Cactus group actions and tableaux

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Introduction and motivation

Motivation

Fix a partition λ, the associated *Schur polynomial* is

$$s_\lambda(\mathbf{x}) = \sum_T \mathbf{x}^T = \sum_T x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

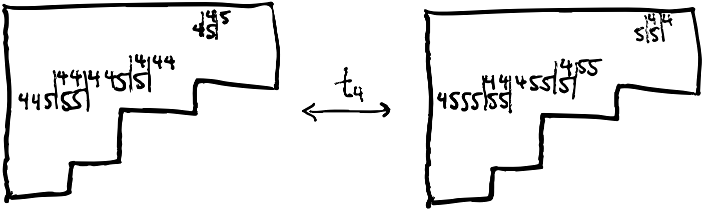
where the summation is over all *column-strict (semi-standard) tableaux* T of shape λ. The exponents $\alpha_1, \alpha_2, \ldots, \alpha_n$ count the number of occurrences of $1, 2, \ldots, n$ in T.

- $\alpha = (\alpha_1, \ldots, \alpha_n)$ is called the *content* of T.

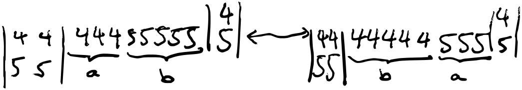
Remark ([BK72])

The *Bender–Knuth involutions* $t_1, \ldots, t_{n-1}$ can be used to show that s_λ is symmetric.

Bender–Knuth involutions



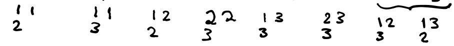
In each row, t_4 swaps the contents of 4 and 5 in the string

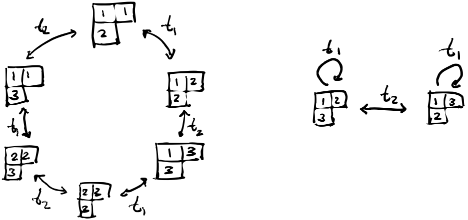


between the columns.

Back to motivation

$$s_{\square\!\square}(x_1,x_2,x_3) = x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_2^2x_3 + x_1x_3^2 + x_2x_3^2 + 2x_1x_2x_3$$

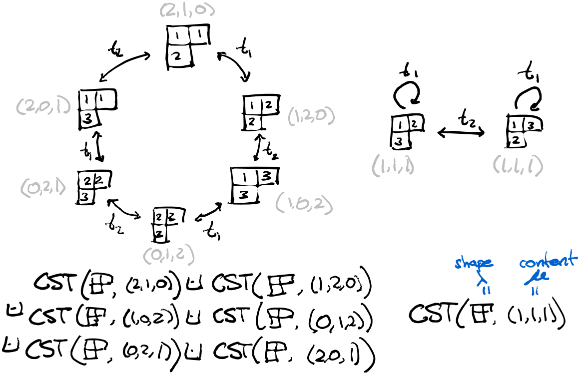




This symmetry in the variables motivated Bender and Knuth to introduce the involutions t_i.

Back to motivation

The right orbit contains the *standard Young tableaux* $\mathrm{SYT}(\lambda)$, the CSTs with shape λ and content $(1, 1, \ldots, 1)$.



Question

What are the Bender–Knuth orbits on $\bigsqcup_{w \in \mathfrak{S}_n} \mathrm{CST}(\lambda, w(\mu))$?

Bender–Knuth orbits on column-strict tableaux

Bender–Knuth orbits on CSTs

Number of orbits of BK action for each shape λ and content μ of size 6:

$\lambda \backslash \mu$	(6)	$(5,1)$	$(4,2)$	$(4,1,1)$	$(3,3)$	$(3,2,1)$	$(3,1,1,1)$	$(2,2,2)$	$(2,2,1,1)$	$(2,1,1,1,1)$	$(1,1,1,1,1,1)$
(6)	1	1	1	1	1	1	1	1	1	1	1
$(5,1)$		1	1	1	1	1	1	1	1	1	1
$(4,2)$			1	1	1	1	1	2	1	1	1
$(4,1,1)$				1		1	1	1	1	1	1
$(3,3)$					1	1	1	1	1	1	1
$(3,2,1)$						1	1	1	1	1	1
$(3,1,1,1)$							1		1	1	1
$(2,2,2)$								1	1	1	1
$(2,2,1,1)$									1	1	1
$(2,1,1,1,1)$										1	1
$(1,1,1,1,1,1)$											1

Reduction to smaller tableaux

Lemma (CKLNPX)

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ *and* $\mu = (\mu_1, \ldots, \mu_m)$.
If $\ell(\lambda) = \ell(\mu)$ *then the BK action on*

$$\bigsqcup_{w \in \mathfrak{S}_m} CST(\lambda, w(\mu))$$

is isomorphic to the BK action on

$$\bigsqcup_{w \in \mathfrak{S}_m} CST(\lambda', w(\mu'))$$

where $\lambda' = (\lambda_1 - 1, \ldots, \lambda_n - 1)$ *and* $\mu' = (\mu_1 - 1, \ldots, \mu_m - 1)$.

Thus, we will only consider $\lambda = \emptyset$ or $\ell(\lambda) < \ell(\mu)$.

Reduction to smaller tableaux

We will only consider $\lambda = \emptyset$ or $\ell(\lambda) < \ell(\mu)$.

$\lambda \backslash \mu$	(6)	(5,1)	(4,2)	(4,1,1)	(3,3)	(3,2,1)	(3,1,1,1)	(2,2,2)	(2,2,1,1)	(2,1,1,1,1)	(1,1,1,1,1,1)
(6)	1	1	1	1	1	1	1	1	1	1	1
(5,1)		1	1	1	1	1	1	1	1	1	1
(4,2)			1	1	1	1	1	2	1	1	1
(4,1,1)				1		1	1	1	1	1	1
(3,3)					1	1	1	1	1	1	1
(3,2,1)						1	1	1	1	1	1
(3,1,1,1)							1		1	1	1
(2,2,2)								1	1	1	1
(2,2,1,1)									1	1	1
(2,1,1,1,1)										1	1
(1,1,1,1,1,1)											1

Transitivity criterion 1

Lemma (CKLNPX)

The set of orbits on $\bigsqcup_{w \in \mathfrak{S}_m} CST(\lambda, w(\mu))$ *is in bijection with the set of orbits on* $CST(\lambda, \mu)$.

Corollary (CKLNPX)

If the Kostka number $K_{\lambda\mu} = 1$*, then the BK action is transitive.*

Transitivity criterion 1

Corollary

If the Kostka number $K_{\lambda\mu} = 1$, then the BK action is transitive.

$\lambda \backslash \mu$	(6)	$(5,1)$	$(4,2)$	$(4,1,1)$	$(3,3)$	$(3,2,1)$	$(3,1,1,1)$	$(2,2,2)$	$(2,2,1,1)$	$(2,1,1,1,1)$	$(1,1,1,1,1,1)$
(6)	1	1	1	1	1	1	1	1	1	1	1
$(5,1)$		1	1	1	1	1	1	1	1	1	1
$(4,2)$			1	1	1	1	1	2	1	1	1
$(4,1,1)$				1		1	1	1	1	1	1
$(3,3)$					1	1	1	1	1	1	1
$(3,2,1)$						1	1	1	1	1	1
$(3,1,1,1)$							1		1	1	1
$(2,2,2)$								1	1	1	1
$(2,2,1,1)$									1	1	1
$(2,1,1,1,1)$										1	1
$(1,1,1,1,1,1)$											1

Transitivity criterion 2

Lemma (CKLNPX)

If $\mu = (\mu', 1)$ and $\ell(\lambda) < \ell(\mu)$, then the BK action is transitive.

λ \ μ	(6)	$(5,1)$	$(4,2)$	$(4,1,1)$	$(3,3)$	$(3,2,1)$	$(3,1,1,1)$	$(2,2,2)$	$(2,2,1,1)$	$(2,1,1,1,1)$	$(1,1,1,1,1,1)$
(6)	1	1	1	1	1	1	1	1	1	1	1
$(5,1)$		1	1	1	1	1	1	1	1	1	1
$(4,2)$			1	1	1	1	1	2	1	1	1
$(4,1,1)$				1		1	1	1	1	1	1
$(3,3)$					1	1	1	1	1	1	1
$(3,2,1)$						1	1	1	1	1	1
$(3,1,1,1)$							1		1	1	1
$(2,2,2)$								1	1	1	1
$(2,2,1,1)$									1	1	1
$(2,1,1,1,1)$										1	1
$(1,1,1,1,1,1)$											1

Transitivity criterion 3

Lemma (CKLNPX)

If $\lambda = (k, 1, 1, \ldots, 1)$, then the BK action is transitive.

λ \ μ	(6)	$(5,1)$	$(4,2)$	$(4,1,1)$	$(3,3)$	$(3,2,1)$	$(3,1,1,1)$	$(2,2,2)$	$(2,2,1,1)$	$(2,1,1,1,1)$	$(1,1,1,1,1,1)$
(6)	1	1	1	1	1	1	1	1	1	1	1
$(5,1)$		1	1	1	1	1	1	**1**	1	1	1
$(4,2)$			1	1	1	1	1	2	1	1	1
$(4,1,1)$				1		1	1	1	1	1	1
$(3,3)$					1	1	1	1	1	1	1
$(3,2,1)$						1	1	1	1	1	1
$(3,1,1,1)$							1		1	1	1
$(2,2,2)$								1	1	1	1
$(2,2,1,1)$									1	1	1
$(2,1,1,1,1)$										1	1
$(1,1,1,1,1,1)$											1

Transitivity criterion 3

Lemma (CKLNPX)

If $\lambda = (k, 1, 1, \ldots, 1)$ then the BK action is transitive.

λ \ μ	(6)	$(5,1)$	$(4,2)$	$(4,1,1)$	$(3,3)$	$(3,2,1)$	$(3,1,1,1)$	$(2,2,2)$	$(2,2,1,1)$	$(2,1,1,1,1)$	$(1,1,1,1,1,1)$
(6)	1	1	1	1	1	1	1	1	1	1	1
$(5,1)$		1	1	1	1	1	1	1	1	1	1
$(4,2)$			1	1	1	1	1	2	1	1	1
$(4,1,1)$				1		1	1	1	1	1	1
$(3,3)$					1	1	1	1	1	1	1
$(3,2,1)$						1	1	1	1	1	1
$(3,1,1,1)$							1		1	1	1
$(2,2,2)$								1	1	1	1
$(2,2,1,1)$									1	1	1
$(2,1,1,1,1)$										1	1
$(1,1,1,1,1,1)$											1

Unexplained transitive cases

Size 7:

$\lambda \backslash \mu$	[3, 2, 2]
[5, 2]	2
[4, 3]	1

Size 8:

$\lambda \backslash \mu$	[4, 2, 2]	[3, 3, 2]	[2, 2, 2, 2]
[6, 2]	2	2	2
[5, 3]	1	2	2
[5, 2, 1]			2
[4, 4]			2
[4, 3, 1]			2
[4, 2, 2]			2
[3, 3, 2]			1

Unexplained transitive cases

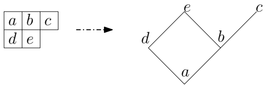
Size 9:

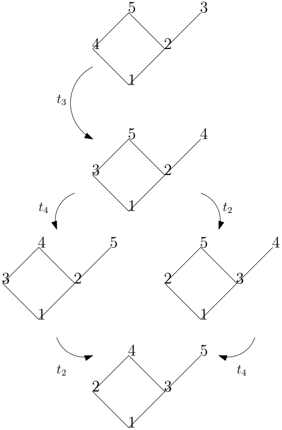
λ \ μ	[5, 2, 2]	[4, 3, 2]	[3, 3, 3]	[3, 2, 2, 2]
[7, 2]	2	2	2	2
[6, 3]	1	2	2	2
[6, 2, 1]				2
[5, 4]		1	1	1
[5, 3, 1]				1
[5, 2, 2]				2
[4, 4, 1]				1
[4, 3, 2]				1

Bender–Knuth involutions on linear extensions of a poset

From standard Young tableaux to LinExt(P)

Observation: We can view an SYT as a linear extension of a Ferrers poset.





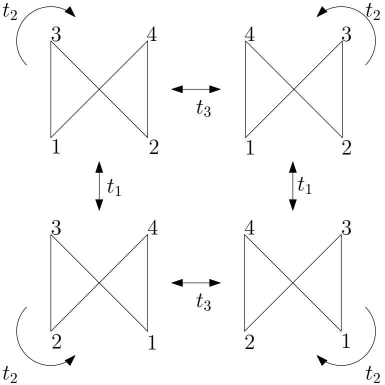
We may extend this to define a Bender–Knuth action on LinExt(P).

From standard Young tableaux to LinExt(P)

Proposition

The BK action on LinExt(P) *is transitive.*

Example:



W_n action on LinExt(P)

The BK action gives an action of the group

$$W_n = \langle t_1, \ldots, t_{n-1} \mid t_i^2 = 1, t_i t_j = t_j t_i \ \ \forall |i-j| \geq 2 \rangle$$

on LinExt(P), which induces a group homomorphism

$$\phi_P : W_n \to \mathfrak{S}_{\mathrm{LinExt}(P)}.$$

Questions

1. What is $\ker \phi_P$?
2. What is $\mathrm{im}\phi_P$?

Braid relations on the t_i

Proposition ([Sta09])

For all posets P, $(t_i t_{i+1})^6 \in \ker \phi_P$ *for all* $1 \leq i \leq n-2$.

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Recall the braid relation: $(\sigma_i \sigma_{i+1})^3 = 1$.

Braid relations on the t_i

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Question: When do we have $(t_i t_{i+1})^3 \in \ker \phi_P$?

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Proposition (CKLNPX)

We have $(t_i t_{i+1})^3 \in \ker \phi_P$ for all $1 \leq i \leq n-2$ if and only if P is a disjoint union of chains.

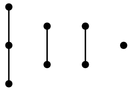

Figure: Disjoint union of chains

When is $t_i \in \ker \phi_P$?

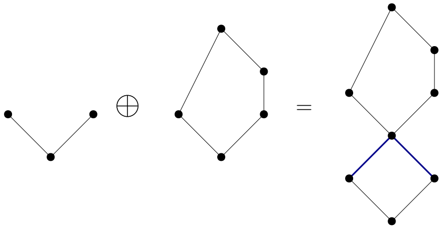

Figure: Ordinal sum of posets

When is $t_i \in \ker \phi_P$?

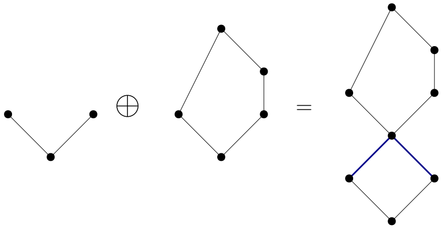

Figure: Ordinal sum of posets

Theorem (CKLNPX)

We have $t_i \in \ker \phi_P$ if and only if $P = P_1 \oplus P_2$, where $|P_1| = i$.

More relations in $\ker \phi_P$

For all posets P, so far we have

1. $t_i^2 \in \ker \phi_P$ for all $1 \leq i \leq n-1$,
2. $(t_i t_j)^2 \in \ker \phi_P$ for all $|i-j| \geq 2$,
3. $(t_i t_{i+1})^6 \in \ker \phi_P$ for all $1 \leq i \leq n-2$.

More relations in $\ker \phi_P$

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Question: Are there any other relations on the t_i that hold for all posets?

More relations in $\ker \phi_P$

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Question: Are there any other relations on the t_i that hold for all posets?

Yes, for example:

$$(t_i t_{i+1} t_{i+2})^{24} \in \ker \phi_P \text{ for all } 1 \leq i \leq n-3,$$
$$(t_i t_{i+1} t_{i+2} t_{i+1})^{30} \in \ker \phi_P \text{ for all } 1 \leq i \leq n-3.$$

The cactus relations

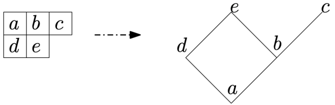
Let P be a poset of size n and let

$$q_\ell := (t_1)(t_2 t_1)\cdots(t_\ell t_{\ell-1}\cdots t_1),$$
$$q_{jk} := q_{k-1} q_{k-j} q_{k-1}.$$

Theorem ([CGP20])

The cactus relations $(t_i q_{jk})^2 = 1$, where $2 \leq i+1 < j < k \leq n$, hold on all column-strict tableaux.

In particular, they hold on standard Young tableaux, and thus on $\mathrm{LinExt}(P)$ for all Ferrers posets P.



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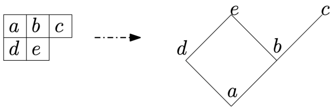
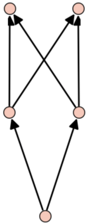
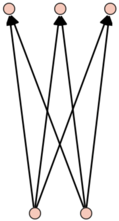


Question: For which posets P do the cactus relations hold on $\mathrm{LinExt}(P)$?

Cactus constructions (CKLNPX)

- If P is cactus, then $1 \oplus P$ and $(1+1) \oplus P$ are cactus.





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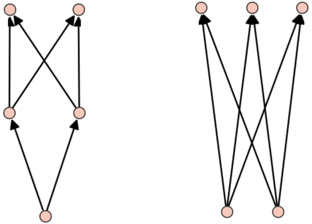
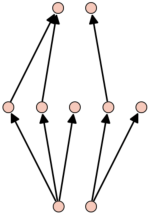
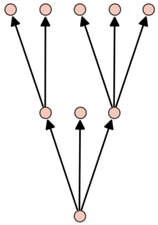


- If P and Q are cactus, then their disjoint union $P + Q$ is cactus.



Cactus constructions (CKLNPX)

- Trees are cactus.



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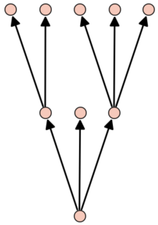
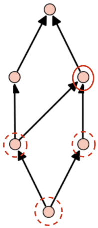
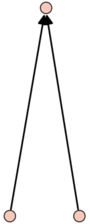
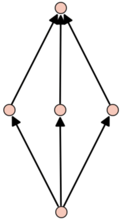
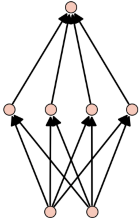
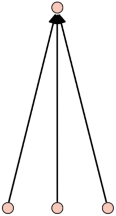
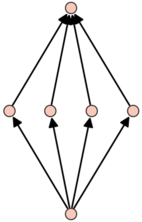
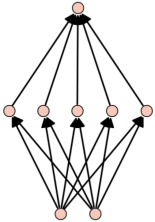


- If P is cactus and I is an order ideal of P, then I is cactus.



Cautionary examples

	$P \oplus 1$	$1 \oplus P \oplus 1$	$(1+1) \oplus P \oplus 1$
Cactus			
Not Cactus			

Optimization for identifying cactus posets

- For the relations $(t_i q_{jk})^2$ with $2 \leq i + 1 < k < j \leq n$, in principle there are $\binom{n-1}{3}$ different (i, j, k)-triples to check.

Optimization for identifying cactus posets

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Proposition (CKLNPX)

- *When* $n = 5$, P *is cactus iff the triple* $(1,3,5)$ *is satisfied;*
- *When* $n = 6$, P *is cactus iff the triples* $(1,4,6)$ *and* $(2,4,6)$ *are satisfied;*
- *When* $n = 7$, P *is cactus iff the triples* $(1,4,7)$ *and* $(2,4,7)$ *are satisfied;*
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Conjecture (CKLNPX)

For each n, there is a "small" set T_n of (i,j,k)-triples such that a poset P of size n is cactus iff P satisfies the cactus relations for all triples in T_n.

Image of ϕ_P

Definition

$H_P := \text{im}\left(\phi_P : W_n \to \mathfrak{S}_{\text{LinExt}(P)}\right).$

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Question

Which posets P are LE-symmetric?

Classifying disconnected LE-symmetric posets

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In general, $P + Q$ is not LE-symmetric.

Proposition (CKLNPX)

The disjoint union $P + Q$ is LE-symmetric if and only if $P = C_n$ is a chain and $Q = \{e\}$ is a singleton.

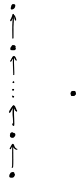

Figure: $P = C_n + 1$

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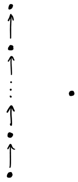

Figure: $P = C_n + 1$

Upshot: We can now look to classify connected LE-symmetric posets.

Group image under ordinal sum

Question: What does $H_{P \oplus Q}$ look like for arbitrary posets P, Q?

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If P and Q are not chains, then $P \oplus Q$ is not LE-symmetric.

Non-series-parallel posets

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The disjoint union $P + Q$ is LE-symmetric if and only if $P = C_n$ is a chain and $Q = \{e\}$ is a singleton.

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These motivate the following definition.

Definition

A *series parallel poset* is a poset that can be constructed from singletons using disjoint union and ordinal sum operations.

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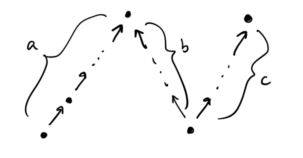
A *series parallel poset* is a poset that can be constructed from singletons using disjoint union and ordinal sum operations.

It is known that non-series parallel posets are characterized as posets that contain an N-poset.

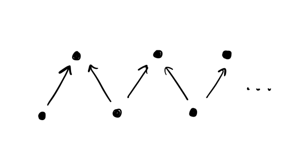
Conjectured LE-symmetric families

Conjecture (CKLNPX)

The following families are LE-symmetric:



N-poset $(a \geq 1, b \geq 2, c \geq 1)$



Zigzag-poset P *with even* $|P|$

- N-shaped posets checked for $a, c \leq 4$ & $b \leq 5$,
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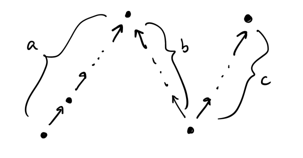

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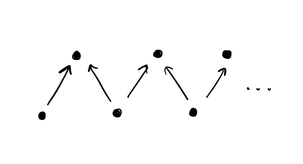

Zigzag-poset P with even $|P|$

- N-shaped posets checked for $a, c \leq 4$ & $b \leq 5$,
- Zigzag-posets P checked for $n \leq 10$,
- Progress on proving N-shaped posets with $a = c = 1$, looking at linear extension graphs of posets.

Conjectured LE-symmetric families

Conjecture (CKLNPX)

The following families of Ferrers posets are LE-symmetric:

1. $[n, n-2]$ *for all* n
2. $[n, 3]$ *for* $n \not\equiv 2 \pmod 4$
3. $[n, 2, 2]$ *for* $n \not\equiv 0 \pmod 4$

Checked computationally for:

1. $n \leq 10$
2. $n \leq 18$
3. $n \leq 16$

The size of the point stabilizer

Definition

For a poset P, define

$$k(P) = |Stab(\ell)| \quad \text{where } \ell \in \text{LinExt}(P).$$

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Intuition: $k(P)$ quantifies how big H_P is.

Posets with $k(P) = 1$

Proposition (CKLNPX)

We have $k(P) = 1$ if and only if P is an ordinal sum of antichains.

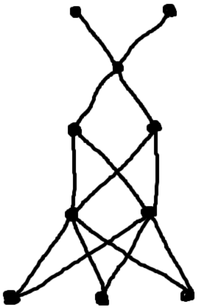

Figure: Ordinal sum of antichains

Possible values of $k(P)$

Proposition (CKLNPX)

There is a poset P with $k(P) = n_1!n_2!\cdots n_r!$, where $n_i \in \mathbb{N}$ for $1 \leq i \leq r$.

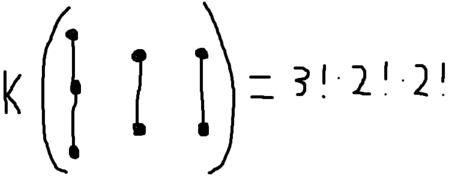

Figure: Poset with $k(P) = 3! \cdot 2! \cdot 2!$

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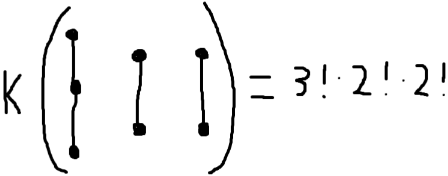

Figure: Poset with $k(P) = 3! \cdot 2! \cdot 2!$

Theorem (CKLNPX)

If $k(P) \notin \{6, 12, 36\} \cup \{2^m \mid m \in \mathbb{N}_0\}$, then $24 \mid k(P)$.

The comparability of P

Definition

For $\ell \in \mathrm{LinExt}(P)$, define

$$c(P, \ell) := |\{i \in [1, n-1] \mid \ell^{-1}(i) < \ell^{-1}(i+1)\}|.$$

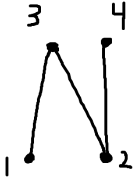

Figure: $c(P, \ell) = 1$

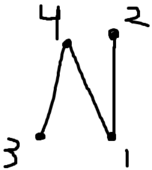

Figure: $c(P, \ell) = 2$

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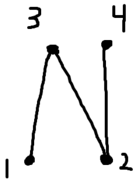

Figure: $c(P,\ell) = 1$

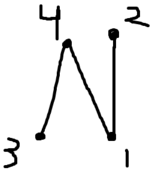

Figure: $c(P,\ell) = 2$

Definition

$c(P) := \max_\ell c(P,\ell)$.

A lower bound for $k(P)$

Theorem (CKLNPX)

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Corollary (CKLNPX)

If P is not a non-trivial ordinal sum, $k(P) \geq 2^{h(P)-1}$, where $h(P)$ is the height of P.

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