Cohen–Macaulayness of ASM Ideals

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Introduction and motivation
Alternating sign matrices

Definition

An alternating sign matrix (ASM) is an $n \times n$ matrix with the following properties:

1. Each entry is taken from the set \{-1, 0, 1\}.
2. The sum of the entries in each row (resp. column) sum to 1.
3. The nonzero entries in a row (resp. column) alternate between 1 and $-1$.

Examples include permutation matrices and

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & -1 & 1 \\
0 & 1 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}.
\]
ASM ideals and ASM varieties

Definition (Weigandt [Wei18])

We can associate an ASM ideal $I_A$ to an ASM $A$, which is a certain generalized determinental, radical ideal. The variety associated to $I_A$ is called the ASM variety.

- $I_w$ for a permutation $w \in S_n$ are called Schubert determinental ideals, first studied by Fulton [Ful92].

Let $S = k[z_{ij}]_{i,j=1}^{n}$, where $k$ is any field.

1. Can talk about whether $S/I_A$ is Cohen–Macaulay,
2. Can talk about when $I_A$ is height unmixed (the associated variety is equidimensional)

Definition

Call an ASM $A$ Cohen–Macaulay (resp. equidimensional) if $I_A$ is Cohen–Macaulay (resp. height unmixed).
Cohen–Macaulayness

**Theorem (Fulton [Ful92])**

Matrix Schubert varieties are Cohen–Macaulay.

**Question**

Are all ASM varieties Cohen–Macaulay?

**Example**

No. For example, if

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & -1 & 1 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix},
\]

then

\[
I_A = (z_{11}, z_{21}, z_{12}z_{31}, z_{31}z_{22}, z_{22}z_{13}),
\]

which is not Cohen–Macaulay.
What is previously known about ASM ideals

An *anti-diagonal initial ideal* in $I_A$ is generated by the product of terms along the antidiagonals of each minor.

**Example**

If

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

then $I_A = \langle z_{11}z_{12}z_{21}z_{22}, z_{11}z_{12}z_{13}, z_{21}z_{22}z_{23}, z_{31}z_{32}z_{33} \rangle$.

**Corollary of Conca–Varbaro [CV20]**

$I_A$ is Cohen–Macaulay if and only if the antidiagonal initial ideal in $I_A$ is Cohen–Macaulay.
What is previously known about ASM ideals

Theorem (Weigandt [Wei18] and Fulton [Ful92]):

$I_A$ has an irredundant prime decomposition

$$I_A = \bigcap_{w \in \text{Perm}(A)} I_w,$$

$I_w$ is prime and the height of $I_w$ is the Coxeter length $\ell(w)$.

- $\text{Perm}(A)$ is a set of permutation matrices associated to $A$ (not defined here).
Previously known implications for Cohen-Macaulayness

\[ \Delta_{\text{in} I_A} \text{ vertex decomposable} \]

\[ \downarrow \]

\[ A \text{ CM} \iff I_A \text{ CM} \iff \text{in} I_A \text{ CM} \iff \Delta_{\text{in} I_A} \text{ CM} \]

\[ \downarrow \]

\[ A \text{ equidim.} \iff I_A \text{ (in } I_A \text{) unmixed} \iff \Delta_{\text{in} I_A} \text{ pure} \]
Obtaining $I_A$ and $\text{in} I_A$ from $A$

Generators of $I_A$ are the determinants of the minors.
Generators of $\text{in} I_A$ are product of anti-diagonal elements in the minors.
Results
A family of non-equidimensional ASMs

Theorem (AF,H,K,L)

Let $A$ be an ASM which satisfies the following properties:

1. $A$ contains the block $B = \begin{bmatrix} 0 & \vdots \\ 1 & -1 \end{bmatrix}$, with the 1 falling in row $r$ and column $c$.
2. All entries of $A$ northwest of the 1 in $B$ are zeros.
3. All essential boxes of $\text{rk} A$ which don’t correspond to a box from $B$ are either rank 0 or rank at least $r - 1$.
4. Column $c$ has no essential boxes.

Then $A$ is not equidimensional.
A family of non-equidimensional ASMs (illustrated)
When is Cohen-Macaulayness preserved?

**Conjecture**

Let \( A \) be a Cohen-Macaulay ASM. Then submatrix of \( A \) which is an ASM is also Cohen-Macaulay.

This has been checked for all ASM’s through ASM 6.

**Question**

How can we add or take away rows and columns from an ASM to preserve Cohen-Macaulayness?
When is Cohen-Macaulayness Preserved?

\[ \begin{align*}
A & \quad \text{CM} \quad A^T \\
A_1 & \quad 0 \quad 1 \\
0 & \quad 0 \\
\end{align*} \]
Diagonal Block Decomposition for ASMs

Conjecture

Let $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ be a block diagonal matrix, where $A_1$ and $A_2$ are ASMs. Then $A$ is CM $\iff$ both $A_1$ and $A_2$ are CM.

We have proved the $\implies$ direction.

Conjecture

Let $A$ be CM and $A' = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$. Then $A$ is CM.

This implies the $\iff$ direction of the previous conjecture.

Theorem (AF,H,K,L)

This is true when ”CM” is replaced with the condition ”height unmixed”.
ASM\(s\) are more complicated than matrix Schubert varieties.

Example

Let \( A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \) and \( \tilde{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \). Then \( A \) is Cohen–Macaulay but \( \tilde{A} \) is not.

- There is an ASM in ASM(5) which is equidimensional but not CM.
- Knutson and Miller used a specific vertex decomposition to show that all matrix Schubert varieties are CM. This does not work for ASMs.

Question

Are all Cohen-Macaulay ASM’s vertex decomposable?
Data Summary
## Counting ASMs

<table>
<thead>
<tr>
<th>dimension</th>
<th># CM ASMs</th>
<th># non-CM ASMs</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>39</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>328</td>
<td>101</td>
</tr>
<tr>
<td>6</td>
<td>4028</td>
<td>3408</td>
</tr>
<tr>
<td>7*</td>
<td>37625</td>
<td>91222</td>
</tr>
</tbody>
</table>

*partial, excludes certain nice classes of CM ASMs.
For all ASMs up through ASM(6), if $A$ is CM then any submatrix of $A$ which is an ASM is also CM.

For all ASMs up through ASM(5), whenever $A$ is Cohen–Macaulay, $\Delta_{\text{in} I_A}$ is vertex decomposable.

For all ASMs up through ASM(5), whenever $A$ is Cohen–Macaulay, then $\frac{1}{A}$ is Cohen–Macaulay.
Aldo Conca and Matteo Varbaro.  
Square-free Gröbner degenerations.  

William Fulton.  
Flags, Schubert polynomials, degeneracy loci, and determinantal formulas.  

Anna Weigandt.  
Prism tableaux for alternating sign matrix varieties.  
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