Counting Whittaker functions using modular linear algebra

Ilani Axelrod-Freed and Veronica Lang
Mentor: Claire Frechette, TA: Carolyn Stephen

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Outline

1. How to count Whittaker functions
2. Patterns in matrices
3. Coroot solutions
4. Coroot to cocharacter
5. Future work

What are Whittaker functions?

- *Special* functions that arise in p-adic number theory and representation theory
- Can be written explicitly using combinatorial data
- Incorporate data about the representations of different groups
- We're interested in *metaplectic* Whittaker functions (on metaplectic covering groups)

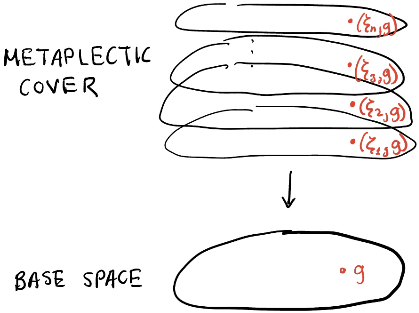
Metaplectic covering groups

Definition

Given a group G and a natural number $n \in \mathbb{N}$, an *n-fold metaplectic covering group* $\widetilde{G}$ is a central extension of G by the n-th roots of unity μ_n. That is, $\widetilde{G}$ is defined by the following short exact sequence:

$$1 \to \mu_n \to \widetilde{G} \to G \to 1.$$

As a set, $\widetilde{G}$ is the set of tuples (ζ, g) where $\zeta \in \mu_n$ and $g \in G$.



Characterizing metaplectic covers

We are looking at metaplectic covers of $G = GL_r(F)$, the general linear group of $r \times r$ matrices over a local field F containing μ_{2n}.

Theorem (Brylinski-Deligne, Frechette)

Every metaplectic cover of $GL_r(F)$ corresponds to a bilinear form $B_{c,d}$ for some $c, d \in \mathbb{Z}$ that acts on $(\mathbf{x}, \mathbf{y}) \in \mathbb{Z}^r \times \mathbb{Z}^r$ by

$$B_{c,d}(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \cdot \begin{pmatrix} c & d & d & \dots & d \\ d & c & d & \dots & d \\ d & d & c & \dots & d \\ \vdots & \vdots & & \ddots & \vdots \\ d & d & d & \dots & c \end{pmatrix} \cdot \mathbf{y}.$$

What is the dimension of the space of Whittaker functions?

The space of Whittaker functions on $GL_r(F)$ is one-dimensional. While Whittaker functions on metaplectic covers behave similarly, there are more of them.

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Theorem (McNamara, Frechette)

Let $\mathfrak{W}$ be the space of Whittaker functions on the metaplectic cover. Then

$$\dim \mathfrak{W} = \frac{n^r}{|\Lambda_{fin}|}, \text{ where}$$

$$\Lambda_{fin} = \{\boldsymbol{x} \in (\mathbb{Z}/n\mathbb{Z})^r : B_{c,d}(\boldsymbol{x}, \boldsymbol{y}) \equiv 0 \pmod{n} \text{ for all } \boldsymbol{y} \in (\mathbb{Z}/n\mathbb{Z})^r\}.$$

Our project goal is to count Λ_{fin}.

Understanding Λ_{fin}

By definition,

$$\Lambda_{fin} = \{\boldsymbol{x} \in (\mathbb{Z}/n\mathbb{Z})^r : B_{c,d}(\boldsymbol{x}, \boldsymbol{y}) \equiv 0 \pmod{n} \text{ for all } \boldsymbol{y} \in (\mathbb{Z}/n\mathbb{Z})^r\}.$$

From the definition of $B_{c,d}(\boldsymbol{x}, \boldsymbol{y})$ we obtain:

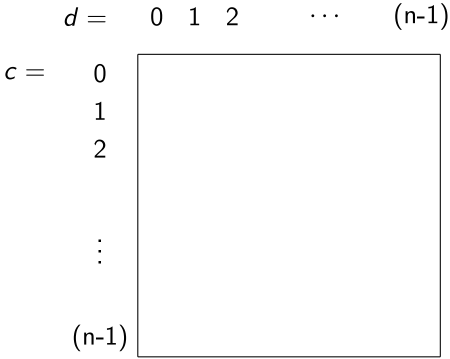
Lemma

$|\Lambda_{fin}|$ *is the number of solutions to the* **the cocharacter equations***:*

$$\begin{pmatrix} c & d & d & \dots & d \\ d & c & d & \dots & d \\ d & d & c & \dots & d \\ \vdots & \vdots & & \ddots & \vdots \\ d & d & d & \dots & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{n}$$

Recording $|\Lambda_{fin}|$ data for different parameters

Fix n and r. Then we will record $|\Lambda_{fin}|$ for every $(c, d) \in \mathbb{Z}_n \times \mathbb{Z}_n$ using a table in the following format:



Patterns in matrices with $r = 2$

n = 1

$$\begin{pmatrix}1\end{pmatrix}$$

n = 2

$$\begin{pmatrix}4 & 1\\ 1 & 2\end{pmatrix}$$

n = 3

$$\begin{pmatrix}9 & 1 & 1\\ 1 & 3 & 3\\ 1 & 3 & 3\end{pmatrix}$$

n = 4

$$\begin{pmatrix}16 & 1 & 4 & 1\\ 1 & 4 & 1 & 4\\ 4 & 1 & 8 & 1\\ 1 & 4 & 1 & 4\end{pmatrix}$$

n = 5

$$\begin{pmatrix}25 & 1 & 1 & 1 & 1\\ 1 & 5 & 1 & 1 & 5\\ 1 & 1 & 5 & 5 & 1\\ 1 & 1 & 5 & 5 & 1\\ 1 & 5 & 1 & 1 & 5\end{pmatrix}$$

n = 6

$$\begin{pmatrix}36 & 1 & 4 & 9 & 4 & 1\\ 1 & 6 & 3 & 2 & 3 & 6\\ 4 & 3 & 12 & 1 & 12 & 3\\ 9 & 2 & 1 & 18 & 1 & 2\\ 4 & 3 & 12 & 1 & 12 & 3\\ 1 & 6 & 3 & 2 & 3 & 6\end{pmatrix}$$

n = 7

$$\begin{pmatrix}49 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 7 & 1 & 1 & 1 & 1 & 7\\ 1 & 1 & 7 & 1 & 1 & 7 & 1\\ 1 & 1 & 1 & 7 & 7 & 1 & 1\\ 1 & 1 & 1 & 7 & 7 & 1 & 1\\ 1 & 1 & 7 & 1 & 1 & 7 & 1\\ 1 & 7 & 1 & 1 & 1 & 1 & 7\end{pmatrix}$$

Some patterns along the diagonal

1000	2	8	2	8	250	8	2	8	2
1	100	5	4	1	4	25	20	1	4
8	2	200	2	40	2	8	50	8	10
1	20	1	100	1	4	5	4	25	4
8	2	8	10	200	2	8	2	40	50
125	4	1	4	1	500	1	4	1	4
8	50	40	2	8	2	200	10	8	2
1	4	25	4	5	4	1	100	1	20
8	10	8	50	8	2	40	2	200	2
1	4	1	20	25	4	1	4	5	100

Some patterns along the diagonal

1000	2	8	2	8	250	8	2	8	2
1	100	5	4	1	4	25	20	1	4
8	2	200	2	40	2	8	50	8	10
1	20	1	100	1	4	5	4	25	4
8	2	8	10	200	2	8	2	40	50
125	4	1	4	1	500	1	4	1	4
8	50	40	2	8	2	200	10	8	2
1	4	25	4	5	4	1	100	1	20
8	10	8	50	8	2	40	2	200	2
1	4	1	20	25	4	1	4	5	100

Definition

Let $d_1 = \gcd(c-d, n)$ be the first diagonal number.
Let $d_2 = \gcd(c+(r-1)d, n)$ be the second diagonal number.

Some patterns along the diagonal

d_2 \ d_1	10	1	2	1	2	5	2	1	2	1
10	1000	2	8	2	8	250	8	2	8	2
1	1	100	5	4	1	4	25	20	1	4
2	8	2	200	2	40	2	8	50	8	10
1	1	20	1	100	1	4	5	4	25	4
2	8	2	8	10	200	2	8	2	40	50
5	125	4	1	4	1	500	1	4	1	4
2	8	50	40	2	8	2	200	10	8	2
1	1	4	25	4	5	4	1	100	1	20
2	8	10	8	50	8	2	40	2	200	2
1	1	4	1	20	25	4	1	4	5	100

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Some patterns along the diagonal

$d_2 \backslash d_1$	10	1	2	1	2	5	2	1	2	1
10	1000	2	8	2	8	250	8	2	8	2
1	1	100	5	4	1	4	25	20	1	4
2	8	2	200	2	40	2	8	50	8	10
1	1	20	1	100	1	4	5	4	25	4
2	8	2	8	10	200	2	8	2	40	50
5	125	4	1	4	1	500	1	4	1	4
2	8	50	40	2	8	2	200	10	8	2
1	1	4	25	4	5	4	1	100	1	20
2	8	10	8	50	8	2	40	2	200	2
1	1	4	1	20	25	4	1	4	5	100

$d_1 = 2, d_2 = 10$

$40 = 2^2 \cdot 10$

Definition

Let $d_1 = \gcd(c - d, n)$ be the first diagonal number.
Let $d_2 = \gcd(c + (r-1)d, n)$ be the second diagonal number.

Back to the cocharacter equations

Definition

The **cocharacter equations**:

$$\begin{aligned} cx_1 + dx_2 + \cdots + dx_r &\equiv 0 \pmod{n} \\ dx_1 + cx_2 + \cdots + dx_r &\equiv 0 \pmod{n} \\ &\vdots \\ dx_1 + dx_2 + \cdots + cx_r &\equiv 0 \pmod{n} \end{aligned}$$

Definition

Let $S_{cochr}(n, r, c, d) = |\Lambda_{fin}|$ be the number of solutions to the cocharacter equations.

Transforming the cocharacter equations

Starting from ...

$$cx_1 + dx_2 + \cdots + dx_r \equiv 0 \pmod{n}$$
$$dx_1 + cx_2 + \cdots + dx_r \equiv 0 \pmod{n}$$
$$\vdots$$
$$dx_1 + dx_2 + \cdots + cx_r \equiv 0 \pmod{n}$$

Add all equations together:

$$(c + (r-1)d)(x_1 + x_2 + \cdots + x_r) \equiv 0 \pmod{n}$$

Subtract row r from row i:

$$(c - d)(x_i - x_r) \equiv 0 \pmod{n}$$

Coroot solutions

Definition

The **coroot equations**:

$$\begin{aligned} (c+(r-1)d)(x_1+\cdots+x_r) &\equiv 0 \\ (c-d)(x_i-x_r) &\equiv 0 \quad \text{for all } 1 \le i \le r-1 \end{aligned}$$

Definition

Let $S_{cort}(n, r, c, d)$ be the number of solutions to the coroot equations.

$S_{cort}(n, r, c, d)$ is a multiple of $S_{cochr}(n, r, c, d)$.

Number of solutions to coroot equations

Theorem (AF, L)

$$S_{cort} = d_1^{r-1} d_2 \cdot \gcd\left(\frac{n}{d_1}, \frac{n}{d_2}, r\right)$$

Recall $d_1 = \gcd(c - d, n)$ and $d_2 = \gcd(c + (r-1)d, n)$.

Characterizing coroot solutions

Let

$$\begin{aligned} y_i &= x_i - x_r \qquad \text{for all } 1 \leq i \leq r-1 \\ z &= x_1 + x_2 + \cdots + x_r \end{aligned}$$

Then we can rewrite the coroot equations as

$$\begin{aligned} (c-d)y_i &\equiv 0 \pmod{n} \qquad \text{for all } 1 \leq i \leq r-1 \\ (c+(r-1)d)z &\equiv 0 \pmod{n} \end{aligned}$$

Characterizing coroot solutions

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$$y_i = x_i - x_r \qquad \text{for all } 1 \leq i \leq r-1$$
$$z = x_1 + x_2 + \cdots + x_r$$

Then we can rewrite the coroot equations as

$$(c-d)y_i \equiv 0 \pmod{n} \qquad \text{for all } 1 \leq i \leq r-1$$
$$(c+(r-1)d)z \equiv 0 \pmod{n}$$

Equivalently:

$$d_1 y_i \equiv 0 \pmod{n} \qquad \text{for all } 1 \leq i \leq r-1$$
$$d_2 z \equiv 0 \pmod{n}$$

Possible values of y and z

The equations $d_1 y_i \equiv 0$ and $d_2 z \equiv 0$ imply:

$y_i =$  $\dfrac{n}{d_1}$ for $1 \leq$  $\leq d_1$

$z =$  $\dfrac{n}{d_2}$ for $1 \leq$  $\leq d_2$

Let $y = y_1 + y_2 + \cdots + y_{r-1} =$  $\dfrac{n}{d_1}$. Then

$$rx_r \equiv z - y \pmod{n}$$

This has $b := \gcd(r, n)$ solutions if b divides $z - y$, and no solutions otherwise.

Counting coroot solutions

Let Fr_b be the fraction of tuples $(y_1, y_2, \ldots, y_{r-1}, z)$ such that $z - y$ is a multiple of b. Then

$$S_{cort} = d_1^{r-1} d_2 \cdot b \cdot Fr_b$$

Lemma

Fr_b evaluates to

$$Fr_b = \frac{\gcd\left(\frac{n}{d_1}, \frac{n}{d_2}, r\right)}{b}$$

Therefore,

$$S_{cort} = d_1^{r-1} d_2 \cdot \gcd\left(\frac{n}{d_1}, \frac{n}{d_2}, r\right)$$

Coroot equations $\not\Longrightarrow$ Cocharacter equations

Let $(x_1, x_2, \dots x_r)^T$ be a solution to the coroot equations. Then

$$\begin{pmatrix} c & d & \dots & d \\ d & c & \dots & d \\ \vdots & & \ddots & \vdots \\ d & d & \dots & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} \equiv k\frac{n}{b} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \pmod{n}$$

Coroot equations $\not\Longrightarrow$ Cocharacter equations

Let $(x_1, x_2, \ldots x_r)^T$ be a solution to the coroot equations. Then

$$\begin{pmatrix} c & d & \ldots & d \\ d & c & \ldots & d \\ \vdots & & \ddots & \vdots \\ d & d & \ldots & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} \equiv k\frac{n}{b} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \pmod{n}$$

Let $M(n, r, c, d)$ be the smallest value of $k\frac{n}{b}$ such that the above equation has a solution for some $(x_1, x_2, \ldots x_r)^T$. Let κ be the k-value corresponding to M.

Then $S_{cort} = S_{cochr} \cdot \frac{n}{M}$.

Roadmap for identifying M

To identify M (and κ) in all cases, we:

1. Find a formula for M when $r = p^\ell$ and $n = p^m$ for a prime p.
2. Identify M when $r = p^\ell$ and $n = ap^m$.
3. Build up to the general case, where $r = p_1^{\ell_1} p_2^{\ell_2} \ldots p_j^{\ell_j}$.

Finding M when r and n are powers of p

If $r = p^\ell$ and $n = p^m$ for some prime p, then what is $M = \kappa \frac{n}{b}$?

The following matrices have $n = 2^3, r = 2^\ell$ and show κ as ℓ increases:

$n = 8, r = 2$

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$n = 8, r = 4$

$$\begin{pmatrix} 4 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$n = 8, r = 8$

$$\begin{pmatrix} 8 & 1 & 2 & 1 & 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 4 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 1 & 2 \\ 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 4 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 & 1 & 1 \end{pmatrix}$$

Identifying M when r is a prime power

Lemma 1 (AF,L)

Let $r = p^\ell$, $n = p^m$, $b = p^\mu$, $c - d = c_0 p^t$, and $d = d_0 p^s$ for some prime p so that $\gcd(c_0, p) = \gcd(d_0, p) = 1$. Then

$$M = p^{\max(m-\mu, \min(t, s+m-t))}$$

Identifying M when r is a prime power

Lemma 1 (AF,L)

Let $r = p^\ell$, $n = p^m$, $b = p^\mu$, $c - d = c_0 p^t$, and $d = d_0 p^s$ for some prime p so that $\gcd(c_0, p) = \gcd(d_0, p) = 1$. Then

$$M = p^{\max(m-\mu, \min(t, s+m-t))}$$

Lemma 2 (AF,L)

Let $n' = ap^m$ where a, p relatively prime and $c' \equiv c \pmod{p^\mu}$, $d' \equiv d \pmod{p^\mu}$. Then

$$M(n', r, c', d') = \kappa \frac{n'}{b} = \kappa \frac{ap^m}{b} = aM(p^m, r, c, d)$$

When $r = p^\ell$, $n = ap^m$

$\mathrm{r} = 2^2, n = 2^2$

$$\begin{pmatrix} 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

$\mathrm{r} = 2^2, n = 12 = 3 \cdot 2^2$

$$\left(\begin{array}{cccc|cccc|cccc} 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\ \hline 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\ \hline 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \end{array}\right)$$

Identifying M when r has different prime powers

r = 2, n = 6

$$\left(\begin{array}{cc|cc|cc} 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

r = 3, n = 6

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 3 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

r = 6, n = 6

$$\begin{pmatrix} 6 & 1 & 2 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Identifying M when r has different prime powers

r = 2, n = 6

$$\left(\begin{array}{cc|cc|cc} 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

r = 3, n = 6

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 3 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

r = 6, n = 6

$$\begin{pmatrix} 6 & 1 & 2 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Conjecture 1 (AF, L)

Let $r = p_1^{\ell_1} p_2^{\ell_2} \dots p_j^{\ell_j}$, and let $M_i = \kappa_i \frac{n}{b_i}$ be the minimum possible value of $k\frac{n}{b_i}$ for $r = p_i^{\ell_i}$. Then

$$M = \gcd(M_1, M_2, \dots, M_j) = \kappa_1 \kappa_2 \dots \kappa_j \frac{n}{b}$$

Main result

Theorem (AF, L)

$$S_{coch} = d_1^{r-1} d_2 \gcd\left(\frac{n}{d_1}, \frac{n}{d_2}, r\right) \cdot \frac{M}{n}$$

The value of M is given by combining Lemmas 1 and 2 and Conjecture 1.

Future Work

- Prove Conjecture 1.
- Use this to better understand a map $\mathfrak{W} \to V$, where V is a particular quantum group module of dimension $\left(\frac{n}{d_1}\right)^r$.
- Generalize for all reductive groups (what we did only works for type A_n)

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