$\mathfrak{S}_n$-equivariant Koszul algebras from the Boolean lattice

2023 Twin Cities REU in Combinatorics & Algebra

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Irreducible Representations of $\mathfrak{S}_n$

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- the **trivial representation** corresponds to $\lambda = \square\square\square\square$.

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- the **trivial representation** corresponds to $\lambda = \ydiagram{4}$.
- the **alternating representation** corresponds to $\lambda = \ydiagram{1,1,1,1}$.

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- the **trivial representation** corresponds to $\lambda = \ydiagram{4}$.
- the **alternating representation** corresponds to $\lambda = \ydiagram{1,1,1,1}$.
- the **reflection representation** corresponds to $\lambda = \ydiagram{3,1}$.

More Specht modules

Specht modules can also be created from **skew partitions** λ/μ, but these are not irreducible representations.

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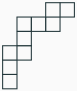
Definition (Ribbon Diagrams)

A **ribbon diagram** is a connected skew shape λ/μ with no 2×2 box that is a subset of the shape.

A ribbon diagram with size $|\lambda/\mu| = n$ can also be described by a composition of n, reading row lengths from top to bottom:

Example



$\longleftrightarrow (2, 3, 1, 2, 1, 1)$

Definition (Restriction)

Let ρ be a representation of a group G and let H be a subgroup of G. The restriction of ρ to H, $\rho|_H$, is the representation of H where for any $h \in H$ we have

$$\rho|_H(h) = \rho(h).$$

Restriction & Branching for $\mathfrak{S}_n$

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- We will mostly be considering the case where $G = \mathfrak{S}_n$ and $H = \mathfrak{S}_{n-1}$.
- Restriction is well-understood in this context for partitions, but is less well understood for skew-shapes.

Free Resolutions

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Definition (Minimal Free Resolution)

A **minimal free resolution** of a module M over a $\Bbbk$-algebra A is a complex

$$\cdots \xrightarrow{\partial_n} F_{n-1} \xrightarrow{\partial_{n-1}} F_{n-2} \xrightarrow{\partial_{n-2}} \cdots \xrightarrow{\partial_1} A \xrightarrow{\partial_0} M$$

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- none of the entries of the matrices for the maps ∂_i are units.

Minimality gives us information about the structure of our module M:

$$\cdots \to F_{n-1} \to \cdots \to F_2 \xrightarrow{\substack{\text{relations} \\ \text{on relations}}} F_1 \xrightarrow{\text{relations}} F_0 \xrightarrow{\text{generators}} M$$

Koszul Complex

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$$\partial_3 = \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix} \quad \partial_2 = \begin{pmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{pmatrix} \quad \partial_1 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

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If a group G acts on our algebra A, then the free modules in the minimal resolution of $\Bbbk$ correspond to representations of G.

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- To any Koszul algebra A, we can associate to it another quadratic algebra called its **Koszul dual**, denoted $A^!$.

$$A = \frac{T(V)}{\langle \mathcal{I} \rangle} \quad \longrightarrow \quad A^! = \frac{T(V^*)}{\langle \mathcal{J} \rangle}$$

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Example

The Koszul dual of $S = \mathbb{k}[x_1, \ldots, x_n]$ is the the exterior algebra $\bigwedge(e_1, \ldots e_n)$.

Koszul algebras

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Theorem (Priddy Complex [Pri70])

The graded components of $A^!$ assemble into the following minimal free resolution:

$$\cdots \to A \otimes (A^!)_3^* \to A \otimes (A^!)_2^* \to A \otimes (A^!)_1^* \to A \otimes (A^!)_0^* \to \mathbb{k} \to 0$$

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This gives us a means of computing the representations of $A^!$ given the representations of A. Moreover, this sequence gives us the following Hilbert series identity:

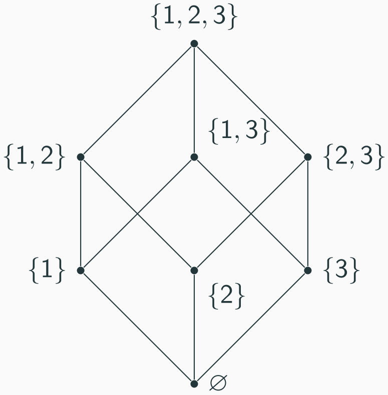
$$\mathrm{Hilb}(A^!, t) = \frac{1}{\mathrm{Hilb}(A, -t)}$$

The Boolean lattice

Definition (Boolean Lattice)

The Boolean lattice $\mathcal{B}_n$ is the set of subsets of $[n]$ ordered by containment.

Example ($\mathcal{B}_3$)



Koszul Algebras from $\mathcal{B}_n$

We consider the following three rings:

Stanley-Reisner ring	$\mathbb{k}[\Delta\mathcal{B}_n] = \dfrac{\mathbb{k}[x_F : F \in \mathcal{B}_n]}{\langle x_F x_G : F, G \text{ incomparable in } \mathcal{B}_n \rangle}$
Chow ring	$\mathrm{Chow}(\mathcal{B}_n) = \dfrac{\mathbb{k}[\Delta\mathcal{B}_n]}{\left\langle \sum_{e \in F} x_F : e \in [n] \right\rangle}$
Colorful ring	$\mathrm{colorful}(\mathcal{B}_n) = \dfrac{\mathbb{k}[\Delta\mathcal{B}_n]}{\langle \sum_{\vert F\vert = i} x_F : i \in [n] \rangle}$

The Stanley-Reisner ring for $\Delta(\mathcal{B}_n)$

Definition (Stanley-Reisner ring)

The Stanley-Reisner ring of the order complex of the Boolean lattice is

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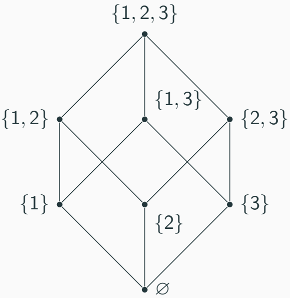
Example ($\Bbbk[\mathcal{B}_3]$)

$$\Bbbk[\Delta\mathcal{B}_3] = \frac{\Bbbk[x_1, x_2, x_3, x_{12}, x_{13}, x_{23}, x_{123}]}{\langle x_1x_2, x_1x_3, x_2x_3, x_{12}x_{13}, x_{12}x_{23}, x_{13}x_{23}, x_1x_{23}, x_2x_{13}, x_3x_{12} \rangle}$$



Chow Ring of $\mathcal{B}_n$

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The Chow ring of $\mathcal{B}_n$ is

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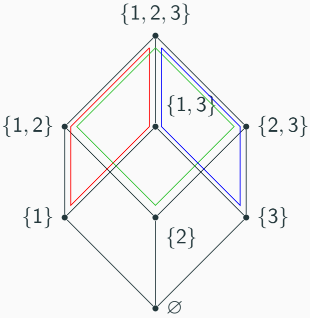
It is possible to obtain quadratic relations by replacing x_e with $x_e - \sum_{e \subsetneq F} x_F$ for all $e \in E$. This is the *atom-free presentation*, which we will use by default.

Example (Boolean lattice on three elements and $\mathrm{Chow}(\mathcal{B}_3)$)



$\mathrm{Chow}(\mathcal{B}_3)$

$$= \frac{\Bbbk[\Delta\mathcal{B}_3]}{\langle x_1 + x_{12} + x_{13} + x_{123},\ x_2 + x_{12} + x_{23} + x_{123},\ x_3 + x_{13} + x_{23} + x_{123} \rangle}$$

Colorful Ring of $\mathcal{B}_n$

Definition (Colorful Ring of $\mathcal{B}_n$)

The colorful ring of $\mathcal{B}_n$ is

$$\mathrm{colorful}(\mathcal{B}_n) := \frac{\Bbbk[\Delta\mathcal{B}_n]}{\langle \sum_{|F|=i} x_F : i \in [n] \rangle}$$

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$$\mathrm{colorful}(\mathcal{B}_n) := \frac{\Bbbk[\Delta \mathcal{B}_n]}{\langle \sum_{|F|=i} x_F : i \in [n] \rangle}$$

no atom-free presentation; choose to remove each $x_{[i]}$

Example

$$\mathrm{colorful}(\mathcal{B}_3) = \frac{\Bbbk[\Delta \mathcal{B}_3]}{\langle x_1 + x_2 + x_3, x_{12} + x_{13} + x_{23}, x_{123} \rangle}$$
$$= \frac{\Bbbk[x_2,\, x_3,\, x_{13},\, x_{23}]}{\langle x_2^2,\, x_3^2,\, x_{13}^2,\, x_{23}^2,\, x_2x_3,\, x_2x_{13},\, x_{13}x_{23},\, x_2x_{23} + x_3x_{23},\, x_3x_{13} + x_3x_{23} \rangle}$$

Combinatorial Interpretation

Example

$$\mathrm{colorful}(\mathcal{B}_3) = \frac{\Bbbk[\Delta\mathcal{B}_3]}{\langle x_1 + x_2 + x_3, x_{12} + x_{13} + x_{23}, x_{123}\rangle}$$

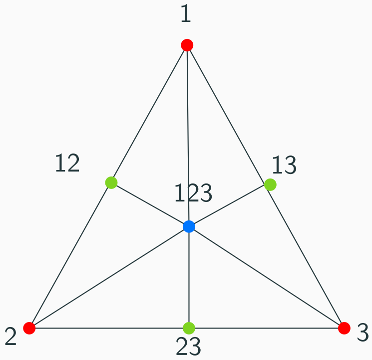


Figure: A 3-coloring of the barycentric subdivision of a 2-simplex

Hilbert Series of $\mathrm{Chow}(\mathcal{B}_n)$ and $\mathrm{colorful}(\mathcal{B}_n)$

The Chow ring and the colorful ring have the same Hilbert series: the dimension of the k-th graded component is given by the Eulerian number $\left\langle {n \atop k} \right\rangle$:

$$\mathsf{Hilb}(\mathrm{Chow}(\mathcal{B}_n), t) = \mathsf{Hilb}(\mathrm{colorful}(\mathcal{B}_n, t)) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle t^k$$

The Eulerian numbers count permutations in $\mathfrak{S}_n$ with k descents, and satisfy the recurrence

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle.$$

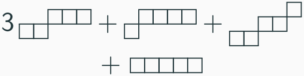
Results

Directions of Study

	$\mathrm{Chow}(\mathcal{B}_n)$	$\mathrm{colorful}(\mathcal{B}_n)$
dims	Eulerian numbers	Eulerian numbers
basis	Feichtner-Yuzvinsky [FY04]	descent monomials ([GS84] [DHKLT23+])
reps	Stembridge [Ste92]	ribbons [DHKLT23+]
reflects branching?	not really [DHKLT23+]	yes [DHKLT23+]
quadratic GB?	yes [Cor23]	yes [DHKLT23+]

	$\mathrm{Chow}(\mathcal{B}_n)^!$	$\mathrm{colorful}(\mathcal{B}_n)^!$
dims	recursive form [DHKLT23+]	recursive form [DHKLT23+]
basis	conj. [DHKLT23+]	TBE
reps	??? [DHKLT23+]	conj. $\oplus$ of ribbons [DHKLT23+]
reflects branching?	TBE	TBE
quadratic GB?	conj. non-quadratic [DHKLT23+]	TBE

Data Table for $\mathrm{Chow}(\mathcal{B}_5)$

degree	basis elements	skew representations	dimension
0	1		1
1	$x_{ij}, x_{ijk}, x_{ijkl}, x_{[5]}$		26
2	$x_{ij}x_{ijkl}, x_{ij}x_{[5]}, x_{ijk}^2, x_{ijk}x_{[5]}, x_{ijkl}^2, x_{[5]}^2$		66
3	$x_{ijk}^2 x_{[5]}, x_{ij}x_{[5]}^2, x_{[4]}^3, x_{[5]}^3$		26
4	$x_{[5]}^4$		1

Graded Components of Dual ($n = 3$)

degree	irreducible representations	dimension
0	$\ydiagram{3}$	1
1	$\ydiagram{2,1} + 2\,\ydiagram{3}$	4
2	$\ydiagram{1,1,1} + 5\,\ydiagram{2,1} + 4\,\ydiagram{3}$	15
3	$7\,\ydiagram{1,1,1} + 19\,\ydiagram{2,1} + 11\,\ydiagram{3}$	56
4	$32\,\ydiagram{1,1,1} + 70\,\ydiagram{2,1} + 37\,\ydiagram{3}$	209

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To be a permutation representation, the graded components should then be expressible in terms of $\ydiagram{2+1,1+1,1}$, $\ydiagram{1+2,1}$, and $\ydiagram{3}$.

Basis for graded components of $\mathrm{Chow}(\mathcal{B}_3)^!$

Theorem (DHKLT23+)

Let M_d be the set of all degree d monomials not in the ideal $\langle G \rangle$ where

$$G = \{z_{123}^2, z_{123}z_{23}^2\}$$

then M_d is a basis for the degree d component of $\mathrm{Chow}(\mathcal{B}_3)^!$.

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$$\mathrm{Chow}(\mathcal{B}_3)^! = \frac{\Bbbk\langle z_{12}, z_{13}, z_{23}, z_{123}\rangle}{\langle z_{123}^2 - z_{12}^2 - z_{13}^2 - z_{23}^2\rangle}$$

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$$(z_{123}^2 - z_{12}^2 - z_{13}^2 - z_{23}^2)z_{123} - z_{123}(z_{123}^2 - z_{12}^2 - z_{13}^2 - z_{23}^2)$$

$$\downarrow$$

$$z_{123}z_{12}^2 + z_{123}z_{13}^2 + z_{123}z_{23}^2 - z_{12}^2z_{123} - z_{13}^2z_{123} - z_{23}^2z_{123}$$

Example: Basis for graded components of $\mathrm{Chow}(\mathcal{B}_3)^!$

$$G = \{z_{123}^2, z_{123}z_{23}^2\}$$

Example

degree	(some) basis elements	dimension
0	1	1
1	$z_{12}, z_{13}, z_{23}, z_{123}$	4
2	$z_{12}^2,\ z_{12}z_{13}, \ldots$ ~~z_{123}^2~~	$4^2 - 1 = 15$
3	$z_{12}^3,\ z_{12}^2 z_{13}, \ldots$ ~~$z_{12}z_{123}^2, z_{13}z_{123}^2, z_{23}z_{123}^2$~~ ~~$z_{123}^3, z_{123}^2 z_{12}, z_{123}^2 z_{13}, z_{123}^2 z_{23}, z_{123}z_{23}^2$~~	$4^3 - 8 = 56$
$\vdots$	$\vdots$	$\vdots$

Conjecture (DHKLT23+)

Let $G = \bigcup_{i=1}^{4} G_i$ *where*

$$G_1 = \{z_F^2 : |F| > 2\}$$
$$G_2 = \{z_G z_H : H \subset G, |G| - |H| > 1, |H| \geq 2\}$$
$$G_3 = \{z_{ijk} z_{jk}^2 : i < j < k\}$$
$$G_4 = \{z_{F\cup ij} z_{F\cup j} z_F : i < j\}$$

Let M_d *be the set of degree* d *monomials not in* $\langle G \rangle$*. Then* M_d *is a basis for the degree* d *component of* $\mathrm{Chow}(\mathcal{B}_n)^!$*.*

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Colorful Basis Combinatorially: Descent Monomials

Definition ([GS84])

For $\sigma \in \mathfrak{S}_n$, we define the descent monomial of σ by

$$\eta(\sigma) = \prod_{\sigma(i+1)<\sigma(i)} x_{\sigma(1)\ldots\sigma(i)}$$

Note that the degree of $\eta(\sigma)$ is the number of descents in σ.

Colorful Basis Combinatorially: Descent Monomials

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Example

For the permutation $17832465 \in \mathfrak{S}_8$ with 3 descents, we get

$$17832465 \mapsto x_{178}x_{1378}x_{1234678},$$

a degree 3 monomial.

Descent Monomials Are a Basis of colorful($\mathcal{B}_n$)

Theorem

$\eta(\mathfrak{S}_n)$ *is a basis of* $\mathrm{colorful}(\mathcal{B}_n)$.

Example (Basis of $\mathrm{colorful}(\mathcal{B}_4)$)

0 descents	$\eta(\sigma)$	1 descent	$\eta(\sigma)$	2 descents	$\eta(\sigma)$	3 descents	$\eta(\sigma)$
1234	1	2134	x_2	2143	x_2x_{124}	4321	$x_4x_{34}x_{234}$
		3124	x_3	3214	x_3x_{23}		
		4123	x_4	3142	x_3x_{134}		
		1324	x_{13}	3241	x_3x_{234}		
		1423	x_{14}	4213	x_4x_{24}		
		2314	x_{23}	4312	x_4x_{34}		
		2413	x_{24}	4132	x_4x_{134}		
		3412	x_{34}	4231	x_4x_{234}		
		1243	x_{124}	1432	$x_{14}x_{134}$		
		1342	x_{134}	2431	$x_{24}x_{234}$		
		2341	x_{234}	3421	$x_{34}x_{234}$		

Colorful Basis

Theorem (DHKLT23+)

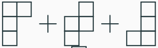
The following set is a quadratic Gröbner basis for the ideal of relations of $\mathrm{colorful}(\mathcal{B}_n)$:

$$\begin{aligned}&\{x_F x_G \mid X, G \text{ incomparable}, X, G \neq [i]\ \forall 1 \leq i \leq n\}\\ \cup&\{x_F^2 \mid F \neq [i]\ \forall 1 \leq i \leq n\}\\ \cup&\{x_G \sum_{|F|=i, F\subset G} x_F \mid [i] \not\subset G, |G| > i, 1 \leq i \leq n\}\\ \cup&\{x_G \sum_{|F|=i, G\subset F} x_F \mid G \not\subset [i], |G| < i, 1 \leq i \leq n\}.\end{aligned}$$

We do not have a conjecture for a basis for the graded components of $(A^!)_i$, but the above result is a first step in this direction!

Reps of $\mathrm{colorful}(\mathcal{B}_n)$: Ribbon Diagrams

The reps of $\mathrm{colorful}(\mathcal{B}_n)_k$ are given by the ribbon diagrams with n boxes and of length $k+1$:

n	degree	representations	dimension
3	0		1
3	1		4
3	2		1
4	0		1
4	1		11
4	2		11
4	3		1

A recurrence on the Eulerian numbers

Recall the recurrence for the Eulerian numbers $\left\langle {n \atop k} \right\rangle$:

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle.$$

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Question: *Can we categorify this recurrence at the level of representations with the Chow ring/colorful ring?*

Can partially do it for the graded components of the Chow ring, but only when $d = 0, 1, n-2, n-1$ for any n.

Example

For $A(n) := \mathrm{Chow}(\mathcal{B}_n)$, we have the short exact sequence

$$0 \to \mathcal{S}^{(n-1,1)/(1)} \otimes A(n-1)_0 \xrightarrow{i} A(n)_1 \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \xrightarrow{q} 2\,\mathcal{S}^{(n-1)} \otimes A(n-1)_1 \to 0.$$

Ribbon Branching

Theorem (Ribbon Branching Rule, DHKLT23+)

Let $\lambda/\mu := (a_1, ..., a_n)$ *be a ribbon and let* $(b_1, ..., b_n) := (\lambda/\mu)^T$*. Then,*

$$\mathcal{S}^{\lambda} \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} = \bigoplus_{i \ | \ a_i > 1} \mathcal{S}^{\lambda - e_i} \oplus \bigoplus_{i \ | \ b_i > 1} \mathcal{S}^{(\lambda^T - e_j)^T}$$

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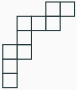
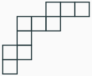
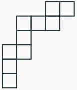
Example

Let $\lambda =$  $= (2, 3, 1, 2, 1, 1)$. Then $\lambda^T =$  $= (3, 3, 1, 2, 1)$.

The restriction of λ from an $\mathfrak{S}_{10}$-representation to a $\mathfrak{S}_9$-representation is given by

 $\Big\downarrow^{\mathfrak{S}_{10}}_{\mathfrak{S}_9} \cong$

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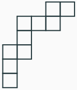
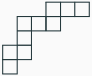
Let $\lambda =$  $= (2,3,1,2,1,1)$. Then $\lambda^T =$  $= (3,3,1,2,1)$.

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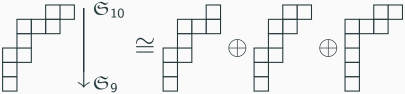

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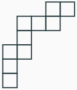
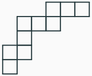
Let $\lambda =$  $= (2, 3, 1, 2, 1, 1)$. Then $\lambda^T =$  $= (3, 3, 1, 2, 1)$.

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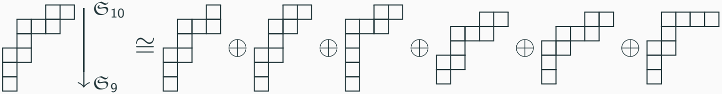

Colorful Ring Branching

Theorem (Colorful Branching Rule, DHKLT23+)

Let $A(n)$ be the ring $\mathrm{colorful}(\mathcal{B}_n)$. Then,

$$A(n)_k \downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \cong (n-k)A(n-1)_{k-1} \oplus (k+1)A(n-1)_k.$$

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Example

$$A(4)_1 = \ydiagram{3,1} \oplus \ydiagram{1+2,2} \oplus \ydiagram{2+1,3} = 2\left(\ydiagram{2,1} \oplus \ydiagram{1+1,2}\right) \oplus 3\,\ydiagram{1,1}.$$

Graded Components of Colorful(B_3) Dual

degree	representations
0	$\ydiagram{3}$
1	$\ydiagram{2,1} + \ydiagram{1+1,2}$
2	$3\,\ydiagram{1,1,1} + 2\left(\ydiagram{2,1} + \ydiagram{1+1,2}\right) + 4\,\ydiagram{3}$
3	$8\,\ydiagram{1,1,1} + 10\left(\ydiagram{2,1} + \ydiagram{1+1,2}\right) + 8\,\ydiagram{3}$
4	$36\,\ydiagram{1,1,1} + 34\left(\ydiagram{2,1} + \ydiagram{1+1,2}\right) + 37\,\ydiagram{3}$

Conjecture (DHKLT23+)

If $A(n) = \mathrm{colorful}(n)$, then $\mathrm{colorful}(n)^!_d$ is expressible in terms of a direct sum of graded components of $A(n)$.

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We would like to thank Ayah and Anastasia for their help and guidance, as well as Vic Reiner for helpful suggestions throughout this project. Thank you also to the UMN Math Department for hosting this REU!

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References

[Pri70] Stewart B. Priddy. "Koszul Resolutions". In: *Transactions of the American Mathematical Society* 152.1 (1970), pp. 39–60. ISSN: 00029947. URL: http://www.jstor.org/stable/1995637 (visited on 07/26/2023).

[GS84] A. M. Garsia and D. Stanton. "Group actions of Stanley-Reisner rings and invariants of permutation groups". In: *Adv. in Math.* 51.2 (1984), pp. 107–201. ISSN: 0001-8708. DOI: 10.1016/0001-8708(84)90005-7. URL: https://doi.org/10.1016/0001-8708(84)90005-7.

[Ste92] John R. Stembridge. "Eulerian numbers, tableaux, and the Betti numbers of a toric variety". In: *Discrete Mathematics* 99.1 (1992), pp. 307–320. ISSN: 0012-365X. DOI: https://doi.org/10.1016/0012-365X(92)90378-S. URL: https://www.sciencedirect.com/science/article/pii/0012365X9290378S.

[FY04] E. M. Feichtner and S. Yuzvinsky. "Chow rings of toric varieties defined by atomic lattices". In: *Invent. Math.* 155.3 (2004), pp. 515–536. ISSN: 0020-9910. DOI: 10.1007/s00222-003-0327-2. URL: https://doi-org.jpllnet.sfsu.edu/10.1007/s00222-003-0327-2.

[Cor23] Basile Coron. "Matroids, Feynman categories, and Koszul duality". In: (2023). arXiv: 2211.12370 [math.CO].