Problem 7: Measuring the space of Metaplectic Whittaker functions

UMN REU

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Contents

1 Introduction

1.1 Motivation

Whittaker functions are special functions that arise in p-adic number theory and representation theory, specifically in the study of automorphic forms over local fields and the study of principal series representations of reductive groups. They can be written as integrals over matrix groups, as generating functions over many different combinatorial objects, and in some cases as partition functions of lattice models. In particular, when this last interpretation is true, it gives a connection between the algebraic structures of the space of Whittaker functions and a quantum group module.

One type of Whittaker functions that have been particularly well-studied are *metaplectic* Whittaker functions, which are Whittaker functions on the principal series representations of *metaplectic covering groups*.

Definition. Given a group G and a natural number $n \in \mathbb{N}$, a *n-fold metaplectic cover* or *n-fold metaplectic covering group* $\widetilde{G}$ is a central extension of G by the n-th roots of unity μ_n. That is, $\widetilde{G}$ is defined by the following short exact sequence:

$$1 \to \mu_n \to \widetilde{G} \to G \to 1.$$

As a set, $\widetilde{G}$ is the set of tuples (ζ, g) where $\zeta \in \mu_n, g \in G$; however, the group structure has a more complicated multiplication rule than just component-wise multiplication.

These groups are named after the first "Metaplectic Group," the unique double cover of the *symplectic* group Sp_{2n} discovered by Weil [6]. In the study of that specific metaplectic cover, it was discovered that general covers of this type inherit much of the interesting representation theory and number theory of their base groups.

One reason for this phenomenon is that if G is a group that is also a topological space, the metaplectic cover is a covering space in the topological sense as well. For example, one such group, which will be the focus of this report, is $G = GL_r(F)$, the general linear group of $r \times r$ matrices over a local field F containing μ_{2n}. In this case, which was first studied by Kazhdan-Patterson [4] and led to a general theory for reductive groups, metaplectic covers have a particularly nice description.

Theorem 1.1 (Brylinski-Deligne [1], Frechette [3])**.** *Every metaplectic cover of $GL_r(F)$ corresponds to a bilinear form $B_{c,d,r}$ for some $c, d \in \mathbb{Z}$ that acts on $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{Z}^r \times \mathbb{Z}^r$ by*

$$B_{c,d,r}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T \cdot \begin{pmatrix} c & d & d & \dots & d \\ d & c & d & \dots & d \\ d & d & c & \dots & d \\ \vdots & \vdots & & \ddots & \vdots \\ d & d & d & \dots & c \end{pmatrix} \cdot \boldsymbol{y}.$$

Principal series representations are a particularly nice class of group representations that can be constructed analogously on both a base group and any of its metaplectic covers. Thus, functions incorporating information from these representations, such as Whittaker functions, which map from $G \to \mathbb{C}$ (or $\widetilde{G} \to \mathbb{C}$ in the metaplectic case), behave very similarly in many respects.

However, on the base group $GL_r(F)$, the space of Whittaker functions for any principal series is one-dimensional [2], while on a metaplectic cover, this is no longer true!

Theorem 1.2 (McNamara [5])**.** *Fix a metaplectic cover $\widetilde{G}$ and let $\mathfrak{W}$ be the space of metaplectic Whittaker functions for a principal series representation on $\widetilde{G}$. Then*

$$\dim(\mathfrak{W}) = \left|\widetilde{T}/H\right|$$

where $\widetilde{T}$ is the preimage in $\widetilde{G}$ of the group of diagonal matrices $T(F)$ and H is the maximal abelian subgroup of $\widetilde{T}$.

Note: the group T of diagonal matrices is denoted T because it is a *torus*, that is, it is isomorphic to $(F^\times)^r$. It is thus abelian. While we will call $\widetilde{T}$ the *metaplectic torus*, it is no longer abelian, nor is it technically a torus, as its elements look like (ζ, t) where $\zeta \in \mu_n$ (where $\mu_n \subsetneq F$) and $t \in T$.

Examining the explicit structures of the groups $\widetilde{T}$ and H for a local field F, we can reduce this to a finite computation.

Theorem 1.3 (McNamara [5], Frechette [3])**.** *For a metaplectic cover $\widetilde{G}$ corresponding to $B_{c,d}$*

$$\left|\widetilde{T}/H\right| = \left|\frac{\mathbb{Z}^r}{\{\boldsymbol{x} \in \mathbb{Z} : B_{c,d}(\boldsymbol{x}, \boldsymbol{y}) \equiv 0 \pmod{n} \text{ for all } \boldsymbol{y} \in \mathbb{Z}^r\}}\right| = \frac{n^r}{|\Lambda_{fin}|},$$

where $\Lambda_{fin} = \{\boldsymbol{x} \in (\mathbb{Z}_n)^r : B_{c,d}(\boldsymbol{x}, \boldsymbol{y}) \equiv 0 \pmod{n} \text{ for all } \boldsymbol{y} \in (\mathbb{Z}_n)^r\}$.

Our goal in this project is to count $|\Lambda_{fin}|$, which will tell us $\dim(\mathfrak{W})$. Counting $|\Lambda_{fin}|$ is equivalent to counting the number of solutions to the cocharacter equations which we introduce in the following section.

1.2 Introducing the cocharacter equations

It is known that $|\Lambda_{fin}|$ is equivalent to the number of solutions to a set of equations called the cocharacter equations, which we now define. We let $\mathbf{1}_r = (1, 1, \ldots, 1)^T$ be the $r \times 1$ column vector with all entries equal to 1, and let $\mathbf{0}_r = (0, 0, \ldots, 0)^T$ be the $r \times 1$ column vector with all entries equal to 0. Let $B_{c,d,r}$ be the $r \times r$ matrix for the bilinear form given in Theorem 1.1.

Definition (Cocharacter equations)**.** For natural numbers $r, n \geq 2$ and constants $c, d \in \mathbb{Z}_n$, we call the following system of equations the **cocharacter equations**:

$$B_{c,d,r}\boldsymbol{x} = \mathbf{0}_r \pmod{n}. \tag{1}$$

Let S_{cochar} be the number of solutions to the cocharacter equations.

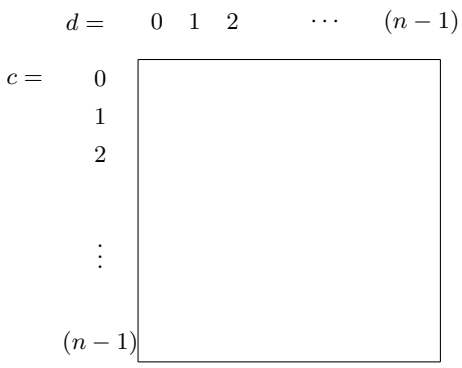
Using Theorem 1.3, we can express elements of Λ_{fin} as solutions to the cocharacter equations. Then Lemma 1.4 follows.

Lemma 1.4. *The number of solutions to the cocharacter equations is* $S_{cochar} = |\Lambda_{fin}|$.

1.3 Diagonal numbers

Looking at the values of S_{cochar} for a fixed r and n as we range over c and d, certain patterns emerge which motivate defining constants which we call the diagonal numbers. These constants will appear in our formulas for S_{cochar}.

For a fixed r and n, one can make a table of the value of $S_{cochar}(n, r, c, d)$ as we vary c and d in the following way:



We can then assign a number to each diagonal of slope -1 based on where it intersects the first row, and each diagonal of slope $r-1$ based on where it intersects the first column. In particular, if a diagonal of slope -1 interesects the first row at d, assign it the value $\ell_1 = \gcd(d, n)$, and if a diagonal of slope $r-1$ intersects the first column at c, assign it the value $\ell_2 = \gcd(n, c)$.

Example. The table for S_{cochar} where $n = 10, r = 3$ is shown in Figure 1.3, with diagonal numbers marked.

In this matrix, the value of every entry is determined by the numbers assigned to the two diagonal on which it falls. In particular we get that $S_{cochar}(10, 3, c, d) = \ell_1^{r-1}\ell_2 = \ell_1^2\ell_2$.

In general, given a random n and r, the value of S_{cochar} will not depend nearly so simply on ℓ_1 and ℓ_2, but they still play an important determining role. This motivates the following definition.

Definition. Let $\ell_1 = gcd(c-d, n)$ be the *first diagonal number* and define $\ell_2 = gcd(c+(r-1)d, n)$ to be the *second diagonal number*.

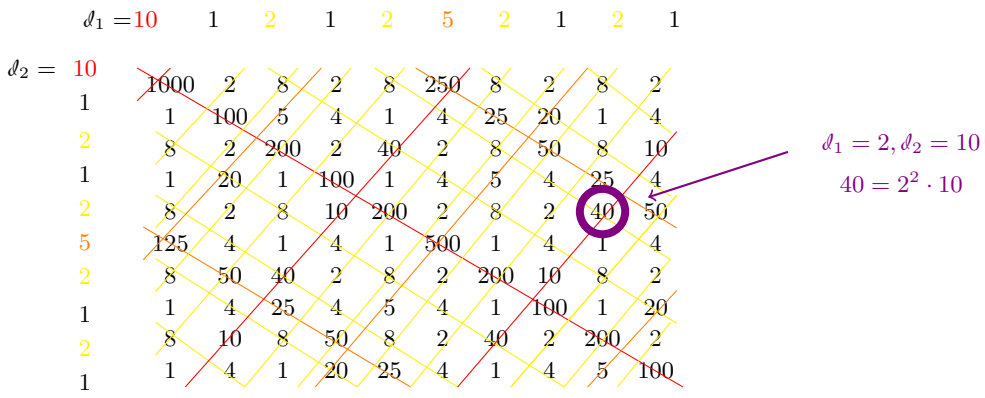


Figure 1: A table showing $S_{cochar}(10, 3, c, d)$ for all $(c, d) \in \mathbb{Z}_{10} \times \mathbb{Z}_{10}$

1.4 Outline and main results

In Section 2 we will review properties of modular arithmetic that we use for later proofs in the paper. In Section 3 we find the total number of solutions to the inhomogenous cocharacter equations, defined as follows:

Definition. Let $A \in \mathbb{Z}_n$, and $\boldsymbol{x} \in (\mathbb{Z}_n)^r$. Then the following are the inhomogenous cocharacter equations:

$$B_{c,d,r}\boldsymbol{x} = A\mathbf{1}_r \pmod{n}. \tag{2}$$

We define S_{inhom}, the number of solutions to the inhomogenous cocharacter equations, to be the number of choices of $\boldsymbol{x}$ which satisfy Equation (2) for some value of A.

In Section 3 we also relate S_{inhom} to S_{cochar} to find the total number of solutions to the cocharacter equations:

Theorem 1.5. *The number of solutions to the cocharacter equations is*

$$S_{cochar}(n, r, c, d) = \ell_1^{r-1} \gcd\left(\ell_2, \frac{n}{\ell_1}\gcd(c, d, n)\right).$$

In Section 4, we find the total number of solutions to the coroot equations, defined as follows.

Definition. The following are the *coroot* equations:

$$(c + (r-1)d)(x_1 + \cdots + x_r) \equiv 0 \pmod{n} \quad \text{for all } 1 \le i \le r-1 \tag{3}$$
$$(c - d)(x_i - x_r) \equiv 0 \pmod{n}. \tag{4}$$

The coroot equations describe the metaplectic covers of $SL_r(F)$, whereas cocharacter equations describe metaplectic covers of $GL_r(F)$. In some cases, the coroot and cocharacter equations are equivalent, and in other cases they are closely related. This motivates us to count the solutions to the coroot equations and relate this number to S_{cochar}.

Theorem 1.6. *The number of solutions to the coroot equations is*

$$S_{coroot}(n, r, c, d) = \mathscr{d}_1^{r-1}\mathscr{d}_2 \gcd\left(\frac{n}{\mathscr{d}_1}, \frac{n}{\mathscr{d}_2}, r\right).$$

In Section 5, we relate S_{cochar} to S_{coroot} by proving the following:

Theorem 1.7. *The number of solutions to the cocharacter equations can also be defined as*

$$S_{cochar} = S_{coroot} \cdot \frac{M}{n}$$

where $M = \operatorname{lcm}\left(\gcd(\mathscr{d}_2, \frac{dn}{\mathscr{d}_1}), \frac{n}{\gcd(n,r)}\right)$ *is a factor of* n *that is described in Section 5.*

We will further prove the following formula for M.

Proposition 1.8. *Let* $r = p_1^{\ell_1}p_2^{\ell_2}\dots p_j^{\ell_j}$ *and* $n = p_1^{m_1}p_2^{m_2}\dots p_j^{m_j}$. *For every* $1 \le i \le j$, *let*

$$(c-d) \equiv c_i p_i^{s_i} \pmod{p_i^{m_i}} \text{ and } d = d_i p_i^{t_i} \pmod{p_i^{m_i}} \text{ for each } 1 \le i \le j$$

so that $0 \le s_i, t_i \le m_i$. *Let* $\mu_i = \min(m_i, \ell_i)$ *and* c_i, d_i *are relatively prime to* p_i. *Then*

$$M = \prod_{i=1}^{j} p_i^{\max(m_i-\mu_i, \min(t_i, s_i+m_i-t_i))}.$$

2 Background on modular arithmetic techniques

Here we provide several lemmas that we will use to solve modular equations throughout this document.

Lemma 2.1. *Suppose* f *divides* n *and*

$$f\ell = fk \pmod{n}. \tag{5}$$

Then it is equivalent to state that

$$\ell = k \pmod{\frac{n}{f}}. \tag{6}$$

If we fix k, *then Equation* (6) *has one solution, while Equation* (5) *has* f *solutions. In particular, if* $\ell = \ell_1$ *is a solution to Equation* (6), *then* $\ell = \ell_1 + j\frac{n}{f}$ *is a solution to Equation* (5) *for any* $0 \le j < f$.

The top line of Figure 2 illustrates an example of Equation (5) for some l and k, where we have grouped all our numbers into blocks of size f. We get the bottom line of Figure 2 by dividing everything in the top line by f. This is now an example of Equation (6).

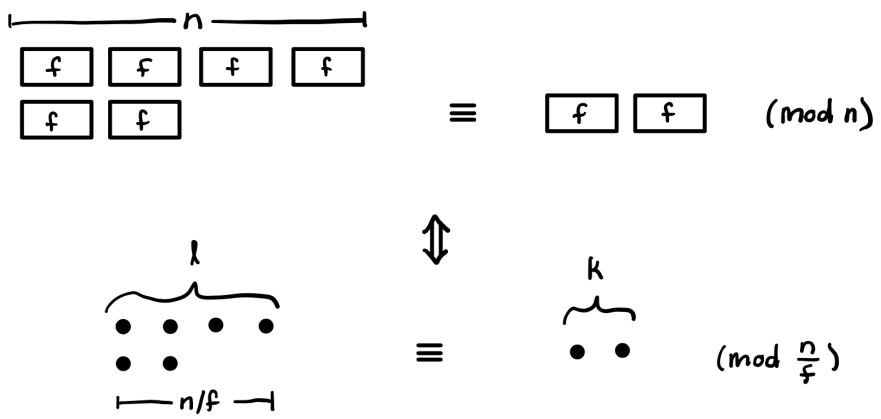


Figure 2: An illustrated example of Lemma 2.1.

Lemma 2.2. *Let $b = \gcd(r, n)$. The equation $rx \equiv a \pmod{n}$ has b solutions if b divides a, and no solutions otherwise.*

Proof. Let $r = sb$. We are counting the solutions to

$$sbx \equiv a \pmod{n} \tag{7}$$

where s and n are relatively prime and b divides n.

If b does not divide a, then taking Equation (7) modulo b gives us

$$sbx \equiv 0 \not\equiv a \pmod{b} \tag{8}$$

which has no solutions. Therefore, Equation (7) can have no solutions in this case.

Suppose b divides a, so that $a = kb$. Then we have

$$sbx \equiv kb \pmod{n}. \tag{9}$$

By Lemma 2.1, Equation (9) is equivalent to

$$sx \equiv k \pmod{\frac{n}{b}}. \tag{10}$$

The numbers $s = \frac{r}{b}$ and $\frac{n}{b}$ must be relatively prime because $b = \gcd(n, r)$. Therefore, s is invertible in $\mathbb{Z}_{\frac{n}{b}}$. We can now rewrite Equation (10) as

$$s^{-1}sx \equiv s^{-1}k \pmod{\tfrac{n}{b}} \tag{11}$$

$$x \equiv s^{-1}k \pmod{\tfrac{n}{b}}. \tag{12}$$

By Lemma 2.1, Equation (9) therefore has solutions $x = s^{-1}_{\bmod \frac{n}{b}} k + j\frac{n}{b}$ for $0 \leq j < b$. This concludes the proof that Equation (7) has b solutions when b divides a, and no solutions otherwise. □

Corollary 2.3. *If $f = \gcd(h, n)$, then the equation $hx \equiv 0 \pmod{n}$ has the same solutions as $fx \equiv 0 \pmod{n}$, which is equivalent to $x \equiv 0 \pmod{\frac{n}{f}}$ and has solutions $x = j\frac{n}{f}$ for some $0 \leq j < f$.*

Proof. This follows from the special case of the proof of Lemma 2.2 where $a = 0$. □

Lemma 2.4. *Let $s_1, s_2 \in \mathbb{Z}_n$. Then for any $x_1, x_2 \in \mathbb{Z}_n$, the minimum possible value that $s_1x_1 + s_2x_2 \pmod{n}$ can have is $\gcd(s_1, s_2, n)$.*

3 Cocharacter solutions and corollaries

In this section we will solve for $S_{cochar}(n, r, c, d)$. We do so by characterizing the set of solutions to the inhomogenous cocharacter equations and identifying the proportion of solutions with $A \equiv 0$. To do this, we will need to identify the smallest nonzero value of A, for which Equation (2) has a solution.

Definition. Let $a(n, r, c, d)$ be the smallest positive integer value for A such that there is a solution to the inhomogenous cocharacter equations.

Note that in most cases a fixed n, r, c, d are implied from context, and we shorten to write a instead of $a(n, r, c, d)$.

In Section 3.1 we show that the solutions to the inhomogenous cocharacter equations fall into $\frac{n}{a}$ equivalence classes, each defined by $A = ka$ for some $1 \leq k \leq \frac{n}{a}$, and that each equivalence class contains the same number of solutions. The solutions to the cocharacter equations are those in the class defined by $A = a\frac{n}{a} = n$. Then

$$S_{cochar}(n, r, c, d) = \frac{S_{inhom}(n, r, c, d)}{n/a} = \frac{a}{n} S_{inhom}(n, r, c, d) \tag{13}$$

because all equivalence classes are of equal size. In Section 3.2 we characterize the solutions to the inhomogenous cocharacter equations, which provides an expression for $S_{inhom}(n, r, c, d)$. We then find a by identifying the solution which minimizes A. This will complete the proof of Theorem 1.5.

In Section 3.3, we state corollaries that provide simpler expressions for S_{cochar} when $c \equiv d \equiv 0$, when $\ell_1 \equiv \ell_2 \equiv 1$, when $d \equiv 0$, and when n and r are relatively prime.

3.1 Equivalence classes of solutions

First, we characterize the set of A for which the inhomogeneous cocharacter equations have a solution.

Lemma 3.1. *The equation $B_{c,d,r}\boldsymbol{x} \equiv A\mathbf{1}_r \pmod{n}$ has a solution if and only if A is a multiple of a.*

Proof. Let $\boldsymbol{x}_a$ be a solution to

$$B_{c,d,r}\boldsymbol{x}_a \equiv a\mathbf{1}_r \pmod{n}.$$

Suppose there exist some positive integer g which is not a multiple of a and a vector $\boldsymbol{x}_g$ so that

$$B_{c,d,r}\boldsymbol{x}_g \equiv g\mathbf{1}_r \pmod{n}.$$

Then $ja < g < (j+1)a$ for some positive integer j. Therefore,

$$\begin{aligned} B_{c,d,r}(\boldsymbol{x}_g - j\boldsymbol{x}_a) &\equiv B_{c,d,r}\boldsymbol{x}_g - jB_{c,d,r}\boldsymbol{x}_a \\ &\equiv (g - ja)\mathbf{1}_r. \end{aligned}$$

This is a contradiction because $0 < g - ja < a$. □

Corollary 3.2. *The value of $a(n, r, c, d)$ divides n.*

This follows from Lemma 3.1 because the equation $B_{c,d,r}\boldsymbol{x} \equiv n\mathbf{1}_r \equiv \mathbf{0}_r \pmod{n}$ has the solution $\boldsymbol{x} = \mathbf{0}_r$.

We now show that solutions to Equation (2) fall into equivalence based on the value of A, and that these equivalences classes are all of the same size.

Proposition 3.3. *Let W_k be the set of solutions to $B_{c,d,r}\boldsymbol{x} \equiv ka\mathbf{1}_r \pmod{n}$ for $1 \le k \le \frac{n}{a}$. Then $|W_k| = |W_1|$ for each $1 \le k \le \frac{n}{a}$.*

Proof. Let $1 \le k \le \frac{n}{a}$ and let $(x_1', x_2', \ldots, x_r')^T \in W_1$. Consider the function

$$\begin{aligned} \phi : W_1 &\to W_k; \\ (x_1, x_2, \ldots, x_r)^T &\mapsto (x_1, x_2, \ldots, x_r)^T + (k-1)(x_1', x_2', \ldots, x_r')^T. \end{aligned}$$

By the linearity of matrices, $\phi((x_1, x_2, \ldots, x_r)^T) \in W_{1+(k-1)} = W_k$. To show that ϕ is a bijection, consider its inverse:

$$\begin{aligned} \phi^{-1} : W_k &\to W_1; \\ (x_1, x_2, \ldots, x_r)^T &\mapsto (x_1, x_2, \ldots, x_r)^T - (k-1)(x_1', x_2', \ldots, x_r')^T. \end{aligned}$$

It follows from the linearity of matrices that if $(x_1, x_2, \ldots, x_r)^T \in W_k$, then $\phi^{-1}((x_1, x_2, \ldots, x_r)^T)$ is in $W_{k-(k-1)} = W_1$. The proof that ϕ and ϕ^{-1} are inverses follows directly from the definitions of the functions.

□

3.2 Characterizing solutions and finding a

We now understand how to find S_{cochar} in terms of S_{inhom} and a, so it remains to find these values. In order to do so, we begin by characterizing the set of solutions to Equation (2), so that we may then count them.

Proposition 3.4. *A vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_r)^T$ solves the inhomogenous cocharacter equations if and only if for each $2 \leq j \leq r$ we have $x_j = x_1 + v_j \frac{n}{\ell_1}$ for some $1 \leq v_j \leq \ell_1$.*

To prove this we will use the following lemma.

Lemma 3.5. *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_r)^T$. Then $\boldsymbol{x}$ solves the inhomogenous cocharacter equations if and only if $cx_1 + dx_j \equiv dx_1 + cx_j \pmod{n}$ for every $2 \leq j \leq r$.*

Proof. Let $2 \leq j \leq r$. The first row of the equation $B_{c,d,r}\boldsymbol{x} \equiv A\mathbf{1}_r \pmod{n}$ tells us that

$$cx_1 + dx_j + \sum_{\substack{2 \leq k \leq r \\ k \neq j}} dx_k \equiv m \pmod{n}$$

while row j tells us that

$$dx_1 + cx_j + \sum_{\substack{2 \leq k \leq r \\ k \neq j}} dx_k \equiv m \pmod{n}.$$

Setting the left-hand sides of these equations equal to each other and then subtracting $\sum_{\substack{2 \leq k \leq r \\ k \neq j}} dx_k$ from both, observe that

$$cx_1 + dx_j \equiv dx_1 + cx_j \pmod{n}.$$

To see the implication in the other direction, let

$$A \equiv cx_1 + dx_j + \sum_{\substack{2 \leq k \leq r \\ k \neq j}} dx_k \pmod{n}$$

and suppose that $cx_1 + dx_j \equiv dx_1 + cx_j \pmod{n}$ for every $2 \leq j \leq r$. Then

$$\begin{aligned} dx_1 + cx_j + \sum_{\substack{2 \leq k \leq r \\ k \neq j}} &\equiv cx_1 + dx_j + \sum_{\substack{2 \leq k \leq r \\ k \neq j}} \\ &\equiv A \end{aligned}$$

for every $2 \leq j \leq r$. Thus, every line of $B_{c,d,r}\boldsymbol{x}$ gives an expression equal to A, which implies that $B_{c,d,r}\boldsymbol{x} \equiv A\mathbf{1}_r \pmod{n}$ as desired. □

We now return to the proof of Proposition 3.4.

Proof. To characterize which $\boldsymbol{x}$ solve the inhomogeneous cocharacter equations, it remains to characterize the solutions to the system of equations given by

$$cx_1 + dx_j \equiv dx_1 + cx_j \pmod{n} \tag{14}$$

for every $2 \leq j \leq r$. If we subtract $dx_1 + cx_j$ from both sides of Equation (14), we obtain

$$(c-d)(x_1 - x_j) \equiv 0 \pmod{n}.$$

Recalling that $\partial_1 = \gcd(c-d, n)$, Lemma 2.3 tells us that this is equivalent to

$$\partial_1(x_1 - x_j) \equiv 0 \pmod{n}. \tag{15}$$

This is true if and only if $x_1 - x_j$ is a multiple of $\frac{n}{\partial_1}$. Letting $x_1 - x_j \equiv v_j \frac{n}{\partial_1}$ for some $1 \leq v_j \leq \partial_1$, we obtain $x_j \equiv x_1 + v_j \frac{n}{\partial_1}$. Thus, the solutions to $B_{c,d,r}\boldsymbol{x} \equiv m\mathbf{1}_r \pmod{n}$ are precisely the vectors of the form

$$x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \frac{n}{\partial_1} \begin{pmatrix} 0 \\ v_2 \\ v_3 \\ \vdots \\ v_r \end{pmatrix}$$

for $0 \leq x_1 < n$ and $1 \leq v_j \leq \partial_1$. Let $\boldsymbol{v} = (0, v_2, v_3, \ldots, v_r)^T$. This is equivalent to the statement of Proposition 3.4. □

Now that we have precisely characterized the set of $\boldsymbol{x}$ which solve the inhomogeneous cocharacteristic equations, we can count the size of this set.

Corollary 3.6. *The number of solutions to the inhomogenous cocharacter equations is*

$$S_{inhom}(n, r, c, d) = n\partial_1^{r-1}.$$

Proof. We have n choices of x_1 and ∂_1 choices for each of $v_2, v_3, \ldots v_r$. Each distinct $(x_1, v_2, \ldots, v_r)$ tuple yields a distinct solution to the inhomogenous cocharacter equations, so there are $n\partial_1^{r-1}$ solutions in total. □

We are now prepared to identify a in terms of n, r, c, and d:

Proposition 3.7. *The minimum positive integer a such that the inhomogenous cocharacter equations have a solution for $A = a$ is $a = \gcd\left(\partial_2, \frac{n}{\partial_1}\gcd(c, d, n)\right)$.*

Note that we can equivalently write a as $a = \gcd\left(\partial_2, \frac{cn}{\partial_1}\right) = \gcd\left(\partial_2, \frac{dn}{\partial_1}\right)$. This follows from $\frac{n}{\partial_1}(c-d) \equiv 0 \pmod{n}$ which is true because ∂_1 divides $c-d$. Therefore, $\frac{dn}{\partial_1} \equiv \frac{cn}{\partial_1} \equiv \frac{n}{\partial_1}\gcd(c, d, n) \pmod{n}$.

Proof. We want to identify the minimum A for which a solution to the inhomogenous cocharacter equations exists. We have already characterized solutions to the inhomogenous cocharacter equations as vectors of the form $\boldsymbol{x} = x_1\mathbf{1}_r + \frac{n}{\ell_1}\boldsymbol{v}$. Substituting this into Equation (2), we see that the right-hand side can equal

$$\begin{aligned}
&\equiv \begin{pmatrix} c & d & d & \dots & d \\ d & c & d & \dots & d \\ d & d & c & \dots & d \\ \vdots & \vdots & & \ddots & \vdots \\ d & d & d & \dots & c \end{pmatrix} \left(x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \frac{n}{\ell_1} \begin{pmatrix} 0 \\ v_2 \\ v_3 \\ \vdots \\ v_r \end{pmatrix} \right) \\
&\equiv x_1(c+(r-1)d)\mathbf{1}_r + \frac{n}{\ell_1} \begin{pmatrix} dv_2 + dv_3 + \cdots + dv_r \\ cv_2 + dv_3 + \cdots + dv_r \\ dv_2 + cv_3 + \cdots + dv_r \\ \vdots \\ dv_2 + dv_3 + \cdots + dv_{r-1} + cv_r \end{pmatrix}.
\end{aligned}$$

Every row of this expression must must equal a constant A. Therefore, looking at the first row,

$$\begin{aligned}
A &\equiv x_1(c+(r-1)d) + \frac{n}{\ell_1}(dv_2 + dv_3 + \cdots + dv_r) \pmod{n} \\
&\equiv x_1(c+(r-1)d) + \frac{dn}{\ell_1}(v_2 + v_3 + \cdots + v_r) \pmod{n}.
\end{aligned}$$

Observe that x_1 and $v_2+v_3+\cdots+v_r$ are both arbitrary constants. Thus, the minimum value A can have is $\gcd\left(c+(r-1)d, \frac{dn}{\ell_1}, n\right)$ which equals $\gcd\left(\ell_2, \frac{dn}{\ell_1}\right)$ because $\ell_2 = \gcd(c+(r-1), n)$. □

Finally, we substitute our values obtained for S_{inhom} and a into Equation (13) to prove Theorem 1.5. We have

$$\begin{aligned}
S_{cochar}(n,r,c,d) &= \frac{a}{n} S_{inhom}(n,r,c,d) \\
&= \frac{1}{n} \cdot \gcd\left(\ell_2, \frac{dn}{\ell_1}\right) \cdot n\ell_1^{r-1} \\
&= \ell_1^{r-1} \gcd\left(\ell_2, \frac{dn}{\ell_1}\right).
\end{aligned}$$

3.3 Corollaries of cocharacter solutions formula

Many properties follow from Theorem 1.5.

Corollary 3.8. *If $d = 0$, then $S_{cochar} = \gcd(c,n)^r$.*

Proof. If $d \equiv 0 \equiv n$, then

$$\begin{aligned} S_{cochar}(n,r,c,d) &= \gcd(c-d,n)^{r-1} \gcd\left(c+(r-1)d, n, \frac{dn}{\gcd(c-d,n)}\right) \\ &= \gcd(c,n)^{r-1} \gcd\left(c, n, \frac{n^2}{\gcd(c,n)}\right) \\ &= \gcd(c,n)^r. \end{aligned}$$

□

Corollary 3.9. *We have* $\dim(\mathfrak{W}) = 1$ *if and only if* $c = d = 0$.

Proof. By Theorem 1.3, this statement is equivalent to saying that $S_{cochar} = n^r$ if and only if $c = d = 0$. The forward direction follows from Corollary 3.8. Now assume $S_{cochar} = n^r$. Then we must have $\ell_1 = n$, so

$$S_{cochar} = n^{r-1} \gcd\left(\ell_2, \frac{n}{n} \gcd(c,d,n)\right).$$

This requires that $c, d \equiv n \pmod{n}$. □

Note that both Corollary 3.8 and Corollary 3.9 are directly verifiable from looking at the cocharacter equations for $d = 0$.

Corollary 3.10. *We have* $\dim(\mathfrak{W}) = n^r$ *if and only if* $\ell_1 = \ell_2 = 1$.

Proof. We will show that $S_{cochar} = 1$ if and only if $\ell_1 = \ell_2 = 1$. To have $S_{cochar} = 1$, we should have $\ell_1^{r-1} = 1$ (implying $\ell_1 = 1$) and $\gcd\left(\ell_2, \frac{dn}{\ell_1}\right) = 1$. Since $\ell_1 = 1$ we have $\gcd(\ell_2, dn) = 1$ which tells us that ℓ_2 must be relatively prime to n. But $\ell_2 = \gcd(c+(r-1)d, n)$ so the only value of ℓ_2 which has no common factors except 1 with n is $\ell_2 = 1$. □

Corollary 3.11. *If* n *and* r *are relatively prime, then* $a = \ell_2$ *which implies that* $S_{cochar} = \ell_1^{r-1}\ell_2$.

Proof. Let $c - d = g_1\ell_1$ and $c + (r-1)d = g_2\ell_2$. Then observe that

$$\begin{aligned} rd &\equiv (c+(r-1)d) - (c-d) \pmod{n} \\ &\equiv g_1\ell_1 - g_2\ell_2 \pmod{n}. \end{aligned}$$

If r and n are relatively prime, then r is invertible in $\mathbb{Z}_n$, so

$$d \equiv r^{-1}(g_1\ell_1 - g_2\ell_2) \pmod{n}.$$

Therefore,

$$\begin{aligned} d\frac{n}{\ell_1} &\equiv r^{-1}(g_1\ell_1 - g_2\ell_2)\frac{n}{\ell_1} \pmod{n} \\ &\equiv (r^{-1}g_1)\ell_1\frac{n}{\ell_1} - \left(r^{-1}g_2\frac{n}{\ell_1}\right)\ell_2 \pmod{n} \\ &\equiv \left(-r^{-1}g_2\frac{n}{\ell_1}\right)\ell_2 \pmod{n}. \end{aligned}$$

which is a multiple of ℓ_2. Therefore,

$$\gcd\left(\ell_2, d\frac{n}{\ell_1}\right) = \ell_2$$

as desired. □

4 Counting solutions to coroot equations

In this section we will count S_{coroot}, the number of solutions to the coroot equations. In doing so, we will prove properties about the form these solutions take on. This will allow us to relate S_{cochar} to S_{coroot} in Section 5.

Let

$$y_i \equiv \quad x_i - x_r \tag{16}$$
$$z \equiv \quad x_1 + \cdots + x_r. \tag{17}$$

We can then rewrite Equations (3) and (4) as follows:

$$(c + (r-1)d)z \equiv 0 \pmod{n} \tag{18}$$
$$(c-d)y_i \equiv 0 \pmod{n} \qquad \text{for all } 1 \le i \le r-1. \tag{19}$$

Lemma 2.3 tells us that Equation (18) holds if and only if z is a multiple of $\frac{n}{\ell_2}$. Similarly, Equation (19) has solutions y_i such that y_i is a multiple of ℓ_1.

Definition. Given a fixed c and d, let Y_{ℓ_1} be the set of integers a in $\mathbb{Z}_n$ such that $y_i = a$ is a solution to Equation (19), and let Z_{ℓ_2} be the set of integers a in $\mathbb{Z}_n$ such that $z = a$ is a solution to Equation (18).

Then Y_{ℓ_1} is the set of multiples of $\frac{n}{\ell_1}$, and $|Y_{\ell_1}| = \ell_1$. Similarly Z_{ℓ_2} is the set of multiples of n/ℓ_2, and $|Z_{\ell_2}| = \ell_2$.

Each set of y_i- and z-values that solve Equations (18) and (19) is an ordered tuple $(y_1, y_2, \ldots, y_{r-1}, z) \in Y_{\ell_1}^{r-1} \times Z_{\ell_2}$, and there are $\ell_1^{r-1} \cdot \ell_2$ such tuples. Given one of these, we search for tuples $(x_1, x_2, \ldots, x_r) \in \mathbb{Z}_n^r$ that satisfy Equations (16) and (17).

Solving for x_i from Equation (16) we get

$$x_i = y_i + x_r. \tag{20}$$

Substituting into Equation (17) and rearranging terms, observe that

$$rx_r = z - \sum_{i-1}^{r-1} y_i \pmod{n}. \tag{21}$$

Let $b = \gcd(n, r)$. Then Lemma 2.2 tells us that Equation (21) has b solutions when $z - \sum_{i-1}^{r-1} y_i$ is a multiple of b and no solutions otherwise.

Lemma 4.1. *Let $(x_1, \cdots, x_r)$ be a solution to equations* (16) *and* (17) *for a tuple $(z, y_1, \ldots, y_r)$ and $(x'_1, \cdots, x'_r)$ be a solution to equations* (16) *and* (17) *for a different tuple $(z', y'_1, \ldots, y'_r) \neq (z, y_1, \ldots, y_r)$. Then $(x_1, \cdots, x_r) \neq (x'_1, \cdots, x'_r)$.*

Proof. Suppose to the contrary that we had $(x_1, \cdots, x_r) = (x'_1, \cdots, x'_r)$. This would imply

$$\begin{aligned} y_i \equiv x_i - x_r &= x'_i - x'_r &&\equiv y'_i \quad \text{for all } 1 \leq i \leq r-1 \\ z \equiv x_1 + \cdots + x_r &= x'_1 + \cdots + x'_r &&\equiv z' \end{aligned}$$

which implies that $(z', y'_1, \ldots, y'_r) = (z, y_1, \ldots, y_r)$, a contradiction. Therefore, $(x_1, \cdots, x_r) \neq (x'_1, \cdots, x'_r)$. □

Now, with the definition of one more function, we can give a convenient way to count the number of solutions to the coroot equations.

Definition. Let $Fr_b(\mathscr{d}_1, \mathscr{d}_2, n)$ be the proportion of $(y_1, \ldots, y_r, z)$ tuples that will yield a valid solution to the coroot equations. In other words,

$$Fr_b = \frac{|\{(y_1, \cdots, y_r, z) \in Y^{r-1} \times Z : z - y_1 - \cdots - y_r \text{ is a multiple of } b\}|}{|\{(y_1, \cdots, y_r, z) \in Y^{r-1} \times Z\}|}.$$

Remark 4.2. *When n and r are relatively prime, $b = 1$ and so any tuple we pick adds to a multiple of b. Thus Fr_b evaluates to* 1*, and so by Theorem 1.6 we have $S_{coroot} = \mathscr{d}_1^{r-1}\mathscr{d}_2$.*

Proposition 4.3. *The function Fr_b evaluates to*

$$Fr_b(\mathscr{d}_1, \mathscr{d}_2, n) = \frac{\gcd\left(\frac{n}{\mathscr{d}_1}, \frac{n}{\mathscr{d}_2}, b\right)}{b}.$$

Proof. Let $k_1 = \gcd(\frac{n}{\mathscr{d}_1}, b)$ so that $b = m_1 k_1$. Then we can show that $y_i \pmod b$ can be any multiple of k_1 in $\mathbb{Z}_b$ with equal probability. Lemma 2.2 tells us that, letting $y_i = j\frac{n}{\mathscr{d}_1}$, the equation

$$y_i \equiv j\frac{n}{\mathscr{d}_1} \quad \equiv g \pmod b$$

has $k_1 = \gcd(\frac{n}{\mathscr{d}_1}, b)$ solutions if $k_1 | g$, and no solutions otherwise. Thus, the values of $y_i \pmod b$ fall into the m_1 equivalence classes $k_1, 2k_1, \ldots, m_1 k_1 = b$ with equal probability.

Let $y = \sum_{i=1}^{r-1} y_i$ be the sum of the y_i's, and let g be a multiple of k_1. We will examine the probability that y will equal g modulo b. If we pick any arbitrary $y_1, y_2, \ldots, y_{r-2}$, we are left with

$$y_{r-1} = g - \sum_{i=1}^{r-2} y_i \pmod b.$$

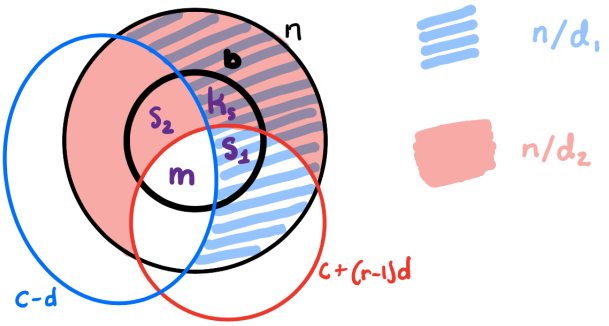


Figure 3: A visualization of our factorization of b, where the overlap of any two circles contains their greatest common divisor.

The right-hand side of this equation defines some equivalence class $\ell k_1 \pmod b$ from which we must choose y_{r-1} to ensure that $y \equiv g \pmod b$. Exactly $\frac{1}{m_1}$ of the possible values of y_{r-1} place us in the correct equivalence class for a given g. Thus, y falls into the equivalence classes $k_1, 2k_1, \ldots, mk_1$ with equal probability.

Similarly, let $k_2 = \gcd(\frac{n}{d_2}, b)$ so that $b = m_2 k_2$. Then Lemma 2.2 tells us that

$$z \equiv j\frac{n}{d_2} \equiv g \pmod b$$

has k_2 solutions if k_2 divides g, and no solutions otherwise. Therefore, $z \pmod b$ can be any of the m_2 multiples of k_2 modulo b with equal probability.

Now let $k_s = \gcd(k_1, k_2)$ so that $k_1 = s_1 k_s$ and $k_2 = s_2 k_s$. Then we have $b = m_1 s_1 k_s = m_2 s_2 k_s$. Letting $m = \gcd(m_1, m_2)$, we write $b = m s_1 s_2 k_s$. Figure 3 provides a visualization of how these factors relate to b, $\frac{n}{d_1}$, and $\frac{n}{d_2}$. We now want to identify the proportion of y- and z-values that satisfy

$$z - y \equiv 0 \pmod b, \tag{22}$$

or equivalently,

$$z \equiv y \pmod{m s_1 s_2 k_s}.$$

Let $y = \alpha k_1 = \alpha s_1 k_s$ and $z = \beta k_2 = \beta s_2 k_s$. Then we want to find the proportion of (α, β) pairs that solve

$$\beta s_2 k_s \equiv \alpha s_1 k_s \pmod{m s_1 s_2 k_s}.$$

We can divide through by k_s:

$$\beta s_2 \equiv \alpha s_1 \pmod{m s_1 s_2}.$$

By Lemma 2.2, the right-hand side must be a multiple of s_2. Since s_1 and s_2 are relatively prime, we must have $\alpha = as_2$ for some a. Exactly $\frac{1}{s_2}$ of the possible α-values are multiples of s_2. Then

$$\beta s_2 \equiv as_1 s_2 \pmod{ms_1 s_2}$$

which has solutions only for $\beta \equiv as_1 \pmod{b}$. Out of the $m_2 = ms_1$ equivalence classes that z can fall into, only the one defined by $\beta = as_1$ works. Therefore,

$$\frac{1}{s_2}\left(\frac{1}{ms_1}\right) = \frac{1}{ms_1 s_2} = \frac{1}{b/k_s} = \frac{k_s}{b} = \frac{\gcd\left(\frac{n}{\ell_1}, \frac{n}{\ell_2}, b\right)}{b}$$

of the choices of y and z solve Equation (22).

□

There are $\ell_1^{r-1}\ell_2$ ordered pairs in $Y^{r-1} \times Z$, and we have shown that exactly $\frac{\gcd\left(\frac{n}{\ell_1}, \frac{n}{\ell_2}, b\right)}{b}$ of these yield solutions to the coroot equations. Any set of y- and z-values that solves $rx_r \equiv z - y \pmod{n}$ will yield b solutions to the coroot equations because x_r can take on b possible values. Therefore, there are

$$\ell_1^{r-1}\ell_2 \cdot \frac{\gcd\left(\frac{n}{\ell_1}, \frac{n}{\ell_2}, b\right)}{b} \cdot b = \ell_1^{r-1}\ell_2 \cdot \gcd\left(\frac{n}{\ell_1}, \frac{n}{\ell_2}, r\right)$$

solutions to the coroot equations, as stated in Theorem 1.6.

5 Relating the cocharacter equation to the coroot equations

In this section, we show how the coroot equations are obtained from the cocharacter equations. We then find the relationship between S_{coroot} and S_{cochar}. To obtain the coroot equations from the cocharacter equations, we can multiply the matrix $B_{c,d,r}$ which defines the cocharacter equations by

$$L_r = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & & 1 & -1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

to obtain

$$L_r B_{c,d,r} = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & & 1 & -1 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix} \begin{pmatrix} c & d & d & \dots & d \\ d & c & d & \dots & d \\ d & d & c & \dots & d \\ \vdots & \vdots & & \ddots & \vdots \\ d & d & d & \dots & c \end{pmatrix}$$

$$= \begin{pmatrix} (c-d) & 0 & \dots & 0 & (d-c) \\ 0 & (c-d) & \dots & 0 & (d-c) \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & & (c-d) & (d-c) \\ c+(r-1)d & c+(r-1)d & \dots & c+(r-1)d & c+(r-1)d \end{pmatrix}.$$

The new system of equations specified by this transformation is

$$L_r B_{c,d,r} \boldsymbol{x} \equiv \mathbf{0}_r \pmod{n} \tag{23}$$

which gives us precisely the coroot equations.

5.1 From coroot to cocharacter

We can apply the following linear transformation to the coroot equations to obtain the cocharacter equations, multiplied by r:

$$\begin{pmatrix} r-1 & -1 & \dots & -1 & 1 \\ -1 & r-1 & \dots & -1 & 1 \\ \vdots & \vdots & \ddots & & \vdots \\ -1 & -1 & & r-1 & 1 \\ -1 & -1 & \dots & -1 & 1 \end{pmatrix} \begin{pmatrix} (c-d) & 0 & \dots & 0 & (d-c) \\ 0 & (c-d) & \dots & 0 & (d-c) \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & & (c-d) & (d-c) \\ c+(r-1)d & c+(r-1)d & \dots & c+(r-1)d & c+(r-1)d \end{pmatrix}$$

$$= r \begin{pmatrix} c & d & d & \dots & d \\ d & c & d & \dots & d \\ d & d & c & \dots & d \\ \vdots & \vdots & & \ddots & \vdots \\ d & d & d & \dots & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{n}. \tag{24}$$

If r and n are relatively prime, then r has an inverse r^{-1} in $\mathbb{Z}_n$. In that case, we can multiply Equation (24) by r^{-1} on both sides to obtain the cocharacter equations. Then the cocharacter and coroot equations are equivalent. The coroot solutions have $\ell_1^{r-1}\ell_2$ solutions in this case, which gives us an additional proof of Corollary 3.11.

More generally, let $b = \gcd(r, n)$. Then we have

$$b\left(\frac{r}{b}\right)\begin{pmatrix} c & d & d & \dots & d \\ d & c & d & \dots & d \\ d & d & c & \dots & d \\ \vdots & \vdots & & \ddots & \vdots \\ d & d & d & \dots & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{b \cdot \frac{n}{b}}.$$

If we divide everything by b, we obtain

$$\left(\frac{r}{b}\right) B_{c,d,r}\boldsymbol{x} = \mathbf{0}_r \pmod{\frac{n}{b}}.$$

The number $\frac{r}{b}$ is relatively prime to $\frac{n}{b}$ and therefore invertible in $\mathbb{Z}_{\frac{n}{b}}$. Multiplying both sides by the inverse of $\frac{r}{b}$, we see that

$$B_{c,d,r}\boldsymbol{x} = \mathbf{0}_r \pmod{\frac{n}{b}}. \tag{25}$$

Remark 5.1. *When r and n are relatively prime $n = b$ and therefore the coroot and cocharacter equations are equivalent.*

Combining Remarks 4.2 and 5.1 gives us an alternative way to see why Corollary 3.11 holds.

If $b \neq 1$, then for any coroot solution $\boldsymbol{x}$, we get

$$B_{c,d,r}\boldsymbol{x} \equiv \frac{n}{b}\boldsymbol{v} \pmod{n} \tag{26}$$

for some vector $\boldsymbol{v}$. We now show that for all coroot solutions, we have $\boldsymbol{v} = k\mathbf{1}_r$ where k is some constant.

Let the ith row of the right-hand side of Equation (25) be

$$w_i \equiv \left(\sum_{j=1}^{i-1} dx_j + cx_i + \sum_{j=i+1}^{r} dx_j\right).$$

Then Equation (25) states that $w_i \equiv 0 \pmod{\frac{n}{b}}$ for each $1 \leq i \leq r$, which implies that each w_i must be a multiple of $\frac{n}{b}$.

Lemma 5.2. *All solutions to the coroot equations have $w_1 = w_2 = \cdots = w_r$.*

Proof. Observe that for any $1 \leq i < r$,

$$w_i - w_r = (c - d)(x_i - x_r).$$

This must equal zero, by Equation (4). Therefore, $w_i = w_r$ for all $1 \leq i < r$. □

We have shown that if $\boldsymbol{x}$ solves the coroot equations, then $\boldsymbol{x}$ solves

$$B_{c,d,r}\boldsymbol{x} = k\frac{n}{b}(\mathbf{1}_r) \pmod{n} \tag{27}$$

where $1 \leq k \leq b$. To see that the converse is true, suppose that $\boldsymbol{x}$ solves Equation (27) and define w_i as above for all $1 \leq i \leq r$. Then

$$\begin{aligned}(c-d)(x_i - x_r) \equiv w_i - w_r &\equiv k\frac{n}{b} - k\frac{n}{b} \equiv 0 \pmod{n} \\ (c+(r-1)d)(x_1 + x_2 + \ldots x_r) \equiv w_1 + w_2 + \cdots + w_r &\equiv r\left(k\frac{n}{b}\right) \equiv 0 \pmod{n}\end{aligned}$$

which gives us the coroot equations. Thus, Equation (27) provides an equivalent formulation of the coroot equations.

In Section 3.1 we showed that Equation (27) has solutions if and only if $k\frac{n}{b}$ is a multiple of a, and that each class of solutions (defined by having the same k) is of the same size. Again, we now want to find the smallest nonzero k for which (27) has a solution.

Definition. Let $\kappa(n, r, c, d)$ be the smallest positive value of k such that there is a solution to Equation (27), and let $M(n, r, c, d) = \kappa(n, r, c, d)\frac{n}{b}$.

Again, it will often be clear from context that we are working with a particular fixed n, r, c, d in which case we will write κ and M instead of $\kappa(n, r, c, d)$ and $M(n, r, c, d)$ for brevity.

We can relate the values of $M(n, r, c, d)$ and $a(n, r, c, d)$ as follows.

$$M = \operatorname{lcm}\left(a, \frac{n}{b}\right) = \operatorname{lcm}\left(\frac{n}{b}, \gcd\left(\ell_2, \frac{dn}{\ell_1}\right)\right). \tag{28}$$

Then there are $\frac{n}{M} = \frac{b}{\kappa}$ equivalence classes of solutions to Equation (27). Exactly one of these equivalence classes—the one given by $k = b$—gives the solutions to the cocharacter equations. Therefore,

$$S_{cochar} = \frac{S_{coroot}}{n/M} = S_{coroot} \cdot \frac{M}{n} = S_{coroot} \cdot \frac{\kappa}{b}. \tag{29}$$

Substituting in our earlier expressions for the values of S_{coroot} and M, we obtain

$$S_{cochar} = \frac{\ell_1^{r-1}\ell_2}{n} \gcd\left(\frac{n}{\ell_1}, \frac{n}{\ell_2}, b\right) \operatorname{lcm}\left(\frac{n}{b}, \gcd\left(\ell_2, \frac{dn}{\ell_1}\right)\right). \tag{30}$$

Although it is not immediately clear from looking at this equation, we know that this formula is equivalent to the one given in Theorem 1.5. One area of future work would be to simplify this expression and more directly understand why it is equivalent to the statement of Theorem 1.5.

In the next section, we conjecture yet another equivalent formula for S_{cochar} by expressing the values of M and κ directly in terms of the prime factors of n, r, c and d.

5.2 Strategy for identifying κ and M

In the following subsections, we identify M (and, equivalently, κ) for all possible values of n, r, c, and d. We identify M in the following three cases:

1. When $r = p^\ell$ and $n = p^m$ for a prime p (Theorem 5.3)
2. When $r = p^\ell$ and $n = n_0 p^m$ for a prime p with p and n_0 relatively prime (Lemma 5.4)
3. When $r = p_1^{\ell_1} p_2^{\ell_2} \dots p_j^{\ell_j}$ and $n = n_0 p_1^{m_1} p_2^{m_2} \dots p_j^{m_j}$ (Conjecture **??**)

In the case where $r = p^\ell$ and $n = p^m$, we prove the following theorem in Section 5.3:

Lemma 5.3. *Fix a prime p so that $r = p^\ell, n = p^m$, and $b = p^\mu$. Let $(c - d) = c_0 p^t$ and $d = d_0 p^s$ where c_0 and d_0 are relatively prime to p. Then*

$$M = p^{\max(m-\mu, \min(t, s+m-t))}. \tag{31}$$

is the smallest positive integer such that there exists a solution $\boldsymbol{x}$ to the coroot equations which satisfies

$$B_{c,d,r}\boldsymbol{x} \equiv M\mathbf{1}_r \pmod{n}. \tag{32}$$

Note that all other solutions $\boldsymbol{v}$ of the coroot equations satisfy $B_{c,d,r}\boldsymbol{v} = C\mathbf{1}_r$ for some C a multiple of M.

Example. We can store κ in tables using the same setup we earlier used to make tables of S_{cochar}. For $n = 8 = 2^3$ and $r = 2^\ell$, the following tables show how κ changes as ℓ increases from 1 to 3. Because M and κ depend on $\mu = \min(\ell, m)$ rather than on ℓ, any κ table for $\ell > 3$ would be identical to the table for $\ell = 3$.

$n = 8, r = 2$

$$\begin{pmatrix} 2&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1 \end{pmatrix}$$

$n = 8, r = 4$

$$\begin{pmatrix} 4&1&1&1&2&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&2&1\\ 1&1&1&1&1&1&1&1\\ 2&1&1&1&2&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&2&1&1&1&1&1\\ 1&1&1&1&1&1&1&1 \end{pmatrix}$$

$n = 8, r = 8$

$$\begin{pmatrix} 8&1&2&1&4&1&2&1\\ 1&1&1&2&1&2&1&2\\ 2&1&2&1&2&1&4&1\\ 1&2&1&1&1&2&1&2\\ 4&1&2&1&4&1&2&1\\ 1&2&1&2&1&1&1&2\\ 2&1&4&1&2&1&2&1\\ 1&2&1&2&1&2&1&1 \end{pmatrix}$$

The entries in these tables are determined by the main diagonals they lie on, which are described by s, and the columns they lie in, which are described by t and index how far down the main diagonal an entry is. In particular, to obtain the matrix for $\ell + 1$, we start with the matrix for ℓ and then multiply a specific fraction of the elements falling on highlighted diagonals of the $\ell + 1$ matrix by p.

To extend the result of Theorem 5.3 to the case where $n = n_0 p^m$ and $r = p^\ell$, we prove the following lemma in Section 5.4:

Lemma 5.4. *Let $n = n_0 p^m$ where n_0, p relatively prime and $c \equiv c' \pmod{p^m}$, $d \equiv d' \pmod{p^m}$ for $0 \le c', d' < p^m$. Then*

$$\kappa(n, r, c, d) = \kappa(p^m, r, c', d') \quad \textit{and} \quad M(n, r, c, d) = n_0 M(p^m, r, c', d').$$

Example. Below are the κ tables for $r = 2^2$ and $n = 2^2$ and $3 \cdot 2^2$:

$r = 2^2, n = 2^2$

$$\begin{pmatrix} 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

$r = 2^2, n = 12 = 3 \cdot 2^2$

$$\left(\begin{array}{cccc|cccc|cccc} 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\ \hline 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \\ \hline 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 & 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 \end{array}\right)$$

When $n = 3 \cdot 2^2$, we obtain the κ table for $n = 2^2$, tessellated in a 3×3 grid. This happens because, as Lemma 5.4 tells us, increasing c or d by a multiple of $p^m = 2^2$ does not change κ.

In Section 5.5 we prove the following lemma:

Lemma 5.5. *Let $r = p_1^{\ell_1} p_2^{\ell_2} \dots p_j^{\ell_j}$, and let $M_i = M(n, p_i^{\ell_i}, c, d)$. Let $b_i = \gcd(n, p_i^{\ell_i})$ and $M_i = \kappa_i \frac{n}{b_i}$. Then*

$$M(n, r, c, d) = \gcd(M_1, M_2, \dots, M_j) = \kappa_1 \kappa_2 \dots \kappa_j \frac{n}{b}.$$

We restate this lemma and combine it with our results from Sections 5.3 and 5.4 to obtain a formula for M in terms of the prime factors of n and r.

Example. Below are the kappa tables for $n = 2 \cdot 3 = 6$ and $r = 2, 3, 6$:

$r = 2, n = 6$

$$\left(\begin{array}{cc|cc|cc} 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

$r = 3, n = 6$

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 3 & 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)$$

$r = 6, n = 6$

$$\begin{pmatrix} 6 & 1 & 2 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The table for $r = 6$ is obtained by multiplying the tables for $r = 2$ and $r = 3$ together elementwise.

5.3 Proof of Lemma 5.3

For a solution $\boldsymbol{x} = (x_1, x_2, \ldots x_r)^T$ to the coroot equations, let $B_{c,d,r}\boldsymbol{x} = C\mathbf{1}_r$. From the last line of the cocharacter equations, we know that

$$\begin{aligned} d(x_1 + \cdots + x_{r-1}) + cx_r &\equiv C \pmod{n} \\ d(x_1 + \cdots + x_{r-1} + x_r) + (c-d)x_r &\equiv C \pmod{n}. \end{aligned}$$

We can now substitute in $z = x_1 + \cdots + x_r$:

$$dz + (c-d)x_r \equiv C \pmod{n}. \tag{33}$$

To find the minimum possible value for C, we will first express Equation (33) in terms of the variables defined in the theorem statement. In particular, we need to express y, z and x_r in terms of these variables.

First we compute y, which can be any multiple of $\frac{n}{\ell_1}$. We have

$$\ell_1 = \gcd(c-d, n) = \gcd(c_0 p^t, p^m) = p^t.$$

Therefore,

$$y = \text{✿}\frac{n}{\ell_1} = \text{✿}p^{m-t}, \tag{34}$$

for $0 \leq$ ✿ $< p^t$. We now get an expression for ℓ_2 which we use to find z. Observe that

$$c + (r-1)d = c - d + rd = c_0 p^t + d_0 p^{\ell+s}.$$

Let $\nu = \min(t, s+\ell)$. Then

$$\begin{aligned} \ell_2 &= gcd(c + (r-1)d, n) = p^\nu \qquad \text{and} \\ z &= \text{✿}\frac{n}{\ell_2} = \text{✿}p^{m-\nu}, \end{aligned} \tag{35}$$

where $0 \leq$ ✿ $< p^\nu$.

We can now solve for x_r. We have that

$$rx_r \equiv z - y \pmod{n} \tag{36}$$

$$p^\ell x_r \equiv \text{✿} \cdot p^{m-\nu} - \text{✿}p^{m-t} \pmod{p^m}. \tag{37}$$

We now consider two different cases.

5.3.1 Case 1: If $\ell \leq m - t$

When $\ell \leq (m-t)$, then p^ℓ divides both terms on the right side of Equation (37) because $m - t \leq m - \nu$. Dividing everything (including n in the modulus) by

p^ℓ, we get that x_r must satisfy the equation

$$\begin{aligned} x_r &\equiv \bigcirc p^{m-\nu-\ell} - p^{m-t-\ell} \pmod{ap^{m-\ell}} \\ &\equiv p^{m-t-\ell}\left(\bigcirc p^{t-\nu} - \bigcirc\right) \pmod{p^{m-\ell}} \end{aligned}$$

because $t \geq \nu$. Then x_r can be any multiple of $p^{m-t-\ell}$. Let

$$x_r \equiv \bigcirc p^{m-t-\ell} \tag{38}$$

where $0 \leq \bigcirc < p^{t+l}$. Given any choice of $\bigcirc$ and any choice of $\bigcirc$, we can always find some value of $\bigcirc$ to make Equation (38) hold.

Substituting the possible values of z and x_r given in Equations (35) and (38) along with the values of d and $c-d$ into Equation (33) and simplifying, we get

$$\begin{aligned} dz + (c-d)x_r &\equiv \kappa\frac{n}{b} \pmod{n} \\ d_0 p^s \bigcirc p^{m-\nu} + c_0 p^t \bigcirc p^{m-t-\ell} &\equiv \kappa p^{m-\ell} \pmod{p^m} \\ \bigcirc \cdot d_0 p^{m-\nu+s} + \bigcirc \cdot c_0 p^{m-\ell} &\equiv \kappa p^{m-\ell} \pmod{p^m}. \end{aligned}$$

Let $\bigcirc = 0$. We then have

$$\bigcirc c_0 p^{m-\ell} \equiv \kappa p^{m-\ell} \pmod{p^m}.$$

Observe that c_0 is relatively prime to p^m and therefore invertible, so we can let $\bigcirc = c_0^{-1}$ and obtain a solution for $\kappa = 1$ and $M = p^{m-\ell} = p^{m-\mu}$.

5.3.2 Case 2: If $\ell \geq m - t$

In this case, p^{m-t} divides all terms in Equation (37), so we can divide all terms by p^{m-t} to yield

$$p^{\ell-m+t} x_r \equiv \bigcirc p^{t-\nu} - \bigcirc \pmod{p^t}.$$

Therefore, x_r can be any number in $\mathbb{Z}_{p^m}$, and $\bigcirc$ can vary simultaneously because we can always pick $\bigcirc$ to obtain the desired x_r. Let $x_r = \bigcirc$.

Substituting the possible values of z given in Equation (35) and $x_r = \bigcirc$ into the right-hand side Equation (33) and simplifying, we get

$$\begin{aligned} dz + (c-d)x_r &\equiv \kappa\frac{n}{b} \pmod{n} \\ d_0 p^s \bigcirc p^{m-\nu} + c_0 p^t \bigcirc &\equiv \kappa p^{m-\mu} \pmod{p^m} \\ \bigcirc \cdot d_0 p^{m+s-\nu} + \bigcirc \cdot c_0 p^t &\equiv \kappa p^{m-\mu} \pmod{p^m}. \end{aligned}$$

The numbers c_0 and d_0 are relatively prime to p^m and therefore invertible in $\mathbb{Z}_{p^m}$. Then we can reparameterize  and to obtain

$$\text{  }p^{s+m-\nu} + \text{}p^t \equiv \kappa p^{m-\mu} \pmod{p^m}.$$

The minimum nonzero value that the right-hand side can have is $p^{\min(s+m-\nu,t)}$, where $\nu = \min(t, \ell + s)$. If $\ell + s < t$, then $p^{s+m-\nu} = p^{m-\ell} = p^{m-\mu}$ and we have a solution for $\kappa = 1$. If $\ell + s > t$, then $p^{\min(s+m-\nu,t)} = p^{\min(s+m-t,t)}$. Then

$$\kappa p^{m-\mu} = M = p^{\max(m-\mu,\min(t,s+m-t))}. \tag{39}$$

Equivalently,

$$\kappa = p^{\max(0,\min(s+\mu-t,t+\mu-m)))}. \tag{40}$$

These formulas also account for case 1, since our assumption that $\ell \leq m - t$ implies that $\ell = \mu$ and $t \leq m - \ell$. Therefore,

$$\max(m - \mu, \min(t, s + m - t)) = m - \mu = m - \ell$$

as desired if $\ell \leq m - t$.

5.4 Proof of Lemma 5.4

Proof. Let $r = p^\ell$ for a prime p and $n = n_0 p^m$ for some n_0 relatively prime to p. To show that $\kappa(n_0 p^m, r, c, d) = \kappa(p^m, r, c', d')$, we first prove that if

$$B_{c',d',r}\boldsymbol{x} \equiv k\frac{p^m}{gcd(p^m, r)}\mathbf{1}_r \pmod{p^m} \tag{41}$$

has a solution then

$$B_{c,d,r}\boldsymbol{x'} \equiv k\frac{n}{gcd(n, r)}\mathbf{1}_r \pmod{ap^m}. \tag{42}$$

does as well. We then show that if Equation (42) has a solution, then Equation (41) does as well.

To do this, we must first write $B_{c,d,r}$ in terms of $B_{c',d',r}$. Observe that $gcd(p^m, r) = gcd(n, r) = b$. Now we first suppose that $\boldsymbol{x}$ is a solution to Equation (41). We show that $\boldsymbol{x'} = n_0\boldsymbol{x}$ is a solution to Equation (42). By Lemma 2.1 we know that

$$B_{c,d,r}(n_0\boldsymbol{x}) \equiv k\left(\frac{n}{b}\right)\mathbf{1}_r \pmod{n}. \tag{43}$$

Observe that $B_{c,d,r}$ differs from $B_{c',d',r}$ in that we add some multiples of p^m to c' and d' to obtain c and d, respectively. Then, for some constants $0 \leq \alpha_c, \alpha_d < n_0$, we have

$$B_{c,d,r} = B_{c',d',r} + \begin{pmatrix} \alpha_c p^m & \alpha_d p^m & \dots & \alpha_d p^m \\ \alpha_d p^m & \alpha_c p^m & \dots & \alpha_d p^m \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_d p^m & \alpha_d p^m & \dots & \alpha_c p^m \end{pmatrix}.$$

Therefore,

$$B_{c,d,r}(n_0\boldsymbol{x}) \equiv B_{c',d',r}(n_0\boldsymbol{x}) + \begin{pmatrix} \alpha_c p^m & \alpha_d p^m & \dots & \alpha_d p^m \\ \alpha_d p^m & \alpha_c p^m & \dots & \alpha_d p^m \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_d p^m & \alpha_d p^m & \dots & \alpha_c p^m \end{pmatrix} (n_0\boldsymbol{x}) \pmod{n}.$$

Multiplying the matrix in the second term by n_0, all entries are now 0 (mod $n_0 p^m$). Thus the entire second term on the right-hand side is 0 (mod n), and we obtain

$$B_{c,d,r}(n_0\boldsymbol{x}) \equiv B_{c',d',r}(n_0\boldsymbol{x}) \equiv n_0 k \frac{p^m}{b}\mathbf{1}_r \equiv k\left(\frac{n}{b}\right)\mathbf{1}_r \pmod{n}.$$

Now suppose that $\boldsymbol{x'}$ is a solution to Equation (42). Then

$$\begin{aligned} B_{c,d,r}(n_0\boldsymbol{x'}) &\equiv n_0 k\left(\frac{n_0 p^m}{b}\right)\mathbf{1}_r \\ &\equiv B_{c',d',r}(n_0\boldsymbol{x'}) + \begin{pmatrix} \alpha_c p^m & \alpha_d p^m & \dots & \alpha_d p^m \\ \alpha_d p^m & \alpha_c p^m & \dots & \alpha_d p^m \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_d p^m & \alpha_d p^m & \dots & \alpha_c p^m \end{pmatrix} (n_0\boldsymbol{x'}) \\ &\equiv B_{c',d',r}(n_0\boldsymbol{x'}) \pmod{n} \\ &\equiv n_0 B_{c',d',r}(\boldsymbol{x'}) \pmod{n_0 p^m}. \end{aligned}$$

Using Lemma 2.1, we can divide by n_0 to obtain

$$B_{c',d',r}(\boldsymbol{x'}) \equiv n_0 k\left(\frac{p^m}{b}\right)\mathbf{1}_r \pmod{p^m}.$$

Since n_0 and p^m are relatively prime, we can multiply by n_0^{-1} to get that

$$\begin{aligned} n_0^{-1} B_{c',d',r}(\boldsymbol{x'}) &\equiv n_0^{-1} n_0 k\left(\frac{p^m}{b}\right)\mathbf{1}_r \pmod{p^m} \\ B_{c',d',r}(n_0^{-1}\boldsymbol{x'}) &\equiv k\left(\frac{p^m}{b}\right)\mathbf{1}_r \pmod{p^m}. \end{aligned}$$

Thus Equations (41) and (42) have solutions for the same set of k-values, which implies that $\kappa(n_0 p^m, p^\ell, c, d) = \kappa(p^m, p^\ell, c, d)$. □

5.5 Case when r is a product of distinct primes

In Lemma 5.5 we stated that if $r = p_1^{\ell_1} p_2^{\ell_2} \dots p_j^{\ell_j}$ and $M_i = M(n, p^\ell c, d)$, then

$$M(n, r, c, d) = \gcd(M_1, M_2, \dots, M_j) = \kappa_1 \kappa_2 \dots \kappa_j \frac{n}{b}.$$

Combining Lemmas 5.3 5.4, and 5.5, we obtain Proposition 1.8 (which in turn implies the statements of each of the three lemmas). To prove Proposition 1.8, we will want to separately examine which power of each p_i is contained in M individually. To do so, the following properties will be helpful:

$$\gcd\left(z, \prod_{i=1}^{j} p_i^{m_i}\right) = \prod_{i=1}^{j} \gcd(z, p_i^{m_i}) \tag{44}$$

$$lcm\left(\prod_{i=1}^{j} p_i^{u_i}, \prod_{i=1}^{j} p_i^{v_i}\right) = \prod_{i=1}^{j} lcm(p_i^{u_i}, p_i^{v_i}) = \prod_{i=1}^{j} p_i^{\max(u_i, v_i)} \tag{45}$$

$$\gcd(a, b) = \gcd(a \pmod b, b). \tag{46}$$

We now prove Proposition 1.8.

Proof. We will prove that Proposition 1.8 follows from the definition of a and the fact that $M = \gcd\left(\frac{n}{b}, a\right)$. For each $1 \leq i \leq j$, let:

$$\begin{aligned} c - d &\equiv c_i p_i^{t_i} \pmod{p_i^{m_i}} \\ d &\equiv d_i p_i^{s_i} \pmod{p_i^{m_i}} \\ r &\equiv r_i p_i^{\mu_i} \pmod{p_i^{m_i}} \end{aligned}$$

where c_i, d_i, and r_i are nonzero and relatively prime to p_i and $0 \leq t_i, s_i, \mu_i \leq m_i$. Note that $\mu_i = \min(m_i, \ell_i)$ so that $b = p_1^{\mu_1} p_2^{\mu_2} \dots p_j^{\mu_j}$. Then

$$\frac{n}{b} = \prod_{i=1}^{j} p_i^{m_i - \mu_i}.$$

Recall that

$$\begin{aligned} a &= \gcd\left(c + (r-1)d, \frac{dn}{\ell_1}, n\right) \\ &= \gcd\left((c-d) + rd, \frac{dn}{\ell_1}, \prod_{i=1}^{j} p_i^{m_i}\right). \end{aligned}$$

For each $1 \leq i \leq j$, let

$$a_i = \gcd\left((c-d) + rd, \frac{dn}{\ell_1}, p_i^{m_i}\right).$$

Note that a_i must be a power of p_i. Then by Equation (44), $a = a_1 a_2 \dots a_j$. Next recall that M is the least common multiple of a and $\frac{n}{b}$. We have identified a way to write a is a product of powers of p_i for each $1 \leq i \leq j$, and $\frac{n}{b} =$

$p_1^{m_1-\mu_1}p_2^{m_2-\mu_2}\dots p_j^{m_j-\mu_j}$ is also a product of powers of $p_1, p_2, \dots p_j$. Therefore, by Equation (45),

$$\begin{aligned} M &= \operatorname{lcm}\left(a, \frac{n}{b}\right) \\ &= \operatorname{lcm}\left(a_1a_2\dots a_j, p_1^{m_1-\mu_1}p_2^{m_2-\mu_2}\dots p_j^{m_j-\mu_j}\right) \\ &= \prod_{i=1}^{j} \operatorname{lcm}(a_i, p_i^{m_i-\mu_i}). \end{aligned} \tag{47}$$

We will next find a_i, but we first need to identify ℓ_1. By Equations (44) and (46), ℓ_1 equals

$$\begin{aligned} \gcd\left(c-d, \prod_{i=1}^{j} p_i^{m_i}\right) &= \prod_{i=1}^{j} \gcd(c-d, p_i^{m_i}) \\ &= \prod_{i=1}^{j} \gcd(c_ip_i^{t_i}, p_i^{m_i}) \\ &= \prod_{i=1}^{j} p_i^{t_i}. \end{aligned}$$

Now we can evaluate a_i, applying Equation (46) to obtain:

$$a_i = \gcd\left((c-d)+rd \pmod{p_i^{m_i}}, d\frac{n}{\ell_1} \pmod{p_i^{m_i}}, p_i^{m_i}\right).$$

We now substitute in the values of $(c-d)$ and d modulo $p_i^{m_i}$. Observe that $\frac{n}{\ell_1} = p_1^{m_1-t_1}p_2^{m_2-t_2}\dots p_j^{m_j-t_j}$, so the only factor of $\frac{n}{\ell_1}$ that can contribute to a_i is $p_i^{m_i-t_i}$. Therefore, we only substitute $p_i^{m_i-t_i}$ for $\frac{n}{\ell_1}$:

$$\begin{aligned} a_i &= \gcd\left(c_ip_i^{t_i} + d_ip_i^{s_i}\cdot r_ip_i^{\mu_i}, d_ip_i^{s_i}p_i^{m_i-t_i}, p_i^{m_i}\right) \\ &= \gcd\left(c_ip_i^{t_i} + r_id_ip_i^{s_i+\mu_i}, d_ip_i^{s_i+m_i-t_i}, p_i^{m_i}\right). \end{aligned}$$

Now we consider three cases. If $t_i < s_i + \mu_i$, then

$$\begin{aligned} a_i &= \gcd\left(p_i^{t_i}(c_i + r_id_ip_i^{s_i+\mu_i-t_i}), d_ip_i^{s_i+m_i-t_i}, p_i^{m_i}\right) \\ &= p_i^{\min(t_i, s_i+m_i-t_i, m_i)} \\ &= p_i^{\min(t_i, s_i+m_i-t_i)} \end{aligned}$$

where we've used the fact that $t_i \le m_i$ in the last step. Therefore,

$$\begin{aligned} \operatorname{lcm}(a_i, p_i^{m_i-\mu_i}) &= \operatorname{lcm}(p_i^{\min(t_i, s_i+m_i-t_i)}, p_i^{m_i-\mu_i}) \\ &= p_i^{\max(m_i-\mu_i, \min(t_i, s_i+m_i-t_i))}. \end{aligned}$$

Next suppose that $t_i > s_i + \mu_i$. Then we have

$$\begin{aligned} a_i &= \gcd\left(p_i^{s_i+\mu_i}(c_i p_i^{t_i-s_i-\mu_i} + r_i d_i), d_i p_i^{s_i+m_i-t_i}, p_i^{m_i}\right) \\ &= p_i^{\min(s_i+\mu_i, s_i+m_i-t_i, m_i)}. \end{aligned}$$

Since $s_i + \mu_i < t_i \leq m_i$, we can rewrite this as follows:

$$a_i = p_i^{\min(t_i, s_i+\mu_i, s_i+m_i-t_i)}.$$

Then

$$\begin{aligned} \operatorname{lcm}(a_i, p_i^{m_i-\mu_i}) &= \operatorname{lcm}(p_i^{\min(t_i, s_i+\mu_i, s_i+m_i-t_i)}, p_i^{m_i-\mu_i}) \\ &= p_i^{\max(m_i-\mu_i, \min(t_i, s_i+\mu_i, s_i+m_i-t_i))}. \end{aligned}$$

Since $t_i > s_i + \mu_i$, we have $s_i + m_i - t_i < m_i - \mu_i$. Therefore,

$$\min(t_i, s_i + \mu_i, s_i + m_i - t_i) < m_i - \mu_i$$

and

$$\operatorname{lcm}(a_i, p_i^{m_i-\mu_i}) = p_i^{m_i-\mu_i} = p_i^{\max(m_i-\mu_i, \min(t_i, s_i+m_i-t_i))}$$

If $t_i = s_i + \mu_i$, then we have:

$$a_i = \gcd\left(p_i^{s_i-\mu_i}(c_i + r_i d_i), d_i p_i^{s_i+m_i-(s_i+\mu_i)}, p_i^{m_i}\right).$$

Note that $c_i + r_i d_i$ may contain additional powers of p_i, so the largest power of t_i in $p_i^{s_i-\mu_i}(c_i + r_i d_i)$ is unknown. Call this power $\heartsuit$ for now. Then

$$a_i = p_i^{\min(\heartsuit, m_i-\mu_i, m_i)} = p_i^{\min(\heartsuit, m_i-\mu_i)}.$$

Therefore,

$$\begin{aligned} \operatorname{lcm}(a_i, p_i^{m_i-\mu_i}) &= \operatorname{lcm}(p_i^{\min(\heartsuit, m_i-\mu_i)}, p_i^{m_i-\mu_i}) \\ &= p_i^{\max(m_i-\mu_i, \min(\heartsuit, m_i-\mu_i))} \\ &= p_i^{m_i-\mu_i}. \end{aligned}$$

Because $s_i + m_i - t_i = m_i - \mu_i$, we can also write this as

$$\operatorname{lcm}(a_i, p_i^{m_i-\mu_i}) = p_i^{\max(m_i-\mu_i, \min(t_i, s_i+m_i-t_i))}$$

which is identical to the expression we obtained in the other two cases.

Substituting this into Equation (47), we obtain

$$M = \prod_{i=1}^{j} p_i^{\max(m_i-\mu_i, \min(t_i, s_i+m_i-t_i))}$$

as desired.

□

6 Future Work

We have obtained two very different expressions for S_{cochar} in Theorem 1.5 and Equation (29), and it currently not obvious that they always give the same number. Connecting the solutions to the inhomogenous cocharacter and coroot equations in more detail would also help us to better reconcile our formulas for S_{cochar}. If we better understood why they are equivalent, that might point towards a simpler formula from S_{cochar}. In addition to the corollaries provided in Section 3.3, there are other corollaries that we would like to obtain by knowing S_{cochar}. For example, we would like to describe all cases when $S_{cochar} = \partial_1^{r-1}\partial_2$.

The work in [3] which motivated this project also points to other open questions. Frechette has a function from $\mathfrak{W}$ to a quantum group module of dimension $\left(\frac{n}{\partial_1}\right)^r$ which is always injective, surjective or bijective. Knowing that $\dim\mathfrak{W} = \frac{n^r}{\partial_1^{r-1}a}$, we could identify when each case occurs by comparing ∂_1 to a. It would also be illuminating to generalize our work here to all reductive groups, as this paper only describes type A_n groups.

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