$, we must have $\ell(p) \in [m]$ for all $p \in P$. Suppose not. Then there exists $q \in Q$ such that $\ell(q) \in [m]$ and there exists $p \in P$ such that $\ell(p) \in \{m+1, \ldots, m+n\}$, but $p < q$ in our ordinal sum, which is a contradiction. Then we take the linear extension $\ell_i \in \mathrm{LinExt}(P)$ where $\ell_i(p) = \ell(p)$ for all $p \in P$ and the linear extension $\ell_j \in \mathrm{LinExt}(Q)$ where $\ell_j(q) = \ell(q) - m$. These must be linear extensions because $\ell(p)$ respects the ordering on P, and if $\ell(q)$ respects the ordering on Q, then subtracting m from all labels still respects the ordering on Q. Then $L(\ell_i, \ell_j) = \ell$, so our map is surjective as desired.

Define $T_P := \langle t_1, \ldots, t_{m-1}\rangle$ and $T_Q := t_{m+1}, \ldots, t_{m+n-1}$ as actions on $\mathrm{LinExt}(P \oplus Q))$. Since the indices are far apart, elements of T_P and T_Q commute, so any group element $t_{i_1}t_{i_2}\ldots t_{i_k}$ can be rewritten as T_PT_Q. Thus $H_{P\oplus Q} \cong T_P \times T_Q$. Note that t_m is in the kernel, since for any linear extension of $P \oplus Q$ we have the element with label m is in P and that with label $m+1$ is in Q, and thus are comparable.

Note that on linear extensions, T_P acts on P and T_P act on Q independently of each other. Then by our bijection above, the action of T_P on $\mathrm{LinExt}(P \oplus Q)$ is isomorphic to the action of T_P on $\mathrm{LinExt}(P)$, which is precisely H_P, and similarly the action of T_Q on $\mathrm{LinExt}(P \oplus Q)$ is isomorphic to the action of T_Q on $\mathrm{LinExt}(Q)$, which is precisely H_Q. □

3.2. **Relations in the linear extension group.** In this section, we find more relations that the generators t_i satisfy in H_P. Then we show that if *convex induced sub-posets* enjoy special properties of the linear extension group, see Subsection 3.2.1.

Let H_n be the free group on $t_1, \ldots, t_{n-1}$ modulo the relations satisfied by every n-element poset. Clearly, the t_i satisfy

$$(t_i)^2 = 1$$
$$(t_i t_j)^2 = 1, \text{ where } |i - j| \geq 2.$$

Furthermore, by examining all linear extensions of all posets of size 3, we can see that H_n satisfies

$$(t_i t_{i+1})^6 = 1.$$

(Stanley notes this in his paper; see [Sta09, Note, p.6]) This relation does not follow from the two above because the braid group satisfies the latter but not the former. There are no other relations satisfied in H_n involving only t_i and t_{i+1}.

It turns out that these are not the only relations satisfied in H_n. Here are some more we have found with the help of a computer:

$$(t_i t_{i+1} t_{i+2})^{24} = 1 \tag{2}$$
$$(t_i t_{i+1} t_{i+2} t_{i+1})^{30} = 1 \tag{3}$$
$$(t_i t_{i+1} t_i t_{i+1} t_{i+2})^{60} = 1 \tag{4}$$
$$(t_i t_{i+1} t_{i+2} t_{i+1} t_{i+2})^{60} = 1 \tag{5}$$
$$(t_i t_{i+1} t_{i+2} t_{i+3})^{840} = 1. \tag{6}$$

None of these relations are implied by the three relations above. Note that these may not be independent relations themselves, though we see no obvious relation that subsumes any subset of the relations above.

3.2.1. *Convex induced sub-posets.* An *induced poset* $Q \subset P$ is a subset of vertices in P such that if $x, y \in Q$ and $x \leq y$ in P, then $x \leq y$ in Q. A *convex induced poset* is an induced poset such that if $x, z \in Q$ and $y \in P$ satisfy $x \leq y \leq z$ in P, then $z \in Q$. We study which relations in the linear extension group of a convex induced subposet remain relations in the larger poset.

Lemma 3.4. *Let $\mathcal{P}_1$ be a convex induced sub-poset of $\mathcal{P}_2$ and $\ell_1 \in \mathrm{LinExt}(\mathcal{P}_1)$. Then, there is an $\ell_2 \in \mathrm{LinExt}(\mathcal{P}_2)$ such that for some $j \in \mathbb{Z}$, $\ell_2(v) = \ell_1(v) + j$ for all $v \in \mathcal{P}_1$.*

Proof. We will augment our poset $\mathcal{P}_2$ by adding additional order relations. First, we add order relations between the vertices in $\mathcal{P}_1$ to make $\mathcal{P}_1$ a total order with the order given by ℓ_1. By the definition of a linear extension, adding such order relations is permissible. Let v_i be the vertex in $\mathcal{P}_1$ such that $\ell_1(v_i) = i$, so that v_i is the minimum vertex in $\mathcal{P}_1$ with these new order relations. Then, for every vertex $v \in \mathcal{P}_2$ incomparable with v_1, add an order relation to make $v < v_1$. Finally, for every vertex $v \in \mathcal{P}_2$ such that $v > v_i$ and $v \notin \mathcal{P}_1$, add an order relation to make $v' < v$ for every $v' \in \mathcal{P}_1$. We claim it is permissible to add such relations. The only way it could not be is if we already had $v' > v$ for some $v' \in \mathcal{P}_1$. Then, $v' \neq v_i$, since $v > v_i$ by assumption. Thus, $v' > v > v_i$. Since $\mathcal{P}_1$ is a convex induced sub-poset containing v' and v_i, $v \in \mathcal{P}_1$, contradicting our assumption. Thus, our augmented poset is a valid poset.

Let ℓ_2 be a linear extension of our augmented poset. Then, $\ell_2(v_1) = j+1$ for some integer j. We claim that $\ell_2(v) = \ell_1(v) + j$ for all $v \in \mathcal{P}_1$. Suppose this is true for v_i. Then we need to show that $\ell_2(v_{i+1}) = \ell_2(v_i)+1$. We have $\ell_2(v_{i+1}) \geq \ell_2(v_i)+1$, since $v_{i+1} > v_i$. If some $w \in \mathcal{P}_2$ is such that $\ell_2(w) = \ell_2(v_i) + 1$, then $w \not\leq v_i$, and there are no vertices incomparable with v_i, so $w > v_i$. If $w \in \mathcal{P}_1$, then $w \geq v_{i+1}$, since we imposed a total order on $\mathcal{P}_1$ with our order relations. And if $w \notin \mathcal{P}_1$, then $w > v_{i+1}$ by our order relations. Therefore, $w = v_{i+1}$ as desired. Since ℓ_2 is also a linear extension of the unaugmented version of $\mathcal{P}_2$, the proof is complete. □

Corollary 3.5. *If $\mathcal{P}_1$ is a convex induced sub-poset of $\mathcal{P}_2$, then $H_{\mathcal{P}_1} \leq H_{\mathcal{P}_2}$. In particular, if $\ell_1 \in \mathrm{LinExt}(\mathcal{P}_1)$ and $\ell_2 \in \mathrm{LinExt}(\mathcal{P}_2)$, then $\mathrm{Stab}(\ell_1) \leq \mathrm{Stab}(\ell_2)$.*

Definition 3.6. A *relation* is an equation $w = 1$, where w is a word with the letters $t_1, \dots, t_{n-1}$. We say a relation is satisfied for a poset if it is satisfied for every linear extension of the poset.

Definition 3.7. A *relation type* is a set of relations $w_i = 1$, where w_i is obtained by translating all the indices of t terms by i, and i is taken over all integers for which the t terms are among $t_1, \dots, t_{n-1}$. Equivalently, a relation type is a set of relations $w_i = 1$ where each w_i is generated by $\langle t_i, t_{i+1}, \dots, t_k \rangle$ where k is constant. We say a relation type is satisfied for a poset if all of its relations are satisfied.

For example, $(t_1t_2)^6 = 1$ is a relation, and the relations $(t_it_{i+1})^6 = 1$ form a relation type.

Lemma 3.8. *If a relation type is satisfied for a poset, then it is also satisfied for every convex induced sub-poset.*

Proof. We prove the contrapositive. Suppose there is a convex induced sub-poset $\mathcal{P}_1 \subseteq \mathcal{P}_2$ for which the relation type fails. That is, some relation w in the relation type fails for some linear extension ℓ_1 of $\mathcal{P}_1$. (Here ℓ_1 is a bijection from vertices to labels.) By Lemma 3.4, for some j, w_j fails for the larger poset. □

3.3. **Braid relation.** Here we show that the Bender–Knuth involutions satisfy the braid relations if and only if the underlying poset is a disjoint union of chains. Then we show that a n-element disjoint union of chains has linear extension group S_n.

Definition 3.9. Call a poset P *LE-braided* if the Bender–Knuth involutions t_i satisfy the braid relation $(t_it_{i+1})^3 = 1$ for all $i = 1, 2, \dots, n-1$.

Proposition 3.10. *The only posets whose linear extension groups satisfy the braid relations with respect to $t_1, \dots, t_{n-1}$ are disjoint unions of chains.*

Proof. One can check that the only posets of size 3 that fail to satisfy $(t_1t_2)^3 = 1$ are the upward facing triangle and the downward facing triangle. The posets not containing one of these two posets are exactly disjoint unions of chains. Any poset that contains one of these two posets as induced sub-posets (and therefore convex induced sub-posets) will fail to satisfy $(t_it_{i+1})^3 = 1$ for some i by Lemma 3.8. Conversely, suppose for the sake of contradiction that a disjoint union of chains fails to satisfy $(t_it_{i+1})^3 = 1$ for some i. Then the induced poset on the elements with labels i, $i + 1$, and $i + 2$ fails to satisfy $(t_1t_2)^3 = 1$, and hence is one of the two posets mentioned above. But a disjoint union of chains does not have such an induced poset, so we have a contradiction. □

Proposition 3.11. *A disjoint union of two or more chains has S_n as its linear extension group (up to isomorphism), where n is the number of elements in the poset.*

Proof. By the previous proposition, we know the linear extension group of a disjoint union of two or more chains is a quotient of S_n. Thus, it suffices to show that the group is not trivial, is not C_2 for $n \geq 3$, and is not S_3 for $n = 4$. Each of these is easily checked. $\square$

Remark 3.12. In general, a disjoint union of two or more chains will have more than n linear extensions, so it will not be LE-symmetric.

3.4. **Properties of LE-symmetric posets.** In this section, we ask which posets are LE-symmetric, i.e. which posets P have $H_P = \mathfrak{S}_{\mathrm{LinExt}(P)}$. We also investigate which poset constructions (duality, ordinal sum, disjoint union) preserve the LE-symmetric property. Then we conjecture that a certain family of posets is LE-symmetric.

Question 3.13. Which posets P are LE-symmetric?

We know that there exist posets P such that $H_P \subsetneq \mathfrak{S}_{\mathrm{LinExt}(P)}$, since Kamnitzer's students gave us a table of data on this question for Ferrers posets F_λ, already containing counterexamples. The smallest countere

Example 3.14. Figure 3 contains a list of LE-symmetric posets of size at most 5, where none can be constructed from another LE-symmetric poset in the list by either Proposition 3.15 or Proposition 3.16.

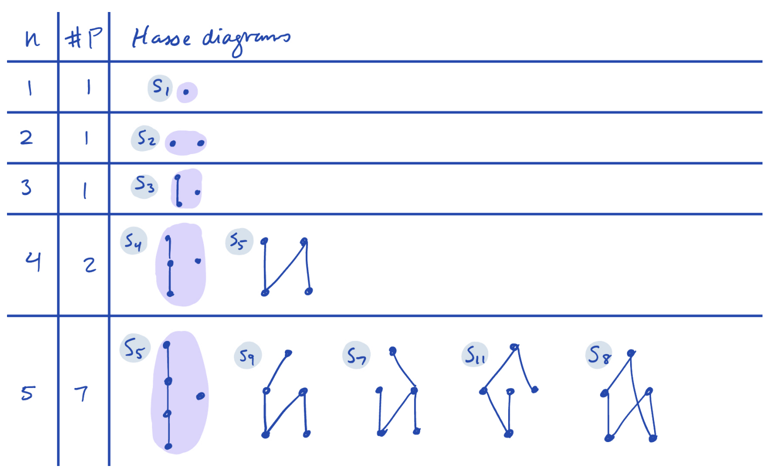


FIGURE 3. Posets P such that $H_P = \mathfrak{S}_{\mathrm{LinExt}(P)}$

Proposition 3.15. *If P is LE-symmetric, then P^* is LE-symmetric.*

Proof. The set $\mathrm{LinExt}(P^*)$ is in bijection with $\mathrm{LinExt}(P)$ by reversing the order of the labels of a given linear extension. It suffices to show that one can generate any transposition of two given linear extensions $f_1, f_2 \in \mathrm{LinExt}(P^*)$. Under the bijection, these correspond to linear extensions $g_1, g_2 \in \mathrm{LinExt}(P)$. Since P is LE-symmetric, there is an element $w \in H_P$ that interchanges g_1 and g_2. The map $H_P \to H_{P*}$ sending t_i acting on LinExt(P) to t_{n-i} acting on LinExt(P^*) for $i = 1, \ldots, n-1$ sends the $w \in H_P$ to a word $w' \in H_{P^*}$ interchanging f_1 and f_2. This suffices for the proof. □

Proposition 3.16. *Let* 1 *be the poset with a single element. If P is LE-symmetric, then $P \oplus 1$ and $1 \oplus P$ are LE-symmetric.*

Proof. The linear extensions of P and $P \oplus 1$ and $1 \oplus P$ are all isomorphic - any linear extension of $P \oplus 1$ is simply a linear extension of P and with the largest element labelled $n + 1$ (and symmetrically for $1 \oplus P$). □

From Proposition 3.16, it thus suffices to classify all posets P without either a unique minimal or maximal element such that P is LE-symmetric. We list some small examples in Figure 3.

Proposition 3.17. *If P and Q are posets such that $P \oplus Q$ is LE-symmetric, then at least one of P, Q is a chain.*

Proof. Let $P \oplus Q$ be LE-symmetric, so that $H_{P \oplus Q}$ is some symmetric group. We have that $H_{P \oplus Q} = H_P \times H_Q$ is a direct product of groups. Then at least one of H_P, H_Q is the trivial group. This happens when at least one of P, Q has only one linear extension, and thus one of P, Q must be a chain. □

One example of an infinite family of LE-symmetric posets is the following.

Lemma 3.18. *If $P = Q + 1$ where Q is a chain, then P is LE-symmetric.*

Proof. Let the chain Q consist of the elements $x_1, \ldots, x_{n-1}$ with relations $x_{n-1} < x_{n-2} < \cdots < x_1$, and let x_n be the extra point.

Note that there are exactly n linear extensions ℓ. To see this, there are n choices for $\ell(x_n)$, and once $\ell(x_n)$ is fixed, we have $\ell(x_1), \ldots, \ell(x_{n-1})$ is also fixed by the total ordering $x_{n-1} < \cdots < x_2 < x_1$. Furthermore, for any i we have that t_i is the identity on all linear extensions except for the linear extension that labels element x_n with i and the corresponding linear extension that labels element x_n with $i+1$, as any other linear extension has labels $i, i+1$ in the chain, and thus they are comparable. Therefore, each t_i acts as a transposition on LinExt(P), and so we generate all of $\mathfrak{S}_{\mathrm{LinExt}(P)}$. □

Question 3.19. Does P LE-symmetric imply $P + 1$ LE-symmetric?

No, $P = 1 + 1$ is LE-symmetric, but $P + 1 = 1 + 1 + 1$ is not LE-symmetric.

Question 3.20. Are the only LE-symmetric posets with two connected components ones in the family above?

The answer is yes. To show this, we will first need a few definitions.

Definition 3.21. Let $K = (A, B)$ denote a partition of $[m + n]$ into sets A and B where $|A| = m$ and $|B| = n$. Throughout, we will simply refer to K as a *partition*.

Let P and Q be posets such that $|P| = m$ and $|Q| = n$.

Definition 3.22. For any partition $K = (A, B)$, let $L(K)$ denote the set of linear extensions of $P + Q$ such that $f(P) = A$ and $f(Q) = B$. We call these sets of linear extensions *partition sets*.

FIGURE 4. Posets P, Q for Example 3.23

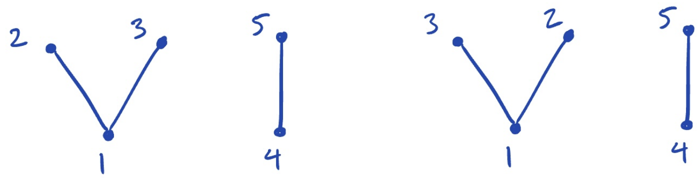


FIGURE 5. $L(K)$ where $K = (\{1, 2, 3\}, \{4, 5\})$

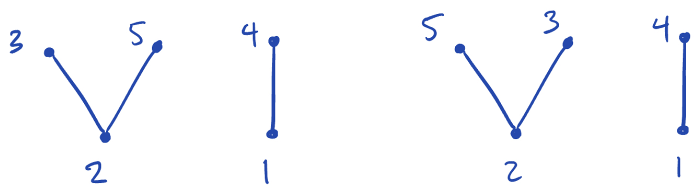


FIGURE 6. $L(K)$ where $K = (\{2, 3, 5\}, \{1, 4\})$

Example 3.23. The figures below illustrates an example of Definition 3.22 when P, Q are the posets in Figure 4.

Lemma 3.24. *We have* $|L(K)| = |\text{LinExt}(P)||\text{LinExt}(Q)|$ *for any partition* K.

Proof. First note that if $K^* = (A^*, B^*)$ is the partition where $A^* = [m]$ and $B^* = \{m+1, m+2, \ldots, m+n\}$, then we have that $L(K^*) \cong L(P \oplus Q)$, and we know by Proposition 3.3 that this is isomorphic to $L(P) \times L(Q)$. Thus $|L(K^*)| = |\text{LinExt}(P)||\text{LinExt}(Q)|$. Then it remains to show that $|L(K_i)| = |L(K^*)|$ for any partition K. Indeed, we have a bijection between $L(K)$ and $L(K^*)$ for any partition $K = (A, B)$. There is a natural bijection from elements A to the set $[m]$ that preserves the relative ordering of elements, and a natural bijection from elements B to the set $\{m+1, m+2, \ldots, m+n\}$. Thus for any linear extension $f \in L(K)$ we map the labels A in P to their relative elements in $[m]$ under this bijection, and symmetrically for B. □

Lemma 3.25. *Let* $T \in \langle t_1, \ldots, t_{m+n-1} \rangle$ *be a word representing a group element. Then for any two linear extensions* f, f' *in the same partition set, then* $T(f), T(f')$ *are in the same partition set.*

Proof. We can decompose T into a product of generators t_i, and so it suffices to show that for any t_i, if f, f' are in the same partition set $L(K)$ where $K = (A, B)$, then $t_i(f)$ and $t_i(f')$ are in the same partition set $L(K')$ for some K'. Indeed, suppose labels $i, i+1$ are both in A or both in B, i.e. both labels appear in P or Q in the linear extensions f, f'. Then both $t_i(f)$ and $t_i(f')$ stay in $L(K)$. If the label i is in P and the label $i+1$ is in Q (or vice versa), then the action of t_i will take both $t_i(f)$ and $t_i(f')$ to $L(K')$ where $K' = (A - \{i\} + \{i+1\}, B - \{i+1\} + \{i\})$. Since f, f'

stay in the same partition set of linear extension after any t_i action, they stay in the same partition set after the action by any group element $T \in \langle t_1, \dots, t_{m+n-1} \rangle$, as desired. □

Lemma 3.26. *If P, Q are posets such that at least one of $|\mathrm{LinExt}(P)|$ and $|\mathrm{LinExt}(Q)|$ is not equal to 1, then $P + Q$ is not LE-symmetric.*

Proof. Using Lemma 3.25, we know that any element of the group maps any entire set of linear extensions $L(K_i)$ to an entire set of linear extensions $L(K_j)$. Unless $|L(K_i)| = 1$, we cannot generate the whole symmetric group on $\mathrm{LinExt}(P + Q)$, since the permutation group is not primitive. In other words, let $f_1, f_2 \in L(K_j)$ and $f_3 \in L(K_i)$, then we can never generate the permutation that takes f_1 to f_3 and f_2 to itself. However, at least one of $|\mathrm{LinExt}(P)|$ and $|\mathrm{LinExt}(Q)|$ is not equal to 1, so indeed $|L(K_i)| \neq 1$ and thus $P + Q$ is not LE-symmetric. □

Theorem 3.27. *The only LE-symmetric posets P with at least two connected components are of the form $P = Q + 1$, where Q is a chain.*

Proof. By Lemma 3.26, we have that if P, Q are not both chains, then $P + Q$ is not LE-symmetric. Thus it suffices to consider when P and Q are both chains. Let $|P| = m$ and $|Q| = n$. By Proposition 3.11, we know that $H_{P+Q} = S_{n+m}$, but there are $\binom{m+n}{n}$ linear extensions. Thus $P + Q$ LE-symmetric implies that $\binom{m+n}{n} = m + n$, which only occurs when $n = 1$ or $Q = 1$. This family is described by Lemma 3.18. □

Aside from the fact that $P + Q$ is not LE-symmetric unless P and Q are a chain and a singleton element, we can say something even stronger about H_{P+Q}. We provide the following upper bound on the size of the permutation group H_{P+Q}.

Proposition 3.28. *Let $|P| = m, |Q| = n$. Then*

$$|H_{P+Q}| \leq (|H_P||H_Q|)^{\binom{m+n}{n}}(m + n)!$$

Proof. We bound $|H_{P+Q}|$ by directly bounding the number of permutations on $\mathrm{LinExt}(P + Q)$ we can obtain. By Lemma 3.25, we have that the total number of permutations on $\mathrm{LinExt}(P + Q)$ is bounded by the number of ways to permute the $\binom{m+n}{n}$ partition sets (treating each partition set as an element) and permute the elements within the partition sets.

For any partition $K = (A, B)$, let H_K be the subgroup of H_P acting on $L(K)$. Then for each permutation of the partition sets, the number of ways to permute within the partition sets is bounded above by $\prod_K |H_K|$.

First, we claim that $|H_K| \leq |H_{K^*}|$, where $K^* = ([m], [m + 1, m + n])$. For any partition set $L(K)$ where $K = (A, B)$, consider any permutation within $L(K)$ generated by a t_i. Without loss of generality, let $i, i+1 \in A$. Then this permutation on $L(K)$ can be achieved by $w^{-1}t_{i^*}w$, where $i^* = |A \cap [i]|$ and $w \in \langle t_1, \dots, t_{m+n-1} \rangle$ is the sequence that maps $L(K^*)$ to $L(K)$. Thus any permutation on a partition set $L(K)$ can be generated from a permutation on $L(K^*)$. The number of ways to permute within $L([m], [m + 1, m + n])$ is the number of ways to permute within $L([m], [m+1, m+n])$ without using t_m. This permutation group is then isomorphic to the $H_P \times H_Q$ with size $|H_P||H_Q|$. Since there are $\binom{m+n}{n}$ partition sets, we have

$$\prod_K |H_K| \leq (|H_P||H_Q|)^{\binom{m+n}{n}}$$

Finally, note that the action of the t_i on the partition sets themselves is isomorphic to the action of t_i the poset $P = C_m + C_n$, i.e. the disjoint union of a chain of size m and a chain of size n. To see this, any partition set $K = (A, B)$ corresponds to the linear extension f on $C_m + C_n$ such that $f(C_m) = A$ and $f(C_n) = B$. By Proposition 3.11, this group is precisely S_{n+m}, so the number of ways to permute the partition sets is given by $|S_{m+n}| = (m+n)!$. Thus the total number of permutations on $\text{LinExt}(P+Q)$ is at most $(|H_P||H_Q|)^{\binom{m+n}{n}}(m+n)!$, as desired. □

Using the bound above, we can characterize how far from LE-symmetric a disjoint union $P + Q$ is. More specifically, we have the following lower bound on the index $[S_{\text{LinExt}(P+Q)} : H_P]$, which follows from Proposition 3.28.

Corollary 3.29. *Let* $|P| = m, |Q| = n, a = |\text{LinExt}(P)|, b = |\text{LinExt}(Q)|$. *Then*

$$[S_{\text{LinExt}(P+Q)} : H_P] \geq (a!)^{(b-2)\binom{m+n}{n}} \left(\left(b\binom{m+n}{n} \right) ! \right)^{a-1}$$

Conjecture 3.30. *For* $n \geq 4$, *let* N_n *be a poset on* n *elements with the relations* $v_1 > v_2 < v_3 < \cdots < v_{n-2} < v_{n-1} > v_n$. *(For example,* N_4 *is the* N*-poset.) Then,* N_n *is LE-symmetric.*

Remark 3.31. The conjecture is true for $4 \leq n \leq 75$.

Remark 3.32. We have $|\text{LinExt}(N_n)| = n^2 - 3n + 1$.

Question 3.33. For $a, c \geq 1$ and $b \geq 2$, let $N_{a,b,c}$ be a poset on $a + b + c$ elements with the relations

$$v_1 > v_2 > \cdots > v_{a+1} < v_{a+2} < \cdots < v_{a+b} > v_{a+b+1} > \cdots > v_{a+b+c}.$$

(For example, $N_{1,n-2,1} = N_n$.) Is $N_{a,b,c}$ LE-symmetric?

Definition 3.34. A poset is *series-parallel* if it can be built from the singleton poset using ordinal sums and disjoint unions.

The class of posets in Conjecture 3.30 is interesting because every non-series-parallel poset contains some N_n as a convex induced sub-poset. Using Proposition 3.3 and Lemma 3.26, we can achieve a classification of LE-symmetric (or even LE-primitive) series-parallel posets. Thus, Conjecture 3.30 may be a first step to understanding the linear extension groups of non-series-parallel posets.

3.5. **Primitive posets.** Here we characterize which disconnected posets are *LE-primitive*, or in other words, have a primitive permutation group.

Definition 3.35. A group G acting on a set X is *primitive* if there is no partition of X that is preserved by G other than the trivial partitions (the partition with a single part, and the partition into singletons).

Definition 3.36. We say a poset is *LE-primitive* if its linear extension group is a primitive permutation group.

Lemma 3.37. *Let* P *be a disjoint union of chains of lengths* $n_1, n_2, \ldots, n_r$. *Then, the stabilizer of any linear extension of* P *is isomorphic to* $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r}$.

Proof. The stabilizers of linear extensions are conjugate to each other, hence isomorphic. So it suffices to consider one linear extension. Let ℓ be the linear extension that assigns the labels $\{n_{j-1} + 1, n_{j-1} + 2, \ldots, n_j\}$ to the jth chain (where here

$n_0 = 0$). Then, $t_i \in \mathrm{Stab}(\ell)$ if $i \neq n_j$ for any j. Recall that by Proposition 3.11, $t_1, \ldots, t_{n-1}$ yield a presentation for the symmetric group S_n, where

$$n = \sum_{j=1}^{r} n_j.$$

Thus, the group generated by $\{t_i \mid i \neq n_j \text{ for any } j\}$ is isomorphic to $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r}$. It remains to show that $S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r}$ exhausts $\mathrm{Stab}(\ell)$, which follows from the following.

$$\begin{aligned} |\mathrm{LinExt}(P)| \cdot \mathrm{Stab}(\ell) &= |H_P| \\ \frac{n!}{n_1! \cdots n_r!} \cdot \mathrm{Stab}(\ell) &= n! \\ \mathrm{Stab}(\ell) &= n_1! \cdots n_r! = |S_{n_1} \times S_{n_2} \times \cdots \times S_{n_r}| \end{aligned}$$

□

Theorem 3.38. *Let P be a poset with at least 2 components. Then P is primitive if and only if P is a disjoint union of 2 chains of different lengths or P is an antichain of size 2.*

Proof. First, if one component of P is not a chain, then P is not primitive by Lemma 3.26. If every component is a chain, then the stabilizer of any linear extension is isomorphic to a direct product of symmetric groups as described in Lemma 3.37. Such a direct product of symmetric groups is a maximal proper subgroup of S_n if and only if $r = 2$ and either $n_1 \neq n_2$ or $n_1 = n_2 = 1$. Since a permutation group is primitive if and only if the stabilizer of any element is a maximal subgroup, this completes the proof. □

3.6. **The stabilizer size of a poset.** In this section, we investigate the order of the stabilizer of any linear extension of P, a parameter we call the *stabilizer size* $k(P)$ of P. We characterize the posets with $k = 1$. Then we find a lower bound for k in terms of another parameter which we call the *comparability* $c(P)$ of P, which is the maximum number of $t_i \in \mathrm{Stab}(\ell)$ over all linear extensions $\ell \in \mathrm{LinExt}(P)$. Then we completely classify posets P with $24 \nmid k(P)$.

We call the linear extension group H_P of P just H when context is clear.

Definition 3.39. For a poset P, its *stabilizer size* $k(P)$ is the k such that

$$|H| = k|\mathrm{LinExt}(P)|.$$

Proposition 3.40. *We have $k(P \oplus Q) = k(P) \cdot k(Q)$.*

Proof. We know the order of linear extension groups is multiplicative:

$$|H_{P \oplus Q}| = |H_P \times H_Q| = |H_P| \cdot |H_Q|.$$

Since the number of linear extensions is also multiplicative, the proposition follows. □

Since H is transitive, we have $k \geq 1$. From the orbit-stabilizer theorem, we also have that k is the size of the stabilizer of any linear extension of P, so $k \in \mathbb{N}$. In fact, something stronger is true.

Proposition 3.41. *We have $k = 1$ or k is even.*

Before we prove it, we need the following lemma. If a poset is the ordinal sum of two or more non-empty posets, we say that it is a non-trivial ordinal sum.

Lemma 3.42. *The operator t_i lies in the kernel of $W_n \to \mathfrak{S}_{\mathrm{LinExt}(P)}$ if and only if one can write P as a nontrivial ordinal sum $P = P_1 \oplus P_2$ where $|P_1| = i$.*

Proof. When $P = P_1 \oplus P_2$ with $|P_1| = i$, every linear extension ℓ has $\ell(P_1) = \{1, 2, \ldots, i\}$ and $\ell(P_2) = \{i+1, i+2, \ldots, n\}$, so $u = \ell^{-1}(i) <_P \ell^{-1}(i+1) = v$, and t_i fixes ℓ.

Conversely, if $P \neq P_1 \oplus P_2$ for any $|P_1| = i$, create a linear extension ℓ having $t_i(\ell) \neq \ell$ as follows. Pick any down-set (order ideal) $I \subset P$ with cardinality $|I| = i$; for example, choosing the inverse image $I := f^{-1}(\{1, 2, \ldots, \ell\})$ for any linear extension f of P will work. Then there exists at least one incomparable pair of elements u, v in P, with u a maximal element of the down-set I, and v a minimal element of up-set $P \setminus I$. Otherwise, every maximal element u of I and minimal element v of $P \setminus I$ would have $u <_P v$, as $v <_P u$ is not allowed given I is a down-set. Therefore, $P = I \oplus (P \setminus I)$, a contradiction.

Once one has found such an incomparable pair u, v, create the linear extension ℓ by labeling $\ell(u) = i, \ell(v) = i + 1$ and using any linear extensions to label $I \setminus \{u\}$ with $\{1, 2, \ldots, i - 1\}$ and $(P \setminus I) \setminus \{v\}$ with $\{i + 2, i + 3 \ldots, n\}$. □

Now we can prove Proposition 3.41.

Proof of Proposition 3.41. By Proposition 3.40, it suffices to consider when P is not a non-trivial ordinal sum. When $k \neq 1$, there is some linear extension ℓ fixed by some t_i. Otherwise, there are no relations in the poset, in which case $k = 1$. Then, the function $x \mapsto t_i \cdot x$ is an involutory permutation of the stabilizer of ℓ with no fixed points, since t_i is not in the kernel. Hence, the stabilizer of ℓ has even size. By the remark above, this means k is even. □

Proposition 3.43. *The posets P satisfying $k(P) = 1$ are exactly the ordinal sums of antichains.*

Proof. By Proposition 3.40, it suffices to prove that the posets P satisfying $k(P) = 1$ that are not non-trivial ordinal sums are exactly antichains. If $k = 1$, the size of the stabilizer of any linear extension of P is of size 1. Suppose for the sake of contradiction that P has some order relation, say $v < w$. We may assume that w covers v. We claim there is a linear extension in which v and w have consecutive labels. Indeed, let I be the ideal consisting of the elements strictly less than w and not equal to v. Then, there is a linear extension ℓ of I. We can extend ℓ by setting $\ell(v) = |I| + 1$ and $\ell(w) = |I| + 2$, from which it can then be extended into a linear extension of all of P.

Now, $t_{|I|+1}$ is in the stabilizer of ℓ. Hence, $t_{|I|+1}$ is in the kernel, contradicting Lemma 3.42.

Finally, one can check that for an antichain, we have $k = 1$. □

Now that we have classified which posets have a given stabilizer-size $k(P) = 1$, we turn to the direction of characterizing what possible values $k(P)$ can take. This motivates the following definition.

Definition 3.44. The *Jordan-Pólya numbers* are numbers that can be expressed as a product of factorials.

Proposition 3.45. *For any Jordan-Pólya number J, we can find a poset P such that $k_P = J$.*

Proof. The proof is a simple calculation. Let $J = n_1!n_2!\dots n_k!$, then $P = C_{n_1} + C_{n_2} + \cdots + C_{n_k}$, where C_{n_i} is a chain of size n_i. Let $N = \sum_{i=1}^{k} n_i$. By Proposition 3.11, we know that $|H_P| = |S_N| = N!$. We also have $|\mathrm{LinExt} P| = \binom{N}{n_1, n_2, \dots, n_k}$, since any linear extension is uniquely defined by the partition of the labels $\{1, 2, \dots, N\}$ to the k chains of size $n_1, \dots, n_k$. Then

$$k_P = \frac{|H_P|}{|\mathrm{LinExt}(P)|} = \frac{N!}{\binom{N}{n_1, n_2, \dots, n_k}} = n_1!n_2!\dots n_k! = J,$$

as desired. □

Conjecture 3.46. *For any poset P, let $k_P = 2^{\alpha_2} \cdot 3^{\alpha_3} \cdot 5^{\alpha_5} \cdot \cdots \cdot p^{\alpha_p}$, where p is the largest prime factor of k_P. Then $\alpha_2 \geq \alpha_3 \geq \alpha_5 \geq \cdots \geq \alpha_p$, or in other words, the exponents of the prime factors are weakly decreasing.*

We now define a poset parameter that gives us useful information about the point-stabilizer, which in particular gives us a lower bound for the stabilizer size.

Definition 3.47. For a poset P and $\ell \in \mathrm{LinExt}(P)$, define

$$c(P, \ell) := |\{i \in [1, |P| - 1] \mid \ell^{-1}(i) < \ell^{-1}(i + 1)\}|.$$

Then, the *comparability* of P is

$$c(P) := \max_{\ell} c(P, \ell).$$

In other words, $c(P)$ is the maximum number of t_i such that $t_i \in \mathrm{Stab}(\ell)$.

Remark 3.48. We have $c(P) = |P| - 1 - j(P)$, where $j(P)$ is the jump number of P.

Definition 3.49. For a group G, let $m(G)$ be the maximum size of an independent set of involutions in G.

Example 3.50. We have $m(S_n) = n - 1$.

Proposition 3.51. *We have $m(G) \leq \log_2(|G|)$.*

Proof. Given an independent set $\{g_1, \dots, g_{m(G)}\} \subseteq G$, we have

$$\langle g_1 \rangle < \langle g_1, g_2 \rangle < \cdots < \langle g_1, g_2, \dots, g_{m(G)} \rangle,$$

where each subgroup is at least twice as large as the previous one by Lagrange's theorem. □

Lemma 3.52. *Let $|P| = n$. If P is not a nontrivial ordinal sum, then the action of any t_i is independent, i.e. $\{t_1, \dots, t_{n-1}\}$ is an independent set.*

Proof. By Lemma 3.42, there is some linear extension f on which t_i is not in the kernel. Let $|P| = n$. Suppose for contradiction that $wf = t_i f$ for some $w \in \langle t_1, t_2, \dots, \hat{t_i}, \dots, t_{n-1} \rangle$. Then consider the order ideal I such that $f(I) = [i]$, i.e. the elements of P with labels $\{1, 2, \dots, i\}$. Then any action that is not t_i preserves this image $f(I)$, so $i + 1 \notin wf$. However, we have $i + 1 \in t_i f(I)$, a contradiction. □

Proposition 3.53. *If P is not a non-trivial ordinal sum, then $c(P) \leq m(\mathrm{Stab}(\ell))$.*

Proof. By Lemma 3.42 and Lemma 3.52, ℓ has at least $c(P)$ independent involutions in $\mathrm{Stab}(\ell)$. ☐

Corollary 3.54. *If P is not a non-trivial ordinal sum, then $k(P) \geq 2^{c(P)}$.*

Proof. This immediate from Propositions 3.51 and 3.53. ☐

Proposition 3.55. *If P is not a non-trivial ordinal sum and $c(P) \geq 2n-1$, then $(\mathbb{Z}/2\mathbb{Z})^n \leq \mathrm{Stab}(\ell)$.*

Proof. If $c(P) \geq 2n-1$, then there are at least $\lceil (2n-1)/2 \rceil = n$ pairwise commuting involutions in $\mathrm{Stab}(\ell)$, which generate $(\mathbb{Z}/2\mathbb{Z})^n$. ☐

Proposition 3.56. *If P is not a non-trivial ordinal sum and $(\mathbb{Z}/2\mathbb{Z})^n \not\leq \mathrm{Stab}(\ell)$, then there are at least $c(P)-n+1$ pairs $t_i, t_{i+1} \in \mathrm{Stab}(\ell)$.*

Proof. Suppose there are at most $c(P)-n$ pairs $t_i, t_{i+1} \in \mathrm{Stab}(\ell)$. Then, there are at least $c(P)-(c(P)-n) = n$ pairwise commuting involutions in $\mathrm{Stab}(\ell)$, which generate $(\mathbb{Z}/2\mathbb{Z})^n$. ☐

Corollary 3.57. *If P is not a non-trivial ordinal sum, $c(P) \geq n$, and $(\mathbb{Z}/2\mathbb{Z})^n \not\leq \mathrm{Stab}(\ell)$, then $S_3 \leq \mathrm{Stab}(\ell)$.*

Proof. By Proposition 3.56, there is at least one pair $t_i, t_{i+1} \in \mathrm{Stab}(\ell)$. If $(t_i t_{i+1})^3 = 1$, then $\langle t_i, t_{i+1} \rangle \cong S_3$. If $(t_i t_{i+1})^6 = 1$, then $\langle t_i, t_{i+1} \rangle \cong \mathbb{Z}/2\mathbb{Z} \times S_3$. ☐

Remark 3.58. Observe that $c(P \sqcup Q) = c(P)+c(Q)$ and $c(P \oplus Q) = c(P)+c(Q)+1$.

Remark 3.59. Observe that a poset P satisfies $c(P) = 0$ if and only if P is an antichain, in which case $k = 1$.

Proposition 3.60. *We have $h(P)-1 \leq c(P) \leq |P| - w(P)$, where $h(P)$ is the height of P and $w(P)$ is the width of P.*

Proof. The first inequality is equivalent to $j(P) \leq |P| - h(P)$. Let P' be a subposet of P obtained by removing a maximum chain C from P, so $|P'| = |P| - h(P)$. We construct $\ell \in \mathrm{LinExt}(P)$ as follows. At each step, if label i can be assigned to the least element in C without a label, then do so; otherwise, assign the label i to any other element for which it is possible. Then, by construction, we have $\ell^{-1}(i) \sim \ell^{-1}(i+1)$ only if $\ell^{-1}(i) \in P'$ or $\ell^{-1}(i+1) \in P'$. Further, if $\ell^{-1}(i) \sim \ell^{-1}(i+1)$ and $\ell^{-1}(i+1) \sim \ell^{-1}(i+2)$, we never have $\ell^{-1}(i), \ell^{-1}(i+2) \in C$ and $\ell^{-1}(i+1) \in P'$. It follows that $j(P) \leq |P| - h(P)$.

We now prove the second inequality, which is equivalent to $j(P) \geq w(P) - 1$. Let A be a maximum antichain in P. Order the elements in A according to any $\ell \in \mathrm{LinExt}(P)$. Then, between any consecutive elements in A, there must be a jump in ℓ, so $j(P) \geq w(P) - 1$. ☐

Proposition 3.61. *A poset P satisfies $c(P) = 1$ if and only if $P = (A_1 \oplus A_2) \sqcup A_3$, where A_i is an antichain and A_1, A_2 are non-empty.*

Proof. It is easy to check that a poset of the form $P = (A_1 \oplus A_2) \sqcup A_3$ satisfies $c(P) = 1$. So suppose that P is a poset satisfying $c(P) = 1$. By Proposition 3.60, $h(P) \leq 2$. In fact, since P is not an antichain, $h(P) = 2$. By the additivity of comparability under disjoint unions, exactly one component P' of P has $c(P') = 1$, so $P = P' \sqcup A$, where A is an antichain. Thus, $h(P') = 2$. Suppose for the sake of contradiction that P' is not an ordinal sum of two non-empty posets. Let v be a maximal element in P'. Since P' is not a non-trivial ordinal sum, there is a child of v, call it v', that is incomparable to some $w \sim v$. Since P' is connected, w has a child w'. Then, there is an $\ell \in \mathrm{LinExt}(P)$ such that $\ell(w') + 1 = \ell(w) < \ell(v') = \ell(v) - 1$, in which case $c(P') \geq 2$, a contradiction. So P' is an ordinal sum of two non-empty posets: $P' = P_1 \oplus P_2$. Since $c(P_1 \oplus P_2) = c(P_1) + c(P_2) + 1 = 1$, we have that P_1 and P_2 are antichains, as desired. □

Proposition 3.62. *If $c(P) = 1$ and $k \neq 1, 2$, then $\mathbb{Z}/2\mathbb{Z} \times S_4 \leq \mathrm{Stab}(\ell)$.*

Proof. By Proposition 3.61, $P = (A_1 \oplus A_2) \sqcup A_3$. Since $k \neq 1$, $|A_3| \geq 1$. Since $k \neq 2$, $|A_1| \geq 2$ or $|A_2| \geq 2$. By duality, we may assume $|A_1| \geq 2$. But then, the poset with $|A_1| = 2$ and $|A_2| = |A_3| = 1$ is a convex induced sub-poset of P. A computation shows that this poset has $Stab(\ell) \cong \mathbb{Z}/2\mathbb{Z} \times S_4$, so the proposition follows by Corollary 3.5. □

Proposition 3.63. *If P is not a non-trivial ordinal sum and not a disjoint union of chains, then $S_4 \leq Stab(\ell)$.*

Proof. We may assume that P is not a non-trivial ordinal sum. If P is a disjoint union of chains, the proposition follows from Lemma 3.37. Thus, we may assume that P either has an up-triangle or a down-triangle as an induced sub-poset. By duality, we may assume P has an up-triangle as an induced sub-poset. That is, there are $u, v, w \in P$ with the only relations among them being $v < u$ and $w < u$. Since P is not a non-trivial ordinal sum, there is an immediate child of u incomparable to some $x \sim u$. Without loss of generality, assume this immediate child is w. If $x \sim v$, then u, v, w, and x form a convex induced sub-poset of P with $Stab(\ell) \cong \mathbb{Z}/2\mathbb{Z} \times S_4$. If $x > v$, then u, v, w, and x form a convex induced sub-poset of P with $Stab(\ell) \cong S_4$. In either case, the proposition follows by Corollary 3.5. □

Corollary 3.64. *If $k \neq 6, 12, 36, 2^n$, then either $(\mathbb{Z}/2\mathbb{Z})^2 \times S_3 \leq Stab(\ell)$ or $S_4 \leq Stab(\ell)$. In particular, $24 \mid k$.*

Proof. We may assume that P is not a non-trivial ordinal sum. If P is a disjoint union of chains, the proposition follows from Lemma 3.37. Otherwise, the proposition follows from Proposition 3.63. □

We can completely classify posets P with $24 \nmid k$. By the multiplicativity of k under ordinal sums, it suffices to classify P that are not a non-trivial ordinal sum. If $24 \nmid k$, then by Proposition 3.63, P is a disjoint union of chains. Then, a classification follows from Lemma 3.37.

4. LE-cactus posets

The *cactus relations* on the group generated by $t_1, \ldots, t_{n-1}$ are $(t_i q_{jk})^2 = 1$, where $q_{ij} = q_{j-1} q_{j-i} q_{j-1}$ and $q_i = t_1(t_2 t_1) \ldots (t_i \ldots t_1)$. Call a poset P *cactus* if P satisfies the cactus relations. Otherwise, P is called *non-cactus.*

In this section, we construct posets that satisfy cactus relations, termed *LE-cactus posets.* In Section 4.1, we provee several properties that hold in LE-cactus posets. Aside from previous results that Ferrers posets are LE-cactus [CGP20, Theorem 1.4], we show that a disjoint union of two LE-cactus posets is LE-cactus (Proposition 4.14) and the ordinal sum of one or two element anti-chain with a LE-cactus poset is LE-cactus (Proposition 4.15 and Proposition 4.16). In Section

4.2, we consider posets that do not satisfy cactus relations and proved that no LE-cactus poset has three bottom elements (Proposition 4.21). Lastly, in Section 4.3, we attempt to construct LE-cactus posets and listed out our conjectures along with examples and counterexamples.

Definition 4.1. Call a poset P *LE-cactus* if the t_i satisfy the cactus relations.

4.1. Cactus group properties on posets.

Definition 4.2. We define *promotion*, denoted δ_i, to be the action of $t_{i-1} \dots t_2 t_1$.

Definition 4.3. We define *evacuation* to be the action of

$$q_{i-1} = t_1(t_2t_1)\dots(t_{i-1}\dots t_1) = \delta_1\delta_2\dots\delta_i.$$

Thus, evacuation q_{i-1} can be thought of as i rounds of promotion.

Note that by this definition of promotion and evacuation is equivalent to the definition of promotion and evacuation in Stanley [Sta09].

Proposition 4.4 ([Sta09])**.** *Evacuation q_i is an involution.*

Corollary 4.5. *$q_{jk} = q_{k-1}q_{k-j}q_{k-1}$ is an involution.*

Thus to show that a poset P is LE-cactus, it suffices to show that the actions t_i and q_{jk} commute for $i + 1 < j < k$.

REU problem 3a (i) asked whether the t_i give a LE-cactus group action (i.e., whether H_P is a quotient of $\mathcal{C}_n$). Son and Matthew's code found some counterexamples.

Example 4.6. The minimal counterexamples appear when $n = 4$:

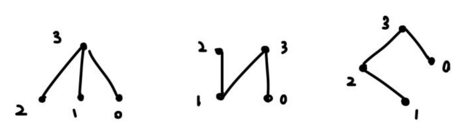


FIGURE 7. Minimal non-cactus posets

Question 4.7. For which P do the t_i give a cactus group action?

Proposition 4.8. *Ferrers posets F_λ are LE-cactus.*

See [CGP20, Theorem 1.4] for a proof.

Lemma 4.9. *A poset P is LE-cactus if and only if every ideal I of P is LE-cactus.*

Recall that in Proposition 3.8, we showed that if a relation type fails on the linear extensions of some convex induced subposet of P, then the relation type fails on P. However, the LE-cactus relations $(t_iq_{jk})^2 = 1$ do not form a relation type. Thus, we cannot say that if some convex induced subposet of P is not LE-cactus, then P is not LE-cactus. However, we have the following condition is sufficient for showing a poset P is not LE-cactus.

Proof of Lemma 4.9. By contrapositive, suppose that there exists an ideal I of P such that I is not LE-cactus. We can construct a linear extension f^* by taking any linear extension on I and on the shifted induced poset $P \setminus I$, where the shifted induced poset is the induced poset $P \setminus I$ with its label adding $m = |I|$. Then, we can construct a linear extension of P such that $\mathrm{im}(f(I)) = [m]$. Moreover, there exists no element $p \in I$, $q \in P \setminus I$ such that $p > q$. Otherwise, it contradicts with I being an ideal.

Since we assumed I is not LE-cactus, there exists indices i, j, k satisfying $2 \leq i + 1 < j < k \leq m - 1$ and $(t_i q_{jk})^2 \neq 1$. Given the construction we have for f^*, $(t_i q_{jk})^2(f) \neq 1$ also holds on linear extension f^*. Thus, since the cactus relations fail on a linear extension of P, they cannot hold on $\mathrm{LinExt}(P)$. □

Thus when classifying which posets are LE-cactus, we can eliminate any poset that contains a not LE-cactus poset as an ideal.

4.1.1. *Cactus relations under disjoint union.* Now we can show that the disjoint union of LE-cactus posets remains LE-cactus. To do so, we first define a bijection that will help us break down showing commutativity on an entire linear extension of $P + Q$.

Definition 4.10. We define a map T from linear extensions to tuples by

$$T(f) = (P^*, Q^*, f(P), f(Q))$$

for any $f \in \mathrm{LinExt}(P + Q)$, where P^*, Q^* are independent linear extensions on P, Q with the same relative ordering of elements as in f, and $f(P), f(Q)$ are defined in Definition 3.22.

Example 4.11. Under T, the union in figure 8 is mapped to the tuple containing P^*, Q^* in figure 9 and $f(P) = \{1, 2, 4, 5, 6, 9\}, f(Q) = \{3, 7, 8, 10\}$.

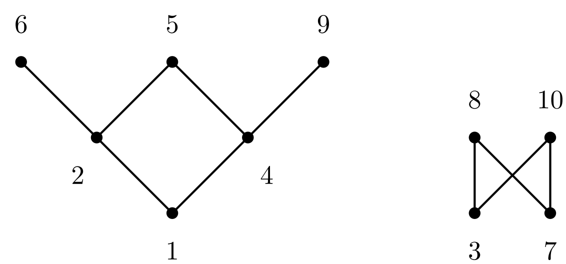


FIGURE 8. $P \sqcup Q$

It is easy to see that T is injective.

We also need the following lemma describing the action of q_{i-1} on $f(P)$ and $f(Q)$.

Lemma 4.12. *Let f be a linear extension of $P + Q$. If element $j \in f(P)$, then*

i) $i - j \in (q_{i-1}f)(P)$ *for all* $i \geq j$

ii) $j \in (q_{i-1}f)(P)$ *for all* $i < j$.

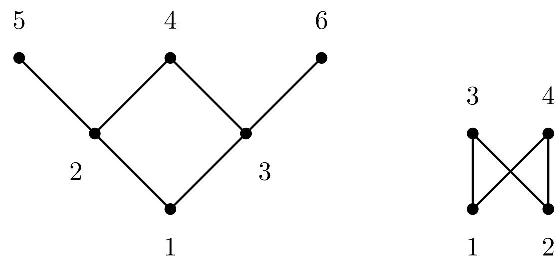


FIGURE 9. P^* and Q^*

In other words, if label j is assigned to an element of P in the linear extension f, then label $i-j$ is assigned to an element of P in the linear extension $q_{i-1}f$ when $i \geq j$, and label j is assigned to an element of P in linear extension $q_{i-1}f$ when $i < j$.

Proof. Consider the action of $q_{i-1} = \delta_1\delta_2 \dots \delta_{i-1}$ on a fixed linear extension f. Observe that by Stanley's definition of promotion, for any δ_j, we only act on either P or Q.

Note that ii) is immediate, as q_{i-1} only acts on elements in $[i]$, so j remains as a label in P.

It remains to show i). If $1 \in f(P)$, then after the first round of promotion $\delta_i = t_{i-1} \dots t_1$, we have $i \in (\delta_i f)(P)$. Since we fix the label i after round δ_i in evacuation, we thus have $i \in (q_i f)(P)$. Similarly, if $2 \in f(P)$, then after two rounds of promotion $\delta_{i-2}\delta_{i-1}$, we have $i-2 \in (\delta_{i-2}\delta_{i-1}f)(P)$, and thus $i-2 \in (q_{i-1}f)(P)$. In general, $j \in f(P)$ where $j < i$, then we have after $j-1$ rounds of promotion that $1 \in (\delta_{i+1-j} \dots \delta_{i-2}\delta_{i-1})f(P)$, since we decrease the label j of by 1 in each round of promotion. Thus, applying δ_{i-j} for round j of promotion gives us $i-j \in (\delta_{i-j} \dots \delta_{i-2}\delta_{i-1})f(P)$. Since the label $i-j$ is frozen for any subsequent rounds of promotion, we have

$$j \in f(P) \Rightarrow i-j \in (q_{i-1}f)(P)$$

as desired. □

We also need the following lemma describing the action of q_{jk} on P^* and Q^*.

Lemma 4.13. *Let $f \in \mathrm{LinExt}(P+Q)$, and let $T(f) = (P^*, Q^*, f(P), f(Q))$. Then*

$$T(q_{jk}f) = (q_{m-n,m}(P^*), q_{jk}(Q^*), (q_{jk}f)(P), (q_{jk}f)(Q)).$$

where $m = |f(P) \cap [k]|$ and $n = |f(P) \cap [j+1,k]| - 1$.

Proof. Note that in each round of promotion during evacuation, we only act on either P or Q, which follows immediately from Stanley's definition. Furthermore, the number of rounds of promotion that act on P is equal to the number of times the label 1 appears in P, which is precisely $|f(P) \cap [k]|$. From this, we see that acting on f by $q_{k-1} = \delta_1 \dots \delta_{k-1}$ gets mapped under T to acting on P^* by q_{m-1}, where $m = |f(P) \cap [k]|$.

Now we consider the action of q_{k-j} on $q_{k-1}f$. By the same argument above, we have the induced action of q_{k-j} on $q_{m-1}P^*$ is equivalent to the action of $q_{n'-1}$ on $q_{m-1}P^*$ where $n' = |(q_{k-1}f)(P) \cap [k-j]|$. Using Lemma 4.12, $e \in f(P) \Rightarrow k-e \in (q_{k-1}f)(P)$. Thus, $n' = |(q_{k-1}f)(P) \cap [k-j]| = |f(P) \cap [j+k+1,k]|$.

Finally, we act again by q_{k-1} on $q_{k-j}q_{k-1}f$. The induced action of q_{k-1} on $q_{n'-1}q_{m-1}P^*$ is equivalent to the action of $q_{m'-1}$ on $q_{n'-1}q_{m-1}P^*$, where $m' = |(q_{k-j}q_{k-1}f)(P) \cap [k]|$. However, by Lemma 4.12, we have that $|(q_{k-j}q_{k-1}f)(P) \cap [k]| = |f(P) \cap [k]|$, so our action of q_{k-1} is really equivalent to the action of q_{m-1} on $q_{n'-1}q_{m-1}P^*$.

Thus we have

$$T(q_{jk}f) = (q_{m-1}q_{n'-1}q_{m-1}(P^*), q_{jk}(Q^*), (q_{jk}f)(P), (q_{jk}f)(Q))$$

where $m = |f(P)\cap[k]|$ and $n' = |f(P)\cap[j+k+1, k]|$, and note that $q_{m-1}q_{n'-1}q_{m-1}$ is equivalent to $q_{m-n,m} = q_{m-1}q_nq_{m-1}$, where $n = n'-1 = |f(P)\cap[j+k+1, k]|-1$. Hence

$$T(q_{jk}f) = (q_{m-n,m}(P^*), q_{jk}(Q^*), (q_{jk}f)(P), (q_{jk}f)(Q))$$

where $m = |f(P) \cap [k]|$ and $n = |f(P) \cap [j + k + 1, k]| - 1$ as desired. □

Now we are ready to prove our main proposition.

Proposition 4.14. *If P and Q are LE-cactus, then $P + Q$ is LE-cactus.*

Proof. Let $f \in \text{LinExt}(P + Q)$ be any linear extension of $P + Q$, and let $T(f) = (P^*, Q^*, f(P), f(Q))$. Since our map T is injective, it suffices to show that t_i and q_{jk} commute with respect to each of $P^*, Q^*, f(P), f(Q)$. First we show that t_i and q_{jk} commute with respect to $f(P)$ and $f(Q)$.

Case 1: Assume without loss of generality that $i, i + 1 \in f(P)$. Then t_i acts as the identity on $f(P)$ and $f(Q)$, so clearly t_i and q_{jk} commute.

Case 2: Assume without loss of generality that $i \in f(P)$ and $i + 1 \in f(Q)$, which will simply denote as $(i, i + 1)$. All other elements are unaffected by t_i, so it suffices to consider where $(i, i + 1)$ gets taken under t_i and q_{jk}. Suppose we act on f by $q_{jk}t_i$. Then t_i takes $(i, i + 1)$ to $(i + 1, i)$. Using Lemma 4.12, since $k - 1 > i$ and $k - j < k - i$, we have

$$(i, i + 1) \overset{t_i}{\to} (i + 1, i) \overset{q_{k-1}}{\to} (k - (i + 1), k - i) \overset{q_{k-j}}{\to} (k - (i + 1), k - i) \overset{q_{k-1}}{\to} (i + 1, i).$$

Now suppose we act f by t_iq_{jk}, which gives

$$(i, i + 1) \overset{q_{k-1}}{\to} (k - i, k - (i + 1)) \overset{q_{k-j}}{\to} (k - i, k - (i + 1)) \overset{q_{k-1}}{\to} (i, i + 1) \overset{t_i}{\to} (i + 1, i).$$

Thus t_i and q_{jk} commute with respect to $f(P), f(Q)$, as desired.

It remains to show that t_i and q_{jk} commute with respect to P^* and Q^*.

Case 1: Assume without loss of generality that $i, i + 1 \in f(P)$. Then t_i does not affect Q^*, so it suffices to show that t_i and q_{jk} commute with respect to P^*. Note that by Lemma 4.13, we have the induced actions of t_iq_{jk} and $q_{jk}t_i$ on P^* are equivalent to $t_{i'}q_{m-n,m}(P^*)$ and $q_{m-n,m}t_{i'}(P^*)$, respectively, where $m = |f(P)\cap[k]|$ and $n = |f(P) \cap [j, k]|$ and $i' = |f(P) \cap [i]|$. Further note that

$$\begin{aligned}(m - n) - i' &= |f(P) \cap [k]| - (|f(P) \cap [j + 1, k]| - 1) - |f(P) \cap [i]| \\ \Rightarrow (m - n) - i' &= |f(P) \cap [j]| - |f(P) \cap [i]| + 1 \geq 2 \\ &\Rightarrow i' + 1 < m - n\end{aligned}$$

where the second to last equation follows from our assumption that $i, i+1$ are both in $f(P)$ and $j > i + 1$. Then $t_{i'}q_{m-n,m}(P^*) = q_{m-n,m}t_{i'}(P^*)$, since P itself is LE-cactus, so t_i and q_{jk} commute on P^* and Q^* in this case.

Case 2: Assume without loss of generality that $i \in f(P)$ and $i+1 \in f(Q)$. Then t_i acts as the identity on P^* and Q^*, as swapping elements $i, i+1$ does not change the relative ordering within $f(P)$ and $f(Q)$. Then clearly t_i and q_{jk} commute.

Thus, t_i and q_{jk} also commute with respect to P^* and Q^*. Hence t_i and q_{jk} commute with respect to the linear extension $f \in \mathrm{LinExt}(P+Q)$, so $P+Q$ is LE-cactus. □

4.1.2. *Cactus relations under ordinal sums.* Now that we have looked at what happens to the cactus relations under ordinal sums, another natural question to ask is when the cactus relations are preserved under taking ordinal sums of posets. We have the following two ordinal sum constructions that preserve the cactus property.

Proposition 4.15. *If P is LE-cactus, then $1 \oplus P$ is LE-cactus.*

Proof. Assume $|P| = n-1$, so $|1 \oplus P| = n$. Since P is LE-cactus, the relations $(t_i q_{jk})^2 = 1$ hold for $3 \le i+1 < j < k \le n$ with $k - j > 1$. Moreover, in any linear extension of $1 \oplus P$, the unique minimum element must be labelled 1 since it is comparable with all other elements, so t_1 is the identity. Since q_{jk} is an involution for all $j < k \le n$, it then holds that $(t_1 q_{jk})^2 = 1$ for all j, k. It then remains to check the relations $(t_i q_{jk})^2$ when $k - j = 1$. In this case, we have

$$q_{jk} = q_{k-1} q_{k-j} q_{k-1} = q_{k-1} q_1 q_{k-1} = q_{k-1} t_1 q_{k-1}.$$

As above, t_1 is the identity, and since q_{k-1} is an involution, q_{jk} is the identity as well. From this the relation $(t_i q_{jk})^2 = 1$ follows. □

Proposition 4.16. *If P is LE-cactus, then $(1+1) \oplus P$ is LE-cactus.*

Proof. Given $i+1 < j < k \le n$, one only need to check $(t_i q_{jk})^2 = 1$ on three cases: $k - j > 2$, $k - j = 2$, and $k - j = 1$.
When $k - j > 2$, the argument is analogous to Proposition 4.15, where this result is implied by P being LE-cactus since t_2 acts trivially. When $k - j = 2$, t_2 is the identity. Therefore,

$$\begin{aligned}(t_i q_{jk}(t_2 t_1)(t_1) q_{jk})^2 &= (t_i q_{jk}(t_2) q_{jk})^2 \\ &= (t_i q_{jk} q_{jk})^2, \text{ (since } t_2 \text{ is the identity)} \\ &= 1\end{aligned}$$

When $k - j = 1$, since t_1 in general commutes with everything except t_2, and t_2 in this case is the identity, so we have

$$(t_i q_{jk} t_2 t_1 q_{jk})^2 = (t_i t_1 q_{jk} q_{jk})^2 = (t_i t_1)^2 = 1.$$

□

4.1.3. *Minimal cactus generators.* We now explore a conjecture about the connection between different relations $(t_i q_{jk})^2$. In particular, we explore whether there are "small" sets of triples (i, j, k) such that a poset P is LE-cactus if and only if P satisfies all relations in this small set.

Definition 4.17. For $2 \le i+1 < j < k \le n$, a poset P of size n is (i,j,k)-*LE-cactus* if $(t_i q_{jk})^2 \in \ker \phi_P$.

Proposition 4.18. *Let P be a poset with $|P| = n$, where $n \in \{5, 6, 7, 8\}$. Then P is LE-cactus if and only if:*

- *When $n = 5$, P is $(1,3,5)$-LE-cactus.*

- *When* $n = 6$*,* P *is* $(2,4,6)$*-LE-cactus and at least one of* $(1,3,6), (1,4,6)$*, or* $(1,5,6)$*-LE-cactus.*
- *When* $n = 7$*,* P *is* $(2,4,7)$*-LE-cactus and is at least one of* $(1,3,7), (1,4,7)$*, or* $(1,6,7)$*-LE-cactus.*
- *When* $n = 8$*,* P *is* $(1,A,8)$*,* $(2,B,8)$*, and* $(3,5,8)$*-LE-cactus for at least one of the following pairs* (A,B)*:*

$$\begin{matrix} (3,4) & (3,5) & (3,6) \\ (4,4) & (4,5) & \\ (5,4) & (5,5) & (5,6) \\ (6,4) & (6,5) & (6,6) \\ (7,4) & (7,5) & \end{matrix}$$

Proof. To obtain this result for a particular $n \in \{5,6,7,8\}$, we used SAGE to generate a list of all pairs (P, C_P), where P is a poset of size n, and C_P consists of all triples (i,j,k) with $2 \le i+1 < j < k \le n$ and $(t_i q_{jk})^2 \in \ker \phi_P$. Let T_n be the set of all such possible triples (i,j,k). For each subset $T' \subset T_n$ of size $1,2,2$, or 3 respectively, we include T' in our list iff $C_P = T_n$ for all P with $T' \subset C_P$. That is, we include T' if for any poset P satisfying each triple in T', P is LE-cactus. □

Question 4.19. Can we find other small sets of (i,j,k)-triples for $n = 6,7,8$ that together imply P is LE-cactus, potentially sets that allow for easier pattern spotting? Can we find similar small sets for $n \ge 9$?

4.2. **Non-LE-cactus poset properties.**

Proposition 4.20. *If* P *is an order ideal of a poset* $Q \supset P$ *and* P *is not LE-cactus, then* Q *is not LE-cactus.*

Proof. This is equivalent to Lemma 4.9. □

Proposition 4.21. *Given any* P*,* $n \oplus P$ *is non-LE-cactus if* $n \ge 3$*.*

Proof. By proposition 4.20, Q is non-LE-cactus if it is an order ideal of a non-LE-cactus poset. Since Example 4.6 shows that $3 \oplus 1$ is non-LE-cactus, $3 \oplus P$ is non-LE-cactus. Similarly, $n \oplus P$ is non-LE-cactus for $n \ge 3$. □

Definition 4.22. Define $P_{\underline{n}} := 1^{n_1} \oplus 1^{n_2} \oplus \cdots \oplus 1^{n_l}$ as the ordinal sum of anti-chains, where 1^{n_i} is the anti-chain of n_i vertices, and $\underline{n} = (n_1, n_2, \ldots, n_\ell)$.

Question: When is $P_{\underline{n}}$ LE-cactus?

Remark 4.23. By observing the data, we know the followings: if $P_{\underline{n}}$ is LE-cactus, under certain (unknown) criteria, $P_{\underline{n}} \oplus 1$ and $P_{\underline{n}} \oplus 2$ will become non-LE-cactus. Likewise, if $P'_{\underline{n}}$ is non-LE-cactus, under certain (unknown) criteria, $1 \oplus P'_{\underline{n}}$ and $2 \oplus P'_{\underline{n}}$ will make it LE-cactus.

We already knew that $P_{\underline{n}}$ $n_i \le 2$ is always LE-cactus, therefore we listed out the ones when $n_i \le 3$ for $l \le 4$ in table 1.

Conjecture 4.24. *Given an antichain* P*,* $1 \oplus P \oplus 1$ *is non-LE-cactus if* P *has size* $l \ge 4$*.*

Remark 4.25. When $n := |P| = 4,5,6,7$, the LE-cactus relations that fail are precisely $(2, \ell, n+2)$ for $4 \le \ell \le n$.

	LE-cactus	non-LE-cactus
$l = 1$	3	N/A
$l = 2$	$(1 \oplus 3)$, $(2 \oplus 3)$	$(3 \oplus 1)$, $(3 \oplus 2)$
		$(3 \oplus 3)$
$l = 3$	$(1 \oplus 1 \oplus 3)$, $(1 \oplus 2 \oplus 3)$	$(3 \oplus 1 \oplus 1)$, $(3 \oplus 1 \oplus 2)$
	$(2 \oplus 1 \oplus 3)$, $(2 \oplus 2 \oplus 3)$	$(3 \oplus 2 \oplus 1)$, $(3 \oplus 2 \oplus 2)$
	$(1 \oplus 3 \oplus 1)$	$(3 \oplus 3 \oplus 1)$
	$(1 \oplus 3 \oplus 2)$	$(3 \oplus 3 \oplus 2)$
	$(2 \oplus 3 \oplus 1)$	$(1 \oplus 3 \oplus 3)$
	$(2 \oplus 3 \oplus 2)$	$(2 \oplus 3 \oplus 3)$
		$(3 \oplus 3 \oplus 3)$
$l = 4$	$(1 \oplus 1 \oplus 1 \oplus 3)$, $(1 \oplus 1 \oplus 2 \oplus 3)$	$(3 \oplus 1 \oplus 1 \oplus 1)$, $(3 \oplus 1 \oplus 1 \oplus 2)$
	$(1 \oplus 2 \oplus 1 \oplus 3)$, $(1 \oplus 2 \oplus 2 \oplus 3)$	$(3 \oplus 1 \oplus 2 \oplus 1)$, $(3 \oplus 1 \oplus 2 \oplus 2)$
	$(2 \oplus 1 \oplus 1 \oplus 3)$, $(2 \oplus 1 \oplus 2 \oplus 3)$	$(3 \oplus 2 \oplus 1 \oplus 1)$, $(3 \oplus 2 \oplus 1 \oplus 2)$
	$(2 \oplus 2 \oplus 1 \oplus 3)$, $(2 \oplus 2 \oplus 2 \oplus 3)$	$(3 \oplus 2 \oplus 2 \oplus 1)$, $(3 \oplus 2 \oplus 2 \oplus 2)$
	$(1 \oplus 1 \oplus 3 \oplus 1)$, $(1 \oplus 1 \oplus 3 \oplus 2)$	$(1 \oplus 3 \oplus 2 \oplus 1)$, $(2 \oplus 3 \oplus 2 \oplus 1)$
	$(1 \oplus 2 \oplus 3 \oplus 1)$, $(1 \oplus 2 \oplus 3 \oplus 2)$	$(1 \oplus 3 \oplus 2 \oplus 2)$, $(2 \oplus 3 \oplus 2 \oplus 2)$
	$(2 \oplus 1 \oplus 3 \oplus 1)$, $(2 \oplus 1 \oplus 3 \oplus 2)$	$(1 \oplus 3 \oplus 1 \oplus 2)$, $(2 \oplus 3 \oplus 1 \oplus 2)$
	$(2 \oplus 2 \oplus 3 \oplus 1)$, $(2 \oplus 2 \oplus 3 \oplus 2)$	
	$(1 \oplus 2 \oplus 3 \oplus 3)$, $(2 \oplus 1 \oplus 3 \oplus 3)$, $(2 \oplus 2 \oplus 3 \oplus 3)$	$(1 \oplus 1 \oplus 3 \oplus 3)$
	$(1 \oplus 3 \oplus 1 \oplus 1)$, $(2 \oplus 3 \oplus 1 \oplus 1)$	

TABLE 1

Conjecture 4.26. *Given an antichain P, $(1 + 1) \oplus P \oplus 1$ is non-LE-cactus if P has size ≥ 5.*

Remark 4.27. When $n := |P| = 5, 6, 7$, the LE-cactus relations that fail are precisely $(3, \ell, n + 3)$ for $5 \leq \ell \leq n$.

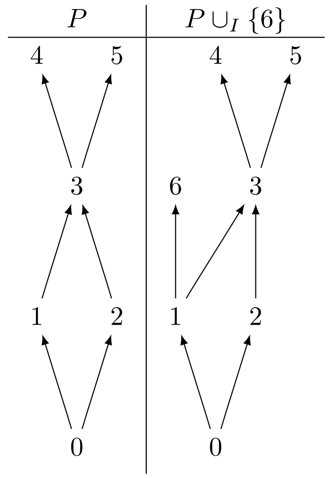
4.3. **Potential LE-cactus constructions.**

Question 4.28. If P is LE-cactus and contains an order ideal I for which $I \oplus \{x\}$ is a rectangular Ferrers poset, is $P \cup_I \{x\}$ LE-cactus[1]?

This is not true in general, as seen by the following example.

Example 4.29. In this example, P is LE-cactus, but $P \cup_I \{6\}$ is not cactus, where I is the order ideal generated by 1.

[1]The notation $P \cup_I \{x\}$ means that x is greater than all elements in I and x is incomparable with elements of P not in I.

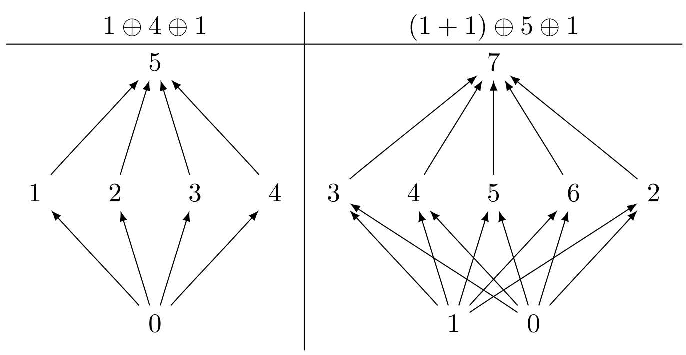


One may wonder what adding a maximal and minimal element does to LE-cactus posets. Proposition 4.15 tells us that adding a minimal element to a LE-cactus poset keeps the poset LE-cactus. However, adding a maximal element to a LE-cactus poset does not always give a LE-cactus poset, as seen by the left poset in Example 4.6. A few related questions are the following:

Question 4.30. If P is LE-cactus, is $1 \oplus P \oplus 1$ LE-cactus? If P is LE-cactus, is $(1+1) \oplus P \oplus 1$ LE-cactus?

This is not true in general, as seen by the following example.

Example 4.31. Neither of the following posets are LE-cactus:



Related to Question 4.30 is the following:

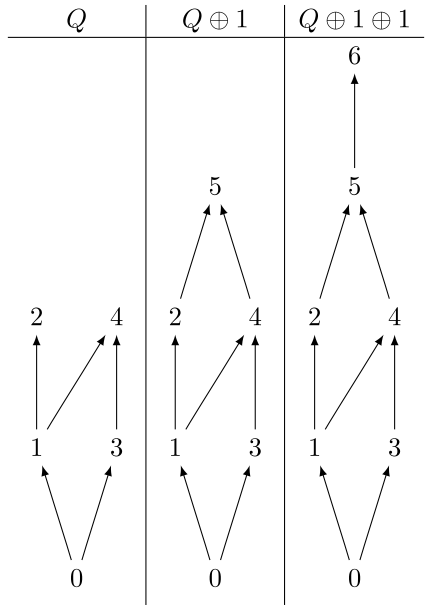
Question 4.32. If P is a Ferrers poset, does Question 4.30 hold?

Question 4.33. Are there sufficient conditions on LE-cactus P so that $P \oplus 1$ is LE-cactus?

Question 4.34. If $Q \oplus 1$ is LE-cactus, do we have $Q \oplus 1 \oplus 1$ is LE-cactus?

This is not always true, as shown in the example below.

Example 4.35. In this case, $Q \oplus 1$ is LE-cactus, but $Q \oplus 1 \oplus 1$ is not LE-cactus.

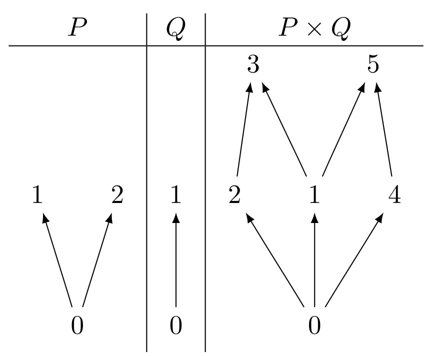


If P and Q are posets, the *Cartesian product poset* $P \times Q$ consists of elements (p, q) for all $p \in P$ and $q \in Q$ such that $(p_1, q_1) \leq (p_2, q_2)$ if and only if $p_1 \leq p_2$ in P and $q_1 \leq q_2$ in Q.

Question 4.36. If P and Q are LE-cactus, is $P \times Q$ LE-cactus?

This is not always true, as shown in the example below.

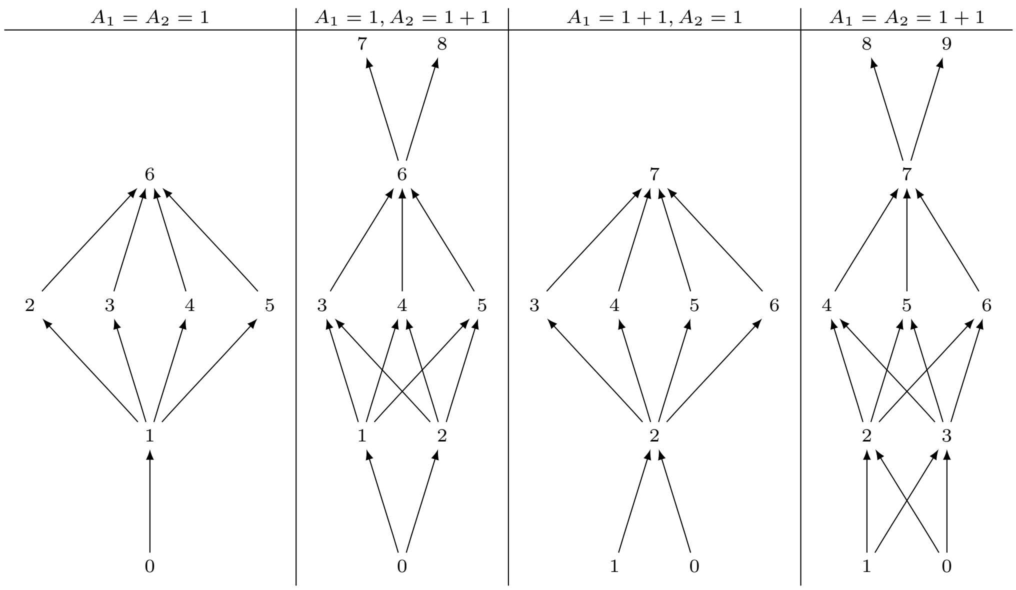
Example 4.37. In this example, P and Q are LE-cactus, but $P \times Q$ is not.



Question 4.38. If $A_1 \oplus A_2 \oplus P$ is LE-cactus, where each A_i is either 1 or $1 + 1$, is $A_2 \oplus P$ LE-cactus?

This does not hold in any of the four cases, as seen in this example:

Example 4.39. In each of the following cases, $A_1 \oplus A_2 \oplus P$ is LE-cactus, but $A_2 \oplus P$ is not LE-cactus. These are of minimal size.



4.4. **Speculation on d-complete posets.** Starting with the unique poset 1 on one element, when one iterates the constructs of disjoint union and $P \mapsto 1 \oplus P$, one produces a family of LE-cactus posets consisting of all *rooted forest posets* (with roots at the bottom). Knuth observed a famous hook-length formula for counting the linear extensions of such forest posets. Knuth was motivated by the family of Ferrers posets for which the Frame-Robinson-Thrall hook-length formula counts their linear extensions, another family of posets that we know are LE-cactus.

Motivated by these two families with hook formulas, R. Proctor introduced the family of *d-complete posets* having such a hook-length formula for their linear extensions– see Proctor and Scoppetta [PS19] and Kim and Yoo [KY19] for definitions and the hook-length formulas. This raises an intriguing question.

Question 4.40. Are all d-complete posets LE-cactus?

Preliminary investigations point to an affirmative answer. For example,

- Small computations with exceptional *minuscule posets* (a subfamily of d-complete posets) indicate that they are LE-cactus.
- It is not too hard to show that all *shifted Ferrers posets*, which form another motivating subfamily of d-complete posets, are all LE-cactus. The idea is to embed standard shifted tableaux (= linear extensions of shifted Ferrers posets) by "doubling" them into what Sagan [Sag87, §4] calls a *shift-symmetric* column-strict tableaux. Then the usual t_i action on the standard shifted tableaux commutes with t_i action on the shift-symmetric tableaux, where one knows that the t_i satisfy cactus relations.

5. Bender–Knuth action on column-strict tableaux

In this section, we return to the Bender–Knuth action on column-strict tableaux. We are particularly interested in finding column-strict tableaux for which the Bender–Knuth action is transitive. We find multiple families in Proposition 5.7, Theorem 5.11, or Theorem 5.13, and provide further data in Subsection 5.2.

The set of all column strict tableaux of shape λ and entries contained in $[n]$ is denoted by $\mathrm{CST}(\lambda, [n])$. We denoted by $\mathrm{CST}(\lambda, \alpha)$ the subset of $\mathrm{CST}(\lambda, [n])$ having *content* $\alpha = (\alpha_1, \ldots, \alpha_n)$, that is, α_i occurrences of i for each $i = 1, 2, \ldots, N$. The Bender-Knuth involutions $t_1, \ldots, t_{n-1}$ are involutions acting on $\mathrm{CST}(\lambda, [n])$ that were defined on Day 6, and send a tableaux T of content α to a tableaux $t_i(T)$ having content

$$s_i(\alpha) = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, a_{\alpha_{i+2}}, \ldots, \alpha_n).$$

Consequently, the t_i also act on each of these sets where the shape λ is fixed, and the content varies through the permutations $w(\alpha)$ of α:

$$\bigsqcup_{w \in \mathfrak{S}_n} \mathrm{CST}(\lambda, w(\alpha)).$$

Definition 5.1. Fix a shape λ and a content α of the same size $N = |\lambda| = |\alpha|$. Call the action of $t_1, \ldots, t_{N-1}$ on the CSTs with shape λ and permutations of a fixed content α the *Bender-Knuth (BK) action*.

Question 5.2. Is this action always transitive, as in the case where $\alpha = (1, 1, \ldots, 1)$?

We do not always get a transitive action, as shown by the below example.

Example 5.3. The column-strict tableaux with shape $\lambda = (4, 2)$ and content $\mu = (2, 2, 2)$ has two orbits, as shown in Figure 10.

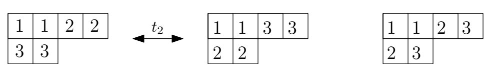


FIGURE 10. Two orbits in the BK action

We can, however, restrict ourselves to the Bender–Knuth action on the column-strict tableaux of content μ. To form a precise statement, recall the infinite Coxeter group

$$W_n = \langle t_1, \ldots, t_{n-1} : t_i^2 = 1 \text{ and } t_i t_j = t_j t_i \text{ if } |i - j| \geq 2 \rangle.$$

The group W_n acts on $\mathfrak{S}_X$, where $X = \bigsqcup_{w \in \mathfrak{S}_n} \mathrm{CST}(\lambda, w(\alpha))$, via the Bender–Knuth involutions. If $p : W_n \to \mathfrak{S}_n$ is the quotient map, let

$$W(\mu) = p^{-1}(\{w \in S_n : w(\mu) = \mu\}),$$

which is the set of subgroup of W_n whose image under p fixes μ.

Proposition 5.4. *The set of W_n-orbits on $\bigsqcup_{w \in \mathfrak{S}_n} \mathrm{CST}(\lambda, w(\mu))$ is in bijection with the set of $W(\mu)$-orbits on $\mathrm{CST}(\lambda, \mu)$.*

Proof. Fix an orbit $\mathcal{O}$ of the W_n-action on $\bigsqcup_{w \in \mathfrak{S}_n} \mathrm{CST}(\lambda, w(\mu))$. Every tableau in $\mathcal{O}$ has some W_n-orbit representative lying in $\mathrm{CST}(\lambda, \mu)$ because the action of W_n on $\mathfrak{S}_n$ is transitive. Whenever two tableaux in $\mathrm{CST}(\lambda, \mu)$ lie in the same W_n-orbit, they actually also lie in the same $W(\mu)$-orbit, because the element that carried one to the other had to fix the content μ. Conversely, if we take an orbit $\mathcal{O}_\mu$ of the $W(\mu)$-action on $\mathrm{CST}(\lambda, \mu)$, the subset of $\bigsqcup_{w \in \mathfrak{S}_n} \mathrm{CST}(\lambda, w(\mu))$ whose representative lies in $\mathcal{O}_\mu$ is a W_n-orbit. $\square$

Lemma 5.5. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_m)$.*
If $\ell(\lambda) = \ell(\mu)$ then the BK action on

$$\bigsqcup_{w \in \mathfrak{S}_m} CST(\lambda, w(\mu))$$

is isomorphic to the BK action on

$$\bigsqcup_{w \in \mathfrak{S}_m} CST(\lambda', w(\mu'))$$

where $\lambda' = (\lambda_1 - 1, \dots, \lambda_n - 1)$ and $\mu' = (\mu_1 - 1, \dots, \mu_m - 1)$.

Proof. If $\ell(\lambda) = \ell(\mu)$ then the first column has to be filled with $1, 2, \dots, \ell(\mu)$. This means that every BK involution fixes the first column, and so we only concern the rest of the CST which has shape $(\lambda_1 - 1, \dots, \lambda_n - 1)$ and content $(\mu_1 - 1, \dots, \mu_m - 1)$. □

Thus, we will only consider either $\lambda = \emptyset$ or $\ell(\lambda) < \ell(\mu)$.

5.1. **Transitivity.**

Definition 5.6. The *Kostka number* is the number of column-strict tableaux with shape λ and content α.

Proposition 5.7. *For the following pairs of (λ, α), the BK action is transitive.*

(1) The Kostka number $K_{\lambda\alpha} = 1$.
(2) The content $\alpha = (1, 1, \dots, 1)$.
(3) Let λ be an $a \times b$ rectangle, with content $\alpha = (b-1, b-1, \dots, b-1, ab \pmod{b-1})$. Then the action of the Bender-Knuth involutions $t_1, \dots, t_{N-1}$, where $N = |\alpha|$, is transitive on $\bigsqcup_{w \in \mathfrak{S}_N} \mathrm{CST}(\lambda, w(\alpha))$

Remark 5.8. The Kostka number $K_{\lambda\alpha} = 1$ has a combinatorial description given by Berenstein-Zelevinski [BZ90] and by Naylor-Vinroot [JNV17]. The combinatorial Kostka number criteria is related to a decomposition of the partition λ into almost rectangular parts, which suggests that there might be a generalization of Case 3 for transitivity. For reference, we write the criteria as written in [JNV17, Theorem 3.1].

Theorem 5.9 ([JNV17]). *Let λ and μ be partitions of n, and suppose $\ell(\mu) = l$. Then $K_{\lambda\mu} = 1$ if and only if there exists a choice of indices $0 = i_0 < i_1 < \cdots < i_t = l$ such that, for $k = 1, \dots, t$, the partitions*

$$\lambda^k = (\lambda_{i_{k-1}+1}, \lambda_{i_{k-1}+2}, \dots, \lambda_{i_k}) \text{ and } \mu^k = (\mu_{i_{k-1}+1}, \mu_{i_{k-1}+2}, \dots, \mu_{i_k}),$$

where we define $\lambda_i = 0$ if $i > \ell(\lambda)$, satisfy the following:

(1) λ^k dominates μ^k, written $\lambda^k \trianglerighteq \mu^k$, which means that for each $j \geq 1$, $\sum_{i \leq j} \lambda_i^k \geq \sum_{i \leq j} \mu_i^k$.
(2) Either $\lambda_{i_{k-1}+1} = \lambda_{i_{k-1}+2} = \cdots = \lambda_{i_k - 1}$ or $\lambda_{i_{k-1}+1} > \lambda_{i_{k-1}+2} = \lambda_{i_{k-1}+3} = \cdots = \lambda_{i_k}$.

What this theorem says is that $K_{\lambda\mu} = 1$ if and only if we can decompose the shape λ into tableaux $(\lambda^1, \mu^1), \dots, (\lambda^t, \mu^t)$ such that each (λ^k, μ^k) has $\lambda^k \trianglerighteq \mu^k$ and the shape λ^k satisfies one of

(1) $\lambda_{i_{k-1}+1} = \lambda_{i_{k-1}+2} = \cdots = \lambda_{i_k}$,
(2) $\lambda_{i_{k-1}+1} > \lambda_{i_{k-1}+2} = \lambda_{i_{k-1}+3} = \cdots = \lambda_{i_k}$ (λ_{i_k} can be zero),

(3) $\lambda_{i_{k-1}+1} = \lambda_{i_{k-1}+2} = \cdots = \lambda_{i_k-1} > \lambda_{i_k}$ (λ_{i_k} can be zero).

Note that in each of the three cases, the shape λ^k of the tableau (λ^k, μ^k) is almost rectangular.

We briefly explain why this condition is equivalent. Let $|\lambda| = \sum_{k\geq 1} \lambda_i$. Since $\sum_{k=1}^t |\lambda^i| = |\lambda| = |\mu| = \sum_{k=1}^t |\mu^i|$, the condition $\lambda^k \trianglerighteq \mu^k$ implies that $|\lambda^k| = |\mu^k|$ for all $k = 1, \ldots, t$. Now we can translate condition (2) in Theorem 5.9 into the above three conditions.

Proof of Proposition 5.7. (1) If $K_{\lambda\alpha} = 1$, then a tableau in $\mathrm{CST}(\lambda, w(\alpha))$ corresponds uniquely to the permutation $w(\alpha)$. Since the action of $\mathfrak{S}_N$ on the content α by left multiplication is transitive, this implies that BK action on the disjoint union $\mathrm{CST}(\lambda, w(\alpha))$ for $w \in \mathfrak{S}_N$ is transitive.

(2) When the content $\alpha = (1, \ldots, 1)$, the tableau $x \in \mathrm{CST}(\lambda, w(\alpha))$ is a Standard Young Tableau. This means that x uniquely corresponds to a linear extension $x \in \mathrm{LinExt}(F_\lambda)$, where F_λ is the Ferrers poset of shape λ. Moreover, the Bender-Knuth action t_i on $\bigsqcup_{w\in\mathfrak{S}_N} \mathrm{CST}(\lambda, \alpha)$ corresponds to the action of t_i on $\mathrm{LinExt}(F_\lambda)$. By Proposition 3.2, the group H_{F_λ} is transitive. This implies that the BK action on $\bigsqcup_{w\in\mathfrak{S}_N} \mathrm{CST}(\lambda, w(\alpha))$ is transitive.

(3) We will prove by induction on a that 1) the BK action is transitive, and 2) we can get any CST with content α from one CST with content α without using t_{N-1}.

The cases where $a = 1$ and $a = 2$ are trivial since $K_{\lambda\alpha} = 1$. Also, when $a = 2$, since there is only 1 CST with content α, we do not need to use t_1 to get all CST with content α.

Suppose the statement is true for $a = k$, consider λ an $(k+1) \times b$ rectangle. Let α be the corresponding content and $N = |\alpha|$. Let $i = ab \mod (b-1)$, and let m be the largest number in column $b - i + 1$, i.e. the leftmost column that contains N, such that $m - 1$ is not in that column.

If $m < N - 1$, we will prove that we can increase m by 1 using t_{m-1}. Since $m - 1$ is not in column $b - i + 1$, and there are $b - 1$ squares in different columns containing m and $m-1$, there is exactly 1 column j that contains $m - 1$ and does not contain m, and every other column contains both $m-1$ and m. Furthermore, the m in column $b-i+1$ and the $m-1$ in column j cannot be in the same row. This is because otherwise the square $(k+1, b-i)$ must contain N (due to column strictness), which is impossible since $b - i + 1$ is the left most column that contains N. Thus, t_{m-1} will swap these two squares, and so we can increase m by 1. Therefore, we can always increase m to $N - 1$. This means that from every CST shape $(k+1) \times b$ with content α, we can get to a CST with $m = N - 1$, and note that we do not have to use t_{N-1} yet.

When $m = N - 1$, then there is $N - 1$ in every other column. Thus, row $k + 1$ only contains N and $N - 1$, and so the first k rows form a CST of shape $k \times b$ with content $\alpha_0 = (b-1, b-1, ..., kb \mod (b-1))$. By the induction hypothesis, we can get all CST of shape α_0 without using $t_{l(\alpha_0)-1}$. This fact is important since we do not interfere with row $k + 1$. And since from every CST shape $(k+1) \times b$ with content α, we can get to a CST with row $k + 1$ only contains N and $N - 1$, we can get from any CST of shape $(k+1) \times b$ with content α to another, so the BK action is transitive. Finally, note that at no point in these steps that we have to use t_{N-1}. This completes the inductive step.

□

Remark 5.10. Case 3 is separate from Case 1. This can be seen by taking λ, α to be a 3×4 rectangle and content $\alpha = (2, 2, 2, 2, 2, 2)$. Then $K_{\lambda\alpha} = 5$. (There are many other similar λ, α).

Theorem 5.11. *For any shape λ, if the content μ is of the form $\mu = (1, \hat{\mu})$, then the BK action is transitive.*

Proof. We induct on the length of the content μ. Our base case is μ has length 2, i.e. $\mu = (1, \mu_2)$ for some positive integer μ_2. In this case, the Kostka number $K_{\lambda\mu} = 1$, so the BK action is transitive by Proposition 5.7 Case 1.

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$. Now suppose $\mu = (1, \mu_2, \ldots, \mu_{i-1}, \mu_i)$ where $i > m$ and, without loss of generality, $\mu_2 \geq \ldots \geq \mu_i$. Define the *greedy tableau* of $\mathrm{CST}(\lambda, \mu)$ to be a tableau formed by putting the numbers $j = i, i-1, i-2, \ldots, 1$ in the tableau by the following process: fill the lowest part λ_r with as many j's as possible, then fill λ_{r-1} with as many j's as possible, and continue until there are no j's left. For example, the greedy tableau with shape $\lambda = (4, 2, 1)$ and content $\mu = (3, 2, 2)$ is shown in Figure 11.

1	1	1	2
2	3		
3			

FIGURE 11. Greedy tableau example

Consider a tableau $T \in \mathrm{CST}(\lambda, \mu)$. Let $S \in \mathrm{CST}(\lambda', \mu')$ be the tableau formed by only considering the entries in T containing the numbers $1, 2, \ldots, i-1$. Let S' be the greedy tableau of $\mathrm{CST}(\lambda', \mu')$. By inductive hypothesis, we can find a word w in the t_j for $j < i-1$ such that when we apply w to T, we obtain a tableau T' containing S'. We show that there is a word in $t_1, \ldots, t_{i-1}$ such that when we apply the word to T', we obtain the greedy tableau of $\mathrm{CST}(\lambda, \tilde{\mu} = (1, \mu_1, \ldots, \mu_i, \mu_{i-1})$. This will show that every tableau can be moved to the greedy tableau, which suffices to show the BK action is transitive.

Let r be the smallest r such that row r contains an $i-1$. We will prove that after a sequence of BK actions, we can get a CST with μ_{i-1} i's in the greedy position, i.e. we fill λ_m with as many i's as possible, then λ_{m-1} with as many i's as possible, and continue until we run out of i's. Starting with T', there are a few cases to consider.

(1) Let k be the leftmost column that contains an $i-1$ in row r. Then column $k-1$ has an $i-1$.
(2) We are not in case (1).
  (2a) For all i in row $(r+1)$, there is an $i-1$ above it.
  (2b) For all $i-1$ in row r, there is an i below it.
  (2c) Neither Case (2a) nor Case (2b) is true.

We prove each case separately.

(1) Applying t_{i-1} is enough for this case, as illustrated in figure 12. Precisely, any i above row r will be changed to $i-1$. Below row r, every i has an $i-1$ above it and so will not be changed to $i-1$. Any $i-1$ that does not have an i below will be changed to i, creating a continuous filling of i. Finally, row r may have a mix of i and $i-1$, but since this is the last row that contains i, this does not affect the greediness of i.

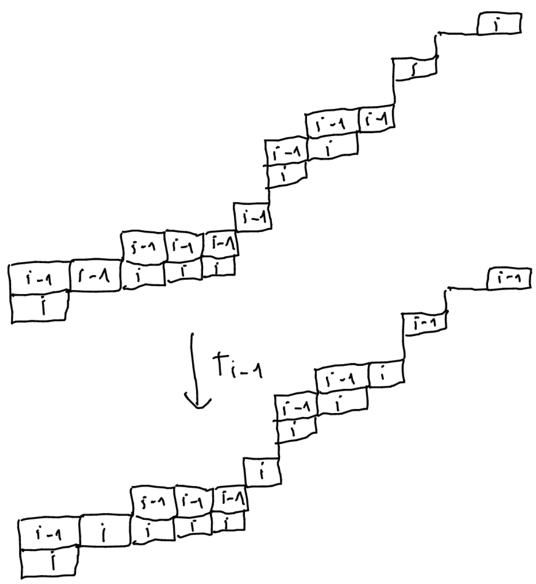


FIGURE 12. Case (1)

(2) From now on, we assume that column $k-1$ does not contain an $i-1$.
  (2a) This case is actually either case (1), or there is no i in row $r+1$. The argument is the same as case (1), and applying t_{i-1} is sufficient.
  (2b) This case is only slightly different from 2a, but applying t_{i-1} is once again enough. In this case, since for all $i-1$ in row r, there is an i below it, t_{i-1} will change all i in row r to $i-1$, as illustrated in figure 14, which makes row $r+1$ the last row containing i. Thus, a mix of i and $i-1$ on row $r+1$ does not affect the greediness.
  (2c) If neither Case 1 nor Case 2 is true, there exists a leftmost i in row $r+1$ that has no $i-1$ above, and a rightmost $i-1$ in row r that has no i. Let the leftmost i be in square $(r+1, a)$ and the rightmost $i-1$ be in square (r, b), where the first entry is the row number and the second entry is the column number in the shape λ. We certainly have $b > a$. We will show that there is a word in $t_1, \ldots, t_{i-1}$ such that the resulting word has a smaller distance $b-a$.
  We can suppose that the entries containing $i-1$ and i in row r and $r+1$ can be split into blocks of the following form shown in Figure 15. Now we perform the following moves shown in Figure 16.

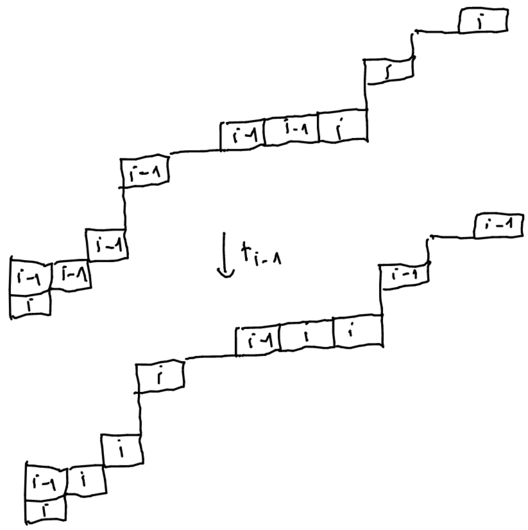


FIGURE 13. Case (2a)

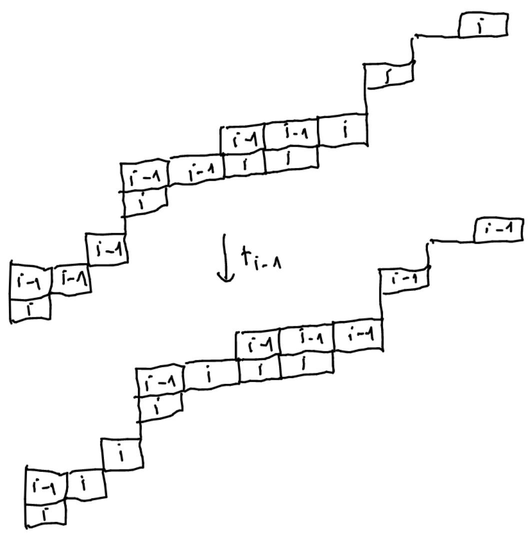


FIGURE 14. Case (2b)

The only move that needs extra explanation is move 2. We are using the fact that, let $C \in \mathrm{CST}(\lambda'', \mu'')$ be the CST formed by considering only the entries in $t_{i-1}(T')$, i.e. the CST after move 1, containing

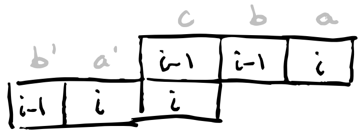


FIGURE 15. Case (2c)

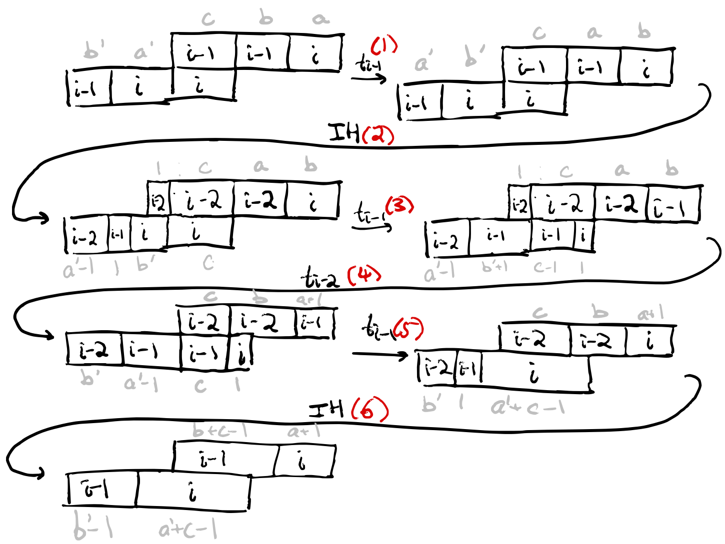


FIGURE 16. Steps for Case (2c)

$1, 2, \ldots, i-1$, there exists a CST with entries $1, 2, \ldots, i-1$, shape λ'' and content $(\mu_2, \ldots, \mu_{i-2}, \mu_i, 1)$ such that there is only one $i-1$ in row $r+1$, and any other entry in C that originally contains an $i-1$ now contains an $i-2$.

Lemma 5.12. *The above CST exists.*

Proof of lemma. We first do promotion on C, i.e. we act $t_1, \ldots t_{i-2}$. Since at first we have only one 1, in the promotion process, t_j will decrease all but one $j+1$ by 1. Connecting the squares that does not decrease during promotion, we get the "promotion path". Note that in each step, this path either goes down 1 row, or goes right some columns (can be zero column).

If this path ends in row $r+1$, then we get the desired CST, since the $i-1$ in this row stays $i-1$, and every other $i-1$ decreases.

Before continuing, we want to make a quick observation that if the promotion path does not end in row $r+1$, then C also has $l(\lambda'') < l(\mu'')$. This is because otherwise, the first column of C is filled with $1, \ldots, i-1$. Then in T', there is either an i on row i, or an i on row $i-1$ that has no $i-1$ above it. The former is impossible since we assume that $l(\lambda) < l(\mu)$, and the latter is impossible since this means

row $r+1$ is row $i-1$, and so the promotion path ends in this row. Thus, $l(\lambda'') < l(\mu'')$, which leads to an important corollary that is the promotion path has to goes right at some point.
Continuing with the proof, suppose the promotion path ends in some row below $r+1$, then we will decrease the $i-1$ at the end of the promotion path by 1. This is only problematic when the $i-1$ has an $i-2$ above it, which means that the promotion path goes up at the end. If this is true, we "fix" by decreasing the above $i-2$ by one, and repeat if until there is no more problem. This is where the above corollary comes in handy. Since the promotion path goes right at some point, we only need to fix up to some t, and this t is not 1.
Now we can add 1 to the rightmost $i-2$ on row $r+1$. Then we pick an $i-3$ that has no $i-2$ below and add 1 to it. The easiest way to pick this $i-3$ is to look at the row right above the above $i-2$. This $i-3$ exists since the content is decrease. We repeat this process until we add 1 to some t, and since $t \neq 1$, we can always do this.
Finally, observe that in the first step, we take away one $i-1$ and add in one t, and in the second step, we add in one $i-1$ and take away one t. Thus, the content stays the same, and we get the desired CST. Figure 17 gives a summary of the above construction.

□

□

Theorem 5.13. *When the tableau has shape* $(n, 1, \dots, 1)$*, the BK action is transitive.*

Proof. We will prove by induction on the length of the content μ. The base case when $l(\mu) = 1$ and $l(\mu) = 2$ is trivial since the Kostka number is 1.

Before going to the inductive case, we want to make a quick observation. Since the tableau has an L-shape, the tableau is uniquely determined by the set of number on the first column. This is because given the set of number of the first column, we have a unique way to fill these number in, and also a unique way to fill the remaining number into the first row.

Now we will prove the inductive case. Specifically, we will prove that given a fixed content $\mu = (\mu_1, \dots, \mu_n)$, we can get from any CST to the CST in which there are μ_n n's, one of which is in the first column. If this is not already true, we will perform the following two steps.

- *Step 1:* Let i be the number in square $(2, 1)$, i.e. the first square on row 2. If $i \neq 2$, we do $t_{i-1}, \dots, t_2$ so that 2 is in $(2, 1)$.
- *Step 2:* We do promotion n times, i.e. P^n.

First of all, step 1 brings 2 into $(2, 1)$, and does not change the number of n. In step 2, since 2 is in $(2, 1)$, during the first promotion, the empty square $(1, 1)$ will move up, and so there is an n in the first column after the first promotion. This n will slide down to $(2, 1)$ before the last promotion. Finally, in the last promotion, this square will slide down to $(1, 1)$, which means that an empty square has to move up, which in turn create an n in the first column after the last promotion.

Furthermore, after step 1, there are μ_n n's, and after n rounds of promotion, there will still be μ_n n's. Thus, we have successfully get to the desired CST. Fixing

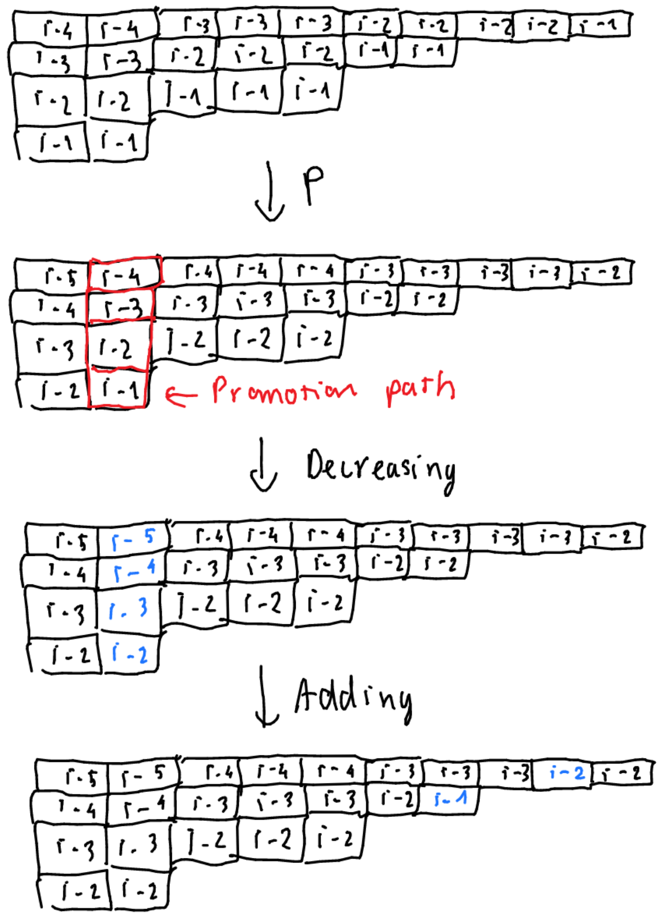


FIGURE 17. Lemma 5.12 summary

the squares containing n, the remaining CST is transitive under BK action by the inductive hypothesis. Therefore, BK action is transitive. □

5.2. **Data on the number of orbits.** Table 2 contains rectangular shapes on the left column, content on the top row, and number of orbits in the box. In Table 1 through Table 6, we put a bold **1** if it is not explained by Proposition 5.7, Theorem 5.11, or Theorem 5.13.

6. Order of promotion on staircase shapes

In this section, we investigate the order of promotion acting on *staircase shapes* with various contents. Haiman [Hai92] proved that the order of promotion on a Young tableau of staircase shape has order $2N$, where $N = \binom{n}{2}$ is the number of blocks in the staircase. In Example 6.2, we provide a column-strict tableau with

	[2, . . .]	[3, . . .]	[4, . . .]	[5, . . .]	[6, . . .]	[7, . . .]	[8, . . .]
[2, 2]	1						
[3, 3]	1	1					
[4, 4]	2	1	1				
[5, 5]	2	**1**	1	1			
[6, 6]	3	2	**1**	1	1		
[7, 7]	3	2	2	**1**	1	1	
[8, 8]	4	**1**	3	**1**	**1**	1	1
[9, 9]	4	3	3	2	**1**	**1**	1
[10, 10]		2	4	3	2	**1**	**1**
[11, 11]			4	3	3	**1**	**1**
[12, 12]			6	3	4	2	**1**
[13, 13]			5	**1**	4	3	2
[14, 14]				2	5	4	3
[15, 15]				5	6	4	4
[16, 16]						4	5
[17, 17]						4	5
[18, 18]							6
[19, 19]							6

Table 2. Number of orbits given shape (left column) and content (top row)

	[2, 2, 2]
[4, 2]	2

Table 3. Number of orbits given shape (left column) and content (top row) for $n = 6$

	[3, 2, 2]
[5, 2]	2
[4, 3]	**1**

Table 4. Number of orbits given shape (left column) and content (top row) for $n = 7$

	[4, 2, 2]	[3, 3, 2]	[2, 2, 2, 2]
[6, 2]	2	2	2
[5, 3]	**1**	2	2
[5, 2, 1]			2
[4, 4]			2
[4, 3, 1]			2
[4, 2, 2]			2
[3, 3, 2]			**1**

Table 5. Number of orbits given shape (left column) and content (top row) for $n = 8$

	[5, 2, 2]	[4, 3, 2]	[3, 3, 3]	[3, 2, 2, 2]
[7, 2]	2	2	2	2
[6, 3]	**1**	2	2	2
[6, 2, 1]				2
[5, 4]		**1**	**1**	**1**
[5, 3, 1]				**1**
[5, 2, 2]				2
[4, 4, 1]				**1**
[4, 3, 2]				**1**

TABLE 6. Number of orbits given shape (left column) and content (top row) for $n = 9$

	[6, 2, 2]	[5, 3, 2]	[4, 4, 2]	[4, 3, 3]	[4, 2, 2, 2]	[3, 3, 2, 2]	[2, 2, 2, 2, 2]
[8, 2]	2	2	2	2	2	2	2
[7, 3]	**1**	2	2	2	2	2	2
[7, 2, 1]					2	2	2
[6, 4]		**1**	2	2	2	2	3
[6, 3, 1]					**1**	**1**	2
[6, 2, 2]					2	2	3
[6, 2, 1, 1]							2
[5, 5]					**1**	2	2
[5, 4, 1]					**1**	**1**	**1**
[5, 3, 2]					**1**	**1**	**1**
[5, 3, 1, 1]							2
[5, 2, 2, 1]							2
[4, 4, 2]					2	**1**	3
[4, 4, 1, 1]							**1**
[4, 3, 3]						**1**	**1**
[4, 3, 2, 1]							2
[4, 2, 2, 2]							2
[3, 3, 3, 1]							**1**
[3, 3, 2, 2]							**1**

TABLE 7. Number of orbits given shape (left column) and content (top row) for $n = 10$

staircase shape and entries in $[N]$ which does not have order $2N$. In Section 6.1, we provide data for the order of promotion for staircases of length at most five.

Definition 6.1. A *staircase shape* λ is a shape that has the form $\lambda = (n, n-1, \ldots, 1)$ for some positive integer n.

Example 6.2. For the staircase shape with $n = 4$, the CST

1	2	3
2	4	
4		

has order $12 = 3N$.

6.1. **Order of promotion data.** Tables 8, 9, 10 show the order of promotion for each content μ. When the order of promotion k is a multiple of $\ell(\mu)$, we show the quotient $k/\ell(\mu)$. When the order of promotion k is not a multiple of $\ell(\mu)$, we show the greatest common divisor $\gcd(k, \ell(\mu))$.

From the tables, it can be seen that there are few cases when the order of promotion is not a multiple of $\ell(\mu)$. The two main cases are $[1,1,1]$ and $[1,2,1,2]$, and the other two are basically the same. Bigger data seems to suggest that there are no other instance except $[4,1,4,1,4,1]$ and $[5,2,5,2,5,2]$. Thus, we make the following conjecture.

Conjecture 6.3. *The order of promotion on a staircase shape diagram and content μ is a multiple of $\ell(\mu)$, except for μ in this list:*

$$[1,1,1], [2,2,2], [1,2,1,2], [2,3,2,3], [4,1,4,1,4,1], [5,2,5,2,5,2].$$

Content (μ)	Order (k)	$\ell(\mu)$	$k/\ell(\mu)$	$\gcd(k, \ell(\mu))$
[1, 1, 1]	2	3		1
[1, 2]	2	2	1	

TABLE 8. Order of promotion on staircase, $n = 3$

Content (μ)	Order (k)	$\ell(\mu)$	$k/\ell(\mu)$	$\gcd(k, \ell(\mu))$
[1, 1, 1, 1, 1, 1]	12	6	2	
[1, 1, 1, 1, 2]	10	5	2	
[1, 1, 1, 3]	8	4	2	
[1, 1, 2, 2]	8	4	2	
[1, 2, 1, 2]	6	4		2
[1, 2, 3]	3	3	1	
[1, 3, 2]	3	3	1	
[2, 2, 2]	2	3		1

TABLE 9. Order of promotion on staircase, $n = 4$

Content (μ)	Order (k)	$\ell(\mu)$	$k/\ell(\mu)$	$\gcd(k, \ell(\mu))$
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]	20	10	2	
[1, 1, 1, 1, 1, 1, 1, 1, 2]	18	9	2	
[1, 1, 1, 1, 1, 1, 1, 3]	16	8	2	
[1, 1, 1, 1, 1, 1, 2, 2]	16	8	2	
[1, 1, 1, 1, 1, 1, 4]	14	7	2	
[1, 1, 1, 1, 1, 2, 1, 2]	48	8	6	
[1, 1, 1, 1, 1, 2, 3]	14	7	2	
[1, 1, 1, 1, 1, 3, 2]	14	7	2	
[1, 1, 1, 1, 2, 1, 1, 2]	48	8	6	
[1, 1, 1, 1, 2, 1, 3]	42	7	6	
[1, 1, 1, 1, 2, 2, 2]	42	7	6	
[1, 1, 1, 1, 2, 4]	12	6	2	
[1, 1, 1, 1, 3, 1, 2]	42	7	6	
[1, 1, 1, 1, 3, 3]	12	6	2	
[1, 1, 1, 1, 4, 2]	12	6	2	

$[1,1,1,2,1,1,1,2]$	112	8	14	
$[1,1,1,2,1,1,3]$	70	7	10	
$[1,1,1,2,1,2,2]$	294	7	42	
$[1,1,1,2,1,4]$	12	6	2	
$[1,1,1,2,2,1,2]$	294	7	42	
$[1,1,1,2,2,3]$	60	6	10	
$[1,1,1,2,3,2]$	12	6	2	
$[1,1,1,3,1,1,2]$	70	7	10	
$[1,1,1,3,1,3]$	36	6	6	
$[1,1,1,3,2,2]$	60	6	10	
$[1,1,1,3,4]$	10	5	2	
$[1,1,1,4,1,2]$	12	6	2	
$[1,1,1,4,3]$	10	5	2	
$[1,1,2,1,1,2,2]$	210	7	30	
$[1,1,2,1,1,4]$	12	6	2	
$[1,1,2,1,2,1,2]$	210	7	30	
$[1,1,2,1,2,3]$	60	6	10	
$[1,1,2,1,3,2]$	60	6	10	
$[1,1,2,2,1,3]$	84	6	14	
$[1,1,2,2,2,2]$	60	6	10	
$[1,1,2,2,4]$	10	5	2	
$[1,1,2,3,1,2]$	60	6	10	
$[1,1,2,3,3]$	10	5	2	
$[1,1,2,4,2]$	10	5	2	
$[1,1,3,1,1,3]$	36	6	6	
$[1,1,3,1,2,2]$	84	6	14	
$[1,1,3,1,4]$	10	5	2	
$[1,1,3,2,1,2]$	60	6	10	
$[1,1,3,2,3]$	30	5	6	
$[1,1,3,3,2]$	10	5	2	
$[1,1,4,1,3]$	10	5	2	
$[1,1,4,2,2]$	10	5	2	
$[1,2,1,2,1,3]$	108	6	18	
$[1,2,1,2,2,2]$	924	6	154	
$[1,2,1,2,4]$	15	5	3	
$[1,2,1,3,3]$	30	5	6	
$[1,2,1,4,2]$	15	5	3	
$[1,2,2,1,2,2]$	84	6	14	
$[1,2,2,1,4]$	10	5	2	
$[1,2,2,2,3]$	40	5	8	
$[1,2,2,3,2]$	50	5	10	
$[1,2,3,1,3]$	30	5	6	
$[1,2,3,2,2]$	50	5	10	
$[1,2,3,4]$	4	4	1	
$[1,2,4,3]$	4	4	1	
$[1,3,1,3,2]$	30	5	6	

$[1,3,2,2,2]$	40	5	8	
$[1,3,2,4]$	4	4	1	
$[1,3,3,3]$	8	4	2	
$[1,3,4,2]$	4	4	1	
$[1,4,2,3]$	4	4	1	
$[1,4,3,2]$	4	4	1	
$[2,2,2,2,2]$	40	5	8	
$[2,2,2,4]$	8	4	2	
$[2,2,3,3]$	8	4	2	
$[2,3,2,3]$	6	4		2

Table 10. Order of promotion on staircase, $n = 5$

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