VIRTUAL RESOLUTIONS OF POINTS IN A PRODUCT OF PROJECTIVE SPACES

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Abstract. We consider finite sets of points in $\mathbb{P}^n \times \mathbb{P}^m$ and describe two methods for producing short virtual resolutions, as introduced by Berkesch, Erman, and Smith [BES20]. First, we describe an explicit virtual resolution for a set X of at least 12 points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^2$; this is a subcomplex of the free resolution for X. Additionally, we extend to $\mathbb{P}^n \times \mathbb{P}^m$ a result of Harada, Nowroozi, and Van Tuyl [HNVT22] for $\mathbb{P}^1 \times \mathbb{P}^1$ by intersecting with a sufficiently high power of one set of variables.

Contents

1. Introduction

For varieties in projective space, minimal free resolutions of the vanishing ideal give important geometric information. In particular, this geometric information includes the fact that all minimal free resolution have length bounded by the dimension of the ambient space. For varieties in multiprojective space, the vanishing ideals have lengths longer than the dimension of the ambient space. Introduced by Berkesch, Erman, and Smith in [BES20], virtual resolutions are shorter than minimal free resolutions and can have length bounded by the dimension of the ambient space. Crucially, virtual resolutions still include important geometric information given by minimal free resolutions. There is growing interest in producing families of short virtual resolutions, see [ABLS20, Dua20, BKLY21, GLLM21, Lop21, Yan21,

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BPC22, HNVT22, KLM+23, BE, BS, BP, FH, HHL]. Notably, virtual resolutions of varieties of points in $\mathbb{P}^1 \times \mathbb{P}^1$ have been studied by [HNVT22, BP].

In [BES20], Berkesch, Erman, and Smith defined a virtual resolution and identified two strategies for finding virtual resolutions for varieties in multiprojective space. The first of these two strategies, which we will call "virtual of a pair", corresponds to a subcomplex of the minimal free resolution obtained using multi-graded regularity, as introduced by Maclagan and Smith in [MS04]. The second strategy, which we will call "intersecting," involves finding the minimal free resolution of the vanishing ideal intersected with a power of the irrelevant ideal.

In this report, we consider finite sets of points $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$. Let $S = \Bbbk[x_0, \ldots, x_n, y_0, \ldots, y_m]$ denote the Cox ring of $\mathbb{P}^n \times \mathbb{P}^m$, where $\Bbbk$ is algebraically closed. The irrelevant ideal is

$$B := \langle x_0, \ldots, x_n \rangle \cap \langle y_0, \ldots, y_m \rangle.$$

Furthermore, let I_X denote the homogeneous vanishing ideal for X.

For points in $\mathbb{P}^1 \times \mathbb{P}^1$, Harada, Nowroozi, and Van Tuyl [HNVT22] used the virtual of a pair and intersecting approaches to describe virtual resolutions for points in $\mathbb{P}^1 \times \mathbb{P}^1$. In [BP], Booms-Peot extended Harada, Nowroozi, and Van Tuyl's results for the virtual of a pair approach and showed that most such virtual resolutions are of Hilbert–Burch type.

Our report extends results similar to [HNVT22] and [BP] to multiprojective spaces of higher dimensions. We present the following three main results:

Theorem (Theorem 3.5)**.** *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ be a finite set of points with generic Hilbert matrix such that $|X| \geq 12$. Assuming that Conjecture 5.4 holds, then there exists a virtual resolution of length 3 which can be explicitly described in terms of $|X|$.*

Theorem (Theorem 8.2)**.** *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^m$. Let s denote the number of unique second coordinates. For all $b \geq s - 1$, the minimal free resolution of $S/(I_X \cap \langle y_0, \ldots, y_m \rangle^b)$ is a virtual resolution of S/I_X of length $m + 1$.*

Theorem (Theorem 9.13)**.** *Let X be a set of points in $\mathbb{P}^n \times \mathbb{P}^m$. Let t denote the number of unique first coordinates. For all $a \geq t-1$, the minimal free resolution of $S/(I_X \cap \langle x_0, \ldots x_n \rangle^a)$ is a virtual resolution of S/I_X of length at most $n + m$ and at least $n + 1$.*

Outline. In Section 2, we introduce general background necessary for the study of virtual resolutions in products of projective spaces. Then in Section 3 we elaborate the particular background necessary for understanding our first method of finding virtual resolutions: virtual of a pair. In Section 4 we prove Proposition 4.3, a result analogous to [Giu92, Proposition 3.3] for $\mathbb{P}^n \times \mathbb{P}^m$. This result shows a relationship between Betti numbers in the minimal free resolution and the Hilbert function for an ideal of points. Next, in Section 5 we prove Theorem 3.5. Then in Section 6 we transition to introduce necessary background to understand our second method of finding virtual resolutions: intersections with the irrelevant ideal. In Section 7 we prove necessary preliminaries which are used in Section 8 to prove Theorem 8.2. Finally, in Section 9 we prove our main result using the intersection approach, Theorem 9.13, in full generality.

2. Background

In this section we will introduce some general background that will be used throughout the entire paper. We introduce more specific background to our two approaches in Section 3 and Section 6.

Recall that $S = \Bbbk[x_0, \ldots, x_n, y_0, \ldots, y_m]$ is the Cox ring of $\mathbb{P}^n \times \mathbb{P}^m$. The ring S is bigraded with $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. In general, a point $P = A \times B \in \mathbb{P}^n \times \mathbb{P}^m$ has the bihomogeneous vanishing ideal

$$I_P = \langle L_{A_1}, \cdots, L_{A_n}, L_{B_1}, \cdots, L_{B_m} \rangle \subset S,$$

where each L_{A_i} and L_{B_j} are linear forms in the x and y variables respectively. Throughout this document we will write $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$ to denote a finite set of points. If

$$X = \{P_1, \cdots, P_s\} = \{A_1 \times B_1, \cdots, A_s \times B_s\},$$

the defining ideal of X is $I_X = I_{P_1} \cap \cdots \cap I_{P_s}$, and we define $\pi_1(X) = \{A_1, \cdots, A_s\} \subseteq \mathbb{P}^n$, $\pi_2(X) = \{B_1, \cdots, B_s\} \subseteq \mathbb{P}^m$.

Definition 2.1. [HNVT22] Let M be a finitely generated S-module, and let

$$\Gamma_B(M) \coloneqq \{m \in M \mid B^t m = 0 \text{ for some } t \in \mathbb{N}\}.$$

Let

$$F_\bullet \coloneqq \cdots \to F_2 \to F_1 \to F_0$$

be a complex of finitely generated free S-modules satisfying the following:

(1) For each $i > 0$ there is some t such that $B^t H_i(F_\bullet) = 0$, and
(2) $H_0(F_\bullet)/\Gamma_B(H_0(F_\bullet)) \cong M/\Gamma_B(M)$.

Then $F_\bullet$ is a *virtual resolution* of M.

Definition 2.2. [HNVT22] For any bihomogeneous ideal $I \subseteq S$, the *Hilbert function* $H_{S/I}$ of S/I is the function $H_{S/I} : (\mathbb{Z}_{\geq 0})^2 \to \mathbb{Z}_{\geq 0}$ defined by

$$H_{S/I}(i, j) := \dim_\Bbbk (S/I)_{i,j} = \dim_\Bbbk S_{i,j} - \dim_\Bbbk I_{i,j}$$

where $M_{i,j}$ denotes the (i, j)-graded component of a bigraded S-module M.

If I_X is the defining ideal of a subvariety X of $\mathbb{P}^n \times \mathbb{P}^m$, denote H_{S/I_X} by H_X. View H_X as an infinite matrix with entries $H_X(i, j)$.

Let R be the Cox ring of $\mathbb{P}^n$. For a subvariety $X \subseteq \mathbb{P}^n$, denote the Hilbert function of R/I_X by $H_X(j) \coloneqq \dim_\Bbbk R_j - \dim_\Bbbk [I_X]_j$, where I_X is the defining ideal of X.

In Section 3 and Section 5, we will also need the notion of the multigraded regularity of a set of points $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$, denoted by $\operatorname{reg}(S/I(X))$. We will not give the definition of multigraded regularity here, because for the purpose of this paper, it suffices to use a characterization of $\operatorname{reg}(S/I(X))$ in terms of H_X from [MS04], which will be stated in Lemma 3.4.

The following is a well known result for the Hilbert function of a finite set of points in $\mathbb{P}^n$.

Lemma 2.3. [VT02, Proposition 1.2] *For a finite set of distinct points $X \subseteq \mathbb{P}^n$ we have $H_X(i) = |X|$ for all $i \geq |X| - 1$.*

In the case of multiprojective spaces, the following result (a special case of [VT02, Proposition 4.2]) gives more information about the "border" of the Hilbert function H_X of a set of points. This will be used in Remark 3.7.

Lemma 2.4. [VT02, Proposition 4.2] *Let X be a set of s distinct points in $\mathbb{P}^n \times \mathbb{P}^m$, and suppose that $\pi_1(X) = \{P_1, \dots, P_{t_1}\}$ is the set of $t_1 \leq s$ distinct first coordinates in X. Fix any $j \in \mathbb{Z}_{\geq 0}$; then for all integers $l \geq t_1 - 1$, we have*

$$H_X(l, j) = \sum_{P_i \in \pi_1(X)} H_{Q_{P_i}}(j)$$

where $H_{Q_{P_i}}$ is the Hilbert function of $Q_{P_i} := \pi_2(\pi_1^{-1}(P_i)) \subseteq \mathbb{P}^m$.

Corollary 2.5. *If $X \subset \mathbb{P}^n \times \mathbb{P}^m$ is a finite set of points with mutually distinct first coordinates, then we have*

$$H_X(s-1, 0) = s$$

Proof. This follows from Lemma 2.3 and Lemma 2.4 since each Q_{P_i} now consists of a single point in $\mathbb{P}^m$. □

3. The virtual resolution of a pair approach

One of the methods given in Berkesch, Erman, and Smith's paper for constructing virtual resolutions is the following theorem ([BES20, Theorem 1.3]):

Theorem 3.1. *Let $S = \Bbbk[x_0, \dots, x_n, y_0, \dots, y_m]$, let M be a finitely generated $\mathbb{Z}^2$-graded B-saturated S-module, and let $d \in \mathbb{Z}^2$ be such that M is d-regular. If G is the free subcomplex of a minimal free resolution F of M consisting of all summands of F in degree at most $d + (n, m)$, then G is a virtual resolution of M.*

As in [BES20], we call the virtual resolution obtained in the above theorem the *virtual resolution of the pair* (M, d).

In [HNVT22], the authors used Theorem 3.1 to construct virtual resolutions of finite sets of points $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. Among other things, they showed that when the points are in sufficiently general position, the virtual resolution obtained by taking $d = (|X| - 1, 0)$ in Theorem 3.1 is always of length 2. In this section we generalize this result to $\mathbb{P}^1 \times \mathbb{P}^2$. The proof in [HNVT22] for the case $\mathbb{P}^1 \times \mathbb{P}^1$ relies on the fact that when X is in sufficiently general position, the Hilbert matrix H_X is uniquely determined by $|X|$ and is of the form $H_X(i, j) = \min\{(i + 1)(j + 1), |X|\}$. Computer experiments in `Macaulay2` [GS] suggest that this form of the "generic" Hilbert matrix can be generalized to $\mathbb{P}^n \times \mathbb{P}^m$:

Conjecture 3.2. For every $s \geq 1$ and all $n, m \geq 1$, there exists a dense open subset $U \subseteq (\mathbb{P}^n \times \mathbb{P}^m)^s$, such that for any $X = \{P_1, ..., P_s\} \subseteq \mathbb{P}^n \times \mathbb{P}^m$, if $(P_1, ..., P_s) \in U$, then $H_X(i, j) = \min\{|X|, T_{i,n} T_{j,m}\}$ for all $i, j \geq 0$, where

$$T_{i,n} := \binom{i+n}{n}$$

is equal to the number of monomials of degree i in $k[x_0, \dots, x_n]$.

Definition 3.3. For a set of points $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$, whenever H_X has the form given in Conjecture 3.2, we say that X has generic Hilbert matrix, and say that X is a set of points in sufficiently general position.

The following lemma is a special case of [MS04, Proposition 6.7].

Lemma 3.4. *Let $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be a finite set of points in sufficiently general position. Then we have*

$$\operatorname{reg}(S/I(X)) = \{(i,j) \mid H_X(i,j) = |X|\}$$

In particular, we always have $(|X|-1, 0) \in \operatorname{reg}(S/I(X))$.

Our main result of the virtual of a pair approach is the following theorem, which will be proved in Section 5. The conclusion also relies on another conjecture (Conjecture 5.4), which is a version of the Minimal Resolution Conjecture. Roughly speaking, this conjecture asserts that when X is a set of points in sufficiently general position, the Betti numbers of X do not overlap with each other. For more details, we refer the reader to the discussion in Section 5 preceding Conjecture 5.4.

Theorem 3.5. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ be a finite set of points with generic Hilbert matrix, and assume $n = |X| \geq 12$. If we assume that Conjecture 5.4 is true, then S/I_X has the following virtual resolution of length 3, obtained by taking $d = (n-1, 0)$ in Theorem 3.1:*

$$0 \to S(-n,-2)^3 \to \begin{matrix} S(-m',-2)^{9-3r'} \\ \oplus \\ S(-m'-1,-2)^{3r'} \\ \oplus \\ S(-n,-1)^3 \end{matrix} \to \begin{matrix} S(-m,-2)^{6-r} \\ \oplus \\ S(-m-1,-2)^{r} \\ \oplus \\ S(-m',-1)^{3-r'} \\ \oplus \\ S(-m'-1,-1)^{r'} \\ \oplus \\ S(-n,0) \end{matrix} \to S \to S/I_X \to 0,$$

where $n = 6m + r = 3m' + r'$ with $0 \leq r \leq 5$, $0 \leq r' \leq 2$. When r (respectively, r') is zero, the term $S(-m-1,-2)^r$ (respectively, $S(-m'-1,-1)^{r'}$ and $S(-m'-1,-2)^{3r'}$) do not appear in the virtual resolution.

Remark 3.6. In the case that $2 \leq n \leq 11$, if we assume Conjecture 5.4, then it is still true that S/I_X has a virtual resolution of length 3 obtained by taking $d = (n-1, 0)$ in Theorem 3.1, and the proof is analogous to Theorem 3.5; however, this virtual resolution can no longer be explicitly given by a general form as in Theorem 3.5

Remark 3.7. Without assuming Conjecture 3.2, if we assume that the points in X have mutually distinct first coordinates, then by Corollary 2.5 and [MS04, Proposition 6.7], we can still conclude that $(|X|-1, 0) \in \operatorname{reg}(S/I(X))$. Therefore we can still obtain a virtual resolution by taking $d = (|X|-1, 0)$ in Theorem 3.1. Moreover, if we assume a version of the Minimal Resolution Conjecture that is stronger than Conjecture 5.4 (see Section 5 for details), then we can still show that the virtual resolution obtained in this way has length 3. However, giving an explicit form of this virtual resolution would require both Conjecture 3.2 and Conjecture 5.4.

4. Difference matrices and Betti numbers

To prove Theorem 3.5, we need to give an explicit description of the Betti numbers of X. When $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, then [Giu92, Proposition 3.3] gives a useful relationship between the Betti numbers of X and the second difference matrix of H_X. The authors of [HNVT22] and [BP] then exploited this relationship in their study of virtual resolutions for sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. We generalize this relationship to arbitrary sets of points in $\mathbb{P}^n \times \mathbb{P}^m$; namely, we show that in $\mathbb{P}^n \times \mathbb{P}^m$, the right matrix to look at for Betti numbers is the difference matrix $(\Delta^C)^{n+1}(\Delta^R)^{m+1}H_X$ defined as follows:

Definition 4.1. Define a partial order on $\mathbb{Z}^2$ by letting $(i,j) \geq (i',j')$ if $i \geq i'$ and $j \geq j'$. For any infinite matrix $M = (M_{ij})$ indexed by $(i,j) \geq (0,0)$, we define the column difference operator Δ^C and row difference operator Δ^R by

$$\Delta^C(M)_{ij} := M_{ij} - M_{i-1,j}$$
$$\Delta^R(M)_{ij} := M_{ij} - M_{i,j-1}$$

where M_{ij} is taken to be 0 if $i < 0$ or $j < 0$.

Remark 4.2. Following [Giu92], we define the first and second difference operators Δ and Δ^2 by

$$\Delta(M)_{ij} = \Delta^C\Delta^R(M)_{ij}$$
$$\Delta^2(M)_{ij} = \Delta(\Delta(M))_{ij}$$

One can check that the operators Δ^C and Δ^R commute, i.e. we have $\Delta^C\Delta^R = \Delta^R\Delta^C$.

The following result relates the alternating sum of the Betti numbers of X to a certain difference matrix of X.

Proposition 4.3. *For any finite collection of points $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$, if we write the minimal free resolution of S/I_X as*

$$0 \to \bigoplus_{i,j\geq 0} S(-i,-j)^{\beta_{n+m+1,(i,j)}} \to \ldots \to \bigoplus_{i,j\geq 0} S(-i,-j)^{\beta_{1,(i,j)}} \to S \to S/I_X \to 0$$

then the Hilbert matrix H of X satisfies

$$\left((\Delta^C)^{n+1}(\Delta^R)^{m+1}H\right)_{ij} = \sum_{r=1}^{n+m+1} (-1)^r \beta_{r,(i,j)}$$

Remark 4.4. It follows from [VT02, Lemma 3.3] that for any finite collection of points $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$, there exists a non-zero-divisor $L \in S$ for S/I_X, such that $\deg(L) = (1,0)$. Consequently $\operatorname{depth}(S/I_X) \geq 1$, so any minimal free resolution of S/I_X has length at most $n+m+1$.

The key to proving this proposition is the following lemma.

Lemma 4.5. *Under the assumptions and notations of Proposition 4.3, we have*

$$H_{ij} = T_{i,n}T_{j,m} + \sum_{(h,k)\leq(i,j)} T_{i-h,n}T_{j-k,m}\left(\sum_{r=1}^{n+m+1}(-1)^r\beta_{r,(h,k)}\right)$$

Remark 4.6. Recall from Conjecture 3.2 that we defined $T_{i,n} := \binom{i+n}{n}$, and we have $T_{i,n}T_{j,m} = \dim_k S_{i,j}$, where $S_{i,j}$ denotes the (i,j)-th graded component of S.

Proof. We abbreviate the minimal free resolution of S/I_X in the statement of Proposition Proposition 4.3 as

$$0 \to F_{n+m+1} \to \dots \to F_1 \to F_0 \to S/I_X \to 0$$

where $F_0 = S$. Since this is an exact sequence in the category of bigraded S-modules, for any $(i,j) \in \mathbb{Z}^2$, we get an exact sequence of $\Bbbk$-vector spaces by restricting to degree (i,j). This implies that

$$\dim_{\Bbbk}(S/I_X)_{i,j} = \sum_{r=0}^{n+m+1} (-1)^r \dim_{\Bbbk}(F_r)_{i,j} \tag{4.1}$$

By Remark 4.6, we have $\dim_{\Bbbk} S_{i,j} = T_{i,n}T_{j,m}$. Since $F_r = \bigoplus_{h,k\geq 0} S(-h,-k)^{\beta_{r,(h,k)}}$ for all $1 \leq r \leq n+m+1$, we have

$$(F_r)_{i,j} = \bigoplus_{(h,k)\leq(i,j)} (S(-h,-k)_{i,j})^{\beta_{r,(h,k)}} = \bigoplus_{(h,k)\leq(i,j)} (S_{i-h,j-k})^{\beta_{r,(h,k)}}$$

and therefore $\dim_{\Bbbk}(F_r)_{i,j} = \sum_{(h,k)\leq(i,j)} T_{i-h,n}T_{j-k,m}\beta_{r,(h,k)}$. Substituting this into Equation (4.1) yields the desired result. $\square$

Using Lemma 4.5, we can now describe the action of Δ^C and Δ^R on H explicitly. For the ease of notation, we will write:

$$B_{hk} := \sum_{r=1}^{n+m+1} (-1)^r \beta_{r,(h,k)}$$

Proposition 4.7. *Under the assumptions and notations of Proposition 4.3, we have*

$$(\Delta^C H)_{ij} = T_{i,n-1}T_{j,m} + \sum_{(h,k)\leq(i,j)} T_{i-h,n-1}T_{j-k,m}B_{hk}$$

Similarly, for Δ^R we have

$$(\Delta^R H)_{ij} = T_{i,n}T_{j,m-1} + \sum_{(h,k)\leq(i,j)} T_{i-h,n}T_{j-k,m-1}B_{hk}$$

Proof. We prove the proposition for Δ^C. The proof for Δ^R is analogous. By Lemma 4.5,

$$H_{ij} = T_{i,n}T_{j,m} + \sum_{(h,k)\leq(i,j)} T_{i-h,n}T_{j-k,m}B_{hk}$$

Here we separate the sum into two parts, one where $h = i$ and $j \leq k$, the other with $(h,k) \leq (i-1,j)$. When $h = i$, $T_{i-h,n} = T_{0,n} = 1$, so

$$H_{ij} = T_{i,n}T_{j,m} + \sum_{k\leq j} T_{j-k,m}B_{ik} + \sum_{(h,k)\leq(i-1,j)} T_{i-h,n}T_{j-k,m}B_{hk}$$

Using Lemma 4.5 again for $H_{i-1,j}$,

$$H_{i-1,j} = T_{i-1,n}T_{j,m} + \sum_{(h,k)\leq(i-1,j)} T_{i-1-h,n}T_{j-k,m}B_{hk}$$

Now by definition of Δ^C, we have

$$
\begin{aligned}
(\Delta^C H)_{ij} &= H_{ij} - H_{i-1,j} \\
&= (T_{i,n} - T_{i-1,n})T_{j,m} + \sum_{k\le j} T_{j-k,m}B_{ik} + \sum_{(h,k)\le(i-1,j)} (T_{i-h,n} - T_{i-1-h,n})T_{j-k,m}B_{hk}
\end{aligned}
$$

Using the binomial identities $T_{i,n} - T_{i-1,n} = T_{i,n-1}$ and $T_{i-h,n} - T_{i-1-h,n} = T_{i-h,n-1}$, the expression above simplifies to:

$$(\Delta^C H)_{ij} = T_{i,n-1}T_{j,m} + \sum_{k\le j} T_{j-k,m}B_{ik} + \sum_{(h,k)\le(i-1,j)} T_{i-h,n-1}T_{j-k,m}B_{hk}$$

Combining the two sums, we get:

$$(\Delta^C H)_{ij} = T_{i,n-1}T_{j,m} + \sum_{(h,k)\le(i,j)} T_{i-h,n-1}T_{j-k,m}B_{hk}. \qquad \square$$

Corollary 4.8. *By repeatedly applying Δ^C and Δ^R on H, we have:*

$$
\begin{aligned}
\big((\Delta^C)^n(\Delta^R)^m H\big)_{ij} &= T_{i,0}T_{j,0} + \sum_{(h,k)\le(i,j)} T_{i-h,0}T_{j-k,0}B_{hk} \\
&= 1 + \sum_{(h,k)\le(i,j)} B_{hk}
\end{aligned}
$$

Proposition 4.3 now follows by applying each of Δ^C and Δ^R for one more time.

Proof of Proposition 4.3. Let $M = (\Delta^C)^n(\Delta^R)^m H$, so that:

$$M_{ij} = 1 + \sum_{(h,k)\le(i,j)} B_{hk}$$

Then we have

$$\Delta^C(M)_{ij} = M_{ij} + M_{i-1,j} = \sum_{(h,k)\le(i,j)} B_{hk} - \sum_{(h,k)\le(i-1,j)} B_{hk} = \sum_{k\le j} B_{ik}$$

And therefore

$$(\Delta^R\Delta^C M)_{ij} = \sum_{k\le j} B_{ik} - \sum_{k\le j-1} B_{ik} = B_{ij}.$$

$\square$

5. Explicit form of a virtual resolution in $\mathbb{P}^1 \times \mathbb{P}^2$

In this section we use the results of Section 4 to prove our main result for finding the virtual resolution of a pair: Theorem 3.5.

Definition 5.1. Given a matrix $M(i,j)$ with indices $i, j \in \mathbb{Z}_{\ge 0}$, we define

$$DM(i,j) = ((\Delta^C)^2(\Delta^R)^3 M)(i,j)$$

for all $(i,j) \neq (0,0)$. We define $DM(0,0) = 0$.

Taking $n = 1$ and $m = 2$ in Proposition 4.3, we get the following:

Proposition 5.2. *For any finite collection of points $X \subseteq \mathbb{P}^1 \times \mathbb{P}^2$, if we write the minimal free resolution of S/I_X as*

$$0 \to \bigoplus_{i,j \geq 0} S(-i,-j)^{\beta_{4,(i,j)}} \to \ldots \to \bigoplus_{i,j \geq 0} S(-i,-j)^{\beta_{1,(i,j)}} \to S \to S/I_X \to 0$$

then for all $i, j \geq 0$, we have

$$DH_X(i,j) = -\beta_{1,(i,j)} + \beta_{2,(i,j)} - \beta_{3,(i,j)} + \beta_{4,(i,j)}$$

The following lemma computes a submatrix of DH_X which will later be used for determining the virtual resolution of S/I_X obtained by taking $d = (|X|-1, 0)$ in Theorem Theorem 3.1.

Lemma 5.3. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ be a collection of points with generic Hilbert matrix (i.e. $H_X(i,j) = \min\{(i+1)\binom{j+2}{2}, |X|\}$), and suppose that $n = |X| \geq 12$. Let H'_X denote the upper-left $(n+1) \times 3$ submatrix of H_X, so that $DH'_X(i,j) = DH_X(i,j)$ for all $0 \leq i \leq n$, $0 \leq j \leq 2$. Write $n = 6m + r = 3m' + r'$, where $0 \leq r \leq 5$, $0 \leq r' \leq 2$. Then*

1. *All the non-zero entries of DH'_X only appear in the rows with $i = m$, $m+1$, m', $m'+1$, or n;*
2. *We have*

$$\begin{bmatrix} DH'_X(m,*) \\ DH'_X(m+1,*) \end{bmatrix} = \begin{bmatrix} 0 & 0 & r-6 \\ 0 & 0 & -r \end{bmatrix}$$

$$\begin{bmatrix} DH'_X(m',*) \\ DH'_X(m'+1,*) \end{bmatrix} = \begin{bmatrix} 0 & r'-3 & 9-3r' \\ 0 & -r' & 3r' \end{bmatrix}$$

and $DH'_X(n,*) = [-1, 3, -3]$, where $M(i,*)$ denotes the i-th row vector of a matrix.

Proof. Recall the definition of the operators Δ and Δ^2 from Remark 4.2. Since $H'_X(0,*) = [1,3,6]$, we have $\Delta H'_X(0,*) = [1,2,3]$, and therefore $DH'_X(0,*) = [0,0,0]$. Note that we have

$$\begin{bmatrix} H'_X(0,*) \\ H'_X(1,*) \\ \vdots \\ H'_X(m-1,*) \\ H'_X(m,*) \\ H'_X(m+1,*) \\ \vdots \\ H'_X(m'-1,*) \\ H'_X(m',*) \\ H'_X(m'+1,*) \\ \vdots \\ H'_X(n-1,*) \\ H'_X(n,*) \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 2 & 6 & 12 \\ \vdots & \vdots & \vdots \\ m & 3m & 6m \\ m+1 & 3m+3 & n \\ m+2 & 3m+6 & n \\ \vdots & \vdots & \vdots \\ m' & 3m' & n \\ m'+1 & n & n \\ m'+2 & n & n \\ \vdots & \vdots & \vdots \\ n & n & n \\ n & n & n \end{bmatrix}$$

which implies that

$$\begin{bmatrix} \Delta H'_X(0,*) \\ \Delta H'_X(1,*) \\ \vdots \\ \Delta H'_X(m-1,*) \\ \Delta H'_X(m,*) \\ \Delta H'_X(m+1,*) \\ \vdots \\ \Delta H'_X(m'-1,*) \\ \Delta H'_X(m',*) \\ \Delta H'_X(m'+1,*) \\ \vdots \\ \Delta H'_X(n-1,*) \\ \Delta H'_X(n,*) \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ \vdots & \vdots & \vdots \\ 1 & 2 & 3 \\ 1 & 2 & r-3 \\ 1 & 2 & -3 \\ \vdots & \vdots & \vdots \\ 1 & 2 & -3 \\ 1 & r'-1 & -r' \\ 1 & -1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Notice in the above matrix that for all $i \geq 1$ such that $i \notin \{m, m+1, m', m'+1, n\}$, we have $\Delta H'_X(i-1,*) = \Delta H'_X(i,*)$, which implies that $DH'_X(i,*) = [0,0,0]$. The conclusion of the lemma now follows from calculating $DH'_X(i,*)$ for $i \in \{m, m+1, m', m'+1, n\}$ using the above matrix. $\square$

Ideally, we would like to say that under certain assumptions (for example, when X has generic Hilbert matrix), the matrix DH_X completely determines all Betti numbers of X. This amounts to saying that a certain version of the Minimal Resolution Conjecture (MRC) holds for sets of points in $\mathbb{P}^1 \times \mathbb{P}^2$. Originally formulated in [Lor93] for points in $\mathbb{P}^n$, the MRC states that for a set of points in sufficiently general position, no overlapping of Betti numbers can happen. Later, [Mus98] generalized the MRC to sets of points in arbitrary projective varieties. For a discussion of the history of the MRC, and the cases in which either it has been proved or counterexamples have been found, we refer the reader to the introduction section of [BP].

In [Giu96], the authors proved that the MRC holds for all sufficiently general sets of points lying on a smooth quadric in $\mathbb{P}^3$ (and hence for points in $\mathbb{P}^1 \times \mathbb{P}^1$); more specifically, [Giu96, Theorem 4.3] says that the first Betti numbers of a set of points $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ are entirely predicted by the matrix $\Delta^2 H_X$. Recent studies of virtual resolutions for points in $\mathbb{P}^1 \times \mathbb{P}^1$, for example, [HNVT22] and [BP], largely rely on this result. However, the techniques that the authors of [Giu96] used are highly specific to the geometry of $\mathbb{P}^3$ and therefore difficult to generalize to higher dimensions.

For the purpose of our study, we state the following weakened version of the Minimal Resolution Conjecture for points in $\mathbb{P}^1 \times \mathbb{P}^2$. Rather than using the full strength of the MRC, we only require that the Betti numbers $\beta_{1,(i,j)}$ are entirely predicted by DH_X, and that the Betti numbers $\beta_{2,(i,j)}$ and $\beta_{3,(i,j)}$ do not overlap. This would suffice for proving Theorem 3.5.

Conjecture 5.4. If $X \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ is a set of points in sufficiently general position, then for every fixed $(i,j) > (0,0)$, the following holds:
(1) We have $\beta_{1,(i,j)} > 0$ if and only if

$$DH_X(i,j) < 0 \text{ and } DH_X(i',j') \leq 0 \text{ for all } (i',j') \leq (i,j)$$

and whenever the above condition holds, we have $\beta_{1,(i,j)} = -DH_X(i,j)$;
(2) At most one of $\beta_{2,(i,j)}$ and $\beta_{3,(i,j)}$ is non-zero.

Remark 5.5. Using Macaulay2, for each $2 \leq s \leq 100$, we generated 50 random sets of s points in $\mathbb{P}^1 \times \mathbb{P}^2$ in sufficiently general position. It has been verified that Conjecture 5.4 holds for all of these examples.

Lemma 5.6. *For any $(i,j) \geq (0,0)$ and $k \in \{2,3,4\}$, if $\beta_{k-1,(r,s)} = 0$ for all $(r,s) < (i,j)$, then $\beta_{k,(i,j)} = 0$.*

Proof. This follows from the fact that in a graded minimal free resolution, any non-zero Betti number $\beta_{k,(i,j)}$ records a syzygy of the syzygies (or minimal generators in the case that $k = 2$) of degree strictly less than (i,j) in homological degree $k-1$. □

Corollary 5.7. *For any $(i,j) > (0,0)$, if $DH_X(i',j') = 0$ for all $(i',j') < (i,j)$, then $\beta_{2,(i,j)} = \beta_{3,(i,j)} = \beta_{4,(i,j)} = 0$, and $\beta_{1,(i,j)} = -DH_X(i,j)$.*

Proof. Assuming that Conjecture 5.4(1) holds, we know that $\beta_{1,(i',j')} = 0$ for all $(i',j') < (i,j)$. The result then follows from repeatedly applying Lemma 5.6 to every point (i',j') in the region $(i',j') < (i,j)$. □

Proof of Theorem 3.5. Since X has generic Hilbert matrix, it follows from [MS04, Proposition 6.7] that we have $(|X|-1,0) \in \operatorname{reg}(S/I_X)$; therefore, by Theorem 3.1, the subcomplex of a minimal free resolution of S/I_X consisting of all summands generated in degree at most $(|X|,2)$ would be a virtual resolution of S/I_X.

First of all, the value of $\beta_{1,(i,j)}$ for all $(0,0) \leq (i,j) \leq (n,2)$ are determined by Conjecture 5.4(1) and Lemma 5.3. Since we always have $DH_X(i,0) = 0$ for all $i < n$ by Lemma 5.3, it follows from Corollary 5.7 that $\beta_{2,(i,0)} = \beta_{3,(i,0)} = \beta_{4,(i,0)} = 0$ for all $i \leq n$. By a similar argument, we have $\beta_{2,(i,1)} = \beta_{3,(i,1)} = \beta_{4,(i,1)} = 0$ for all $i \leq m'$. Lemma 5.6 then implies that $\beta_{3,(m'+1,1)} = \beta_{4,(m'+1,1)} = 0$; since $\beta_{1,(m'+1,1)} = DH_X(m'+1,1) = -r' < 0$, Proposition 5.2 forces $\beta_{2,(m'+1,1)} = 0$. By a similar reasoning, we can show inductively that $\beta_{2,(i,1)} = \beta_{3,(i,1)} = \beta_{4,(i,1)} = 0$ for all $i \leq n-1$, and it follows from Lemma 5.6, Proposition 5.2, and Lemma 5.3 that $\beta_{2,(n,1)} = 3$ and $\beta_{3,(n,1)} = \beta_{4,(n,1)} = 0$.

Using the same kind of argument, we can show that $\beta_{2,(i,2)} = \beta_{3,(i,2)} = \beta_{4,(i,2)} = 0$ for all $i < m'$. Lemma 5.6 then implies that $\beta_{3,(m',2)} = \beta_{4,(m',2)} = 0$, and Conjecture 5.4(1) implies that $\beta_{1,(m',2)} = 0$ (since $DH_X(m',2) = 9 - 3r' > 0$); therefore by Proposition 5.2 we have $\beta_{2,(m',2)} = 9 - 3r'$. By Lemma 5.6, we have $\beta_{4,(m'+1,2)} = 0$, and since $DH_X(m'+1,2) = 3r' \geq 0$ we also have $\beta_{1,(m'+1,2)} = 0$; by Conjecture 5.4(2), only one of $\beta_{2,(m'+1,2)}$ and $\beta_{3,(m'+1,2)}$ can be non-zero, and Proposition 5.2 implies that the non-zero one must be $\beta_{2,(m'+1,2)} = 3r'$. A similar argument shows that $\beta_{2,(i,2)} = \beta_{3,(i,2)} = \beta_{4,(i,2)} = 0$ for all $m'+2 \leq i \leq n-1$, $\beta_{2,(n,2)} = \beta_{4,(n,2)} = 0$, and $\beta_{3,(n,2)} = 3$. The theorem now follows from Theorem 3.1. □

It is worth noting however, that the virtual of a pair approach does not always yield a short virtual resolution. Specifically, for other corners of $\operatorname{reg}(S/I(X))$ the length of the corresponding virtual resolution may be longer than the dimension of the ambient space. In [BP], Booms-Peot showed that in $\mathbb{P}^1 \times \mathbb{P}^1$ under certain conditions virtual of a pair yields a virtual resolution of length 3. The same issue persists in $\mathbb{P}^1 \times \mathbb{P}^2$, in the following example we will show a virtual resolution of length 4 corresponding to performing virtual of a pair for 31 points in sufficiently general position.

Example 5.8. For $X \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ with $|X| = 31$ with points in sufficiently general position the minimal free resolution is enormous, with 12 summands in homological degree 2. It has therefore been omitted. Shown here is the virtual resolution corresponding to performing virtual of a pair for the regularity pair $(2, 4)$.

$$0 \to S(-3,-6)^6 \to \begin{matrix} S(-3,-5)^{24} \\ \oplus \\ S(-2,-6)^{15} \end{matrix} \to \begin{matrix} S(-3,-4)^{26} \\ \oplus \\ S(-2,-5)^{32} \\ \oplus \\ S(-1,-6)^{8} \end{matrix} \to \begin{matrix} S(-3,-3)^{9} \\ \oplus \\ S(-2,-4)^{14} \\ \oplus \\ S(-1,-5)^{11} \end{matrix} \to S \to S/I_X \to 0.$$

$\diamond$

6. The intersection approach

We now transition to considering a second method for constructing short virtual resolutions. This section introduces the big ideas of the intersection approach, then Section 7 discusses specific preliminaries used in Section 8. Finally, we prove an even more general result in Section 9.

In [BES20], authors Berkesch, Erman, and Smith proved the existence of short virtual resolutions using a second method which uses the minimal free resolution of a slightly different module than the original. We cite here a special case of their result:

Theorem 6.1. [BES20, Theorem 4.1] *Let X be a finite set of points in $\mathbb{P}^n \times \mathbb{P}^m$. Then for all $a \gg 0$, the minimal free resolution of $S/(I_X \cap \langle x_0, \dots, x_n \rangle^a)$ is a virtual resolution of S/I_X of length $m + n$*

In [HNVT22], Harada, Nowroozi, and Van Tuyl improve on Theorem 6.1, by giving a specific bound on a. Before introducing their result we recall that for a finite set of points $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$ we define $\pi_1(X)$ to be the projection map onto first coordinates and $\pi_2(X)$ to be the projection map onto second coordinates. The main result of [HNVT22, Section 4] follows:

Theorem 6.2. [HNVT22, Theorem 4.2] *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. For all $a \geq |\pi_1(X)| - 1$, the minimal free resolution of $S/(I_X \cap \langle x_0, x_1 \rangle^a)$ is a virtual resolution of S/I_X of length two.*

The goal of Section 8 is to generalize [HNVT22, Theorem 4.2], from $\mathbb{P}^1 \times \mathbb{P}^1$ to an analogous statement for $\mathbb{P}^1 \times \mathbb{P}^m$. The main result of Section 8, Theorem 8.2 considers a set of points X and the ideal $I_X \cap \langle y_0, \dots, y_m \rangle^b$, with $b \geq |\pi_2(X)| - 1$ in $\mathbb{P}^1 \times \mathbb{P}^m$ (of which $\mathbb{P}^1 \times \mathbb{P}^1$ is a subcase).

While in [HNVT22] the authors intersect with the x-variables, in order to make use of the simpler vanishing ideals of points in $\mathbb{P}^1$, we will instead intersect with the y-variables. In Section 9, we return to intersecting witht he x-variables again. We now introduce necessary background which will be used in the proof of Theorem 8.2 and Theorem 9.13.

Lemma 6.3. [HNVT22, Theorem 2.3] *Let I be a B-saturated bihomogeneous ideal of S. If J is a bihomogeneous ideal of S with $J : B^\infty = I$, then the minimal free resolution of S/J is a virtual resolution of S/I.*

Proof. The proof of this lemma in [HNVT22] is written for $\mathbb{P}^1 \times \mathbb{P}^1$, but their argument does not rely on dimensions specifically. $\square$

Definition 6.4. For a finite set of points $X \subset \mathbb{P}^n \times \mathbb{P}^m$, let $\ell = |\pi_1(X)|$, and $t = |\pi_2(X)|$. Label the elements

$$\pi_1(X) = \{A_1, \dots, A_\ell\} \text{ and } \pi_2(X) = \{B_1, \dots, B_t\}$$

such that $|\pi_1^{-1}(A_k)| \geq |\pi_1^{-1}(A_{k+1})|$ and $|\pi_2^{-1}(B_k)| \geq |\pi_2^{-1}(B_{k+1})|$. For each $A_k \in \pi_1(X)$, let $\alpha_k = |\pi_1^{-1}(A_k)|$. Then $\alpha_X = (\alpha_1, \dots, \alpha_\ell)$. Similarly, for each $B_k \in \pi_2(X)$, let $\beta_k = |\pi_2^{-1}(B_k)|$. Then $\beta_X = (\beta_1, \dots, \beta_\ell)$.

Note that β_X is a partition of $|X|$, so $\beta_X^* = (\beta_1^*, \dots, \beta_{\beta_1}^*)$, will denote the conjugate partition obtained by transposing the rows and columns in the corresponding Young diagram. In particular, $\beta_i^* = |\{j \in \{1, \dots, t\} \mid \beta_j \geq i\}|$.

Definition 6.5. For brevity, define

$$\langle \mathbf{x} \rangle = \langle x_0, \dots, x_n \rangle \text{ and } \langle \mathbf{y} \rangle = \langle y_0, \dots, y_m \rangle$$

7. Intersection preliminaries for $\mathbb{P}^1 \times \mathbb{P}^m$

The following section gives results necessary for Theorem 8.2 which we will prove in the following section. In particular, many of the results in this section are generalizations of results about $\mathbb{P}^1 \times \mathbb{P}^1$ from [GVT15] and [HNVT22] to analogous statements for $\mathbb{P}^1 \times \mathbb{P}^m$.

Proposition 7.1. [VT02, Proposition 3.5] *Let $X \subset \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ be a finite set of distinct points, let H_X be the Hilbert function of S/I_X. Let $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the jth position. Then for any $\underline{i} = (i_1, \dots, i_k) \in \mathbb{Z}^2_{\geq 0}$ we have*

$$H_X(\underline{i}) \leq H_X(\underline{i} + e_j)$$

for all $1 \leq j \leq k$, in other words, if we increase one "coordinate" and fix all of the others, the Hilbert function is weakly increasing.

Lemma 7.2. [GVT15, Lemma 3.25] *If X is a set of s points in $\mathbb{P}^1$, then the Hilbert function of X is given by*

$$H_X(i) = \begin{cases} i+1 & i < s, \\ s & i \geq s. \end{cases}$$

The following result is a refinement of [VT02, Proposition 3.2].

Proposition 7.3. [VT02, Proposition 3.2] *Let $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be a finite set of distinct points with Hilbert function H_X. The sequence $\{H_X(0,j)\}_{j\in\mathbb{Z}_{\geq 0}}$ is the Hilbert function of $\pi_2(X)$ in $\mathbb{P}^m$.*

The following lemma is a generalization of [HNVT22, Lemma 2.5], which proves the analogous statement for points on $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 7.4. *Let X be a set of points in $\mathbb{P}^n \times \mathbb{P}^m$ with coordinate ring S/I_X, then the ideal I_X is B-saturated.*

Proof. Note that for each point $P \in X$, I_P is B-saturated. $I_X = \cap_{P\in X} I_P$ and the intersection of saturated ideals is saturated. Let $P \in X$. Certainly I_P is contained in $I_P\colon B^\infty = \cup_{t\geq 1} I_P\colon B^t$. For the reverse containment, let $f \notin I_P$. We show that $f \notin I_P\colon B^\infty$. Since $f \notin I_P$, $f(P) \neq 0$. But $V(B) = \varnothing$, so for any point $Q \in \mathbb{P}^n \times \mathbb{P}^m$, there exists $g_Q \in B$ such that $g_Q(Q) \neq 0$. So there exists $g_P \in B$ with $g_P(P) \neq 0$, so that $(fg_P^r)(P) = f(P)(g_P(P))^r \neq 0$. Hence $fg_P^r \notin I_P$ for all $r \geq 1$ so that $f \notin I_P\colon B^\infty$ as needed. □

Remark 7.5. By an appropriate change of coordinates, we can assume without loss of generality that for $X \subseteq \mathbb{P}^1 \times \mathbb{P}^m$ the non-zero divisor of Remark 4.4 is x_0, that is, we can assume that if $P = A \times B \in X$, then $A \neq [0:1]$.

Lemma 7.6. *Let $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be a finite set of distinct points all sharing a first coordinate $A \in \mathbb{P}^n$. That is*

$$X = \{A \times B_1, A \times B_2, \ldots, A \times B_t\}.$$

Note that $t = |\pi_2(X)|$. If $j \geq t-1$ we have $H_X(i,j) = t$.

Proof. By Proposition 7.3 we know that $\{H_X(0,j)\}_{j\in\mathbb{Z}_{\geq 0}}$ is the Hilbert function of $\pi_2(X)$ in $\mathbb{P}^m$. By Lemma 2.3 we know that for a set of points $P \in \mathbb{P}^m$ for $h \geq |P| - 1$ we have $H_X(h) = |P|$. So $\{H_X(0,j)\}_{j\in\mathbb{Z}_{\geq 0}} = t$ for $j \geq t-1$. Furthermore, by Proposition 7.1, we know for $i \geq 0$ we have $H_X(i,j) \geq H_X(0,j) = t$. Since $t = |X|$ in this case, and $|X|$ is the maximum value attained for H_X for X a set of points in multiprojective space, it follows that for $j \geq t-1$, and $i \in \mathbb{Z}_{\geq 0}$ we have $H_X(i,j) = t$. □

The following lemma is a generalization of [GVT15, Lemma 3.26].

Lemma 7.7. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^m$ be a finite set of distinct points of the form*

$$X = \{X_1, \ldots, X_\ell\} = \{A_1 \times B, A_2 \times B, \ldots, A_\ell \times B\},$$

where $A_i = \pi_1(X_i)$ and $B = \pi_2(X)$. Then

$$H_X = \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 2 & 2 & 2 & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \ell-1 & \ell-1 & \ell-1 & \cdots \\ \ell & \ell & \ell & \cdots \\ \ell & \ell & \ell & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Proof. By changing coordinates, we may assume without loss of generality that $B = [1 : 0 : \cdots : 0]$. Then

$$I_X = \bigcap_{j=1}^{\ell} I(A_j \times B) = \bigcap_{j=1}^{\ell} \langle L_{A_j}, y_1, \ldots, y_m\rangle = \langle L_{A_1} \cdots L_{A_\ell}, y_1, \ldots, y_m\rangle,$$

where $L_{A_j} = a_{j,1}x_0 - a_{j,0}x_1$ for $A_j = [a_{j_0} : a_{j_1}]$.

Let $R = \Bbbk[x_0, x_1, y_0]$ with $\deg(x_0) = \deg(x_1) = (1, 0)$ and $\deg(y_0) = (0, 1)$, and let $F = L_{A_1} \cdots L_{A_\ell}$. It follows that

$$S/I_X \cong R/\langle F\rangle,$$

and we also obtain the short exact sequence

$$0 \to R(-\ell, 0) \xrightarrow{\cdot F} R \to R/\langle F\rangle \to 0.$$

Since the Hilbert function is additive on short exact sequences we know that $H_R(i, j) = H_{R(-\ell,0)}(i, j) + H_{R/\langle F\rangle}(i, j)$. So $H_{S/I_X} = H_R(i, j) - H_{R(-\ell,0)}(i, j)$. Furthermore, note that $H_{R_{(-\ell,0)}}(i, j) = H_R(i - \ell, j)$, so when $i < \ell$ this Hilbert function vanishes. Since $\dim_{\Bbbk} R_{i,j} = i + 1$ for all $(i, j) \in \mathbb{Z}^2_{\geq 0}$, when $i < \ell$, we have $H_{S/I_X} = i + 1$. Additionally, when $i \geq t$ we have $H_{S/I_X} = i + 1 - (i - \ell + 1) = \ell$. $\square$

The following lemma is a generalization of [GVT15, Lemma 3.28].

Lemma 7.8. *Let $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$ be a finite set of distinct points with $|\pi_2(X)| = t$. Suppose that $L \in S_{(0,1)}$ is a nonzero divisor on S/I_X. Then $\langle \mathbf{y}\rangle^t \subseteq \langle I_X, L\rangle$.*

Proof. Since L is a nonzero divisor, we have the short exact sequence

$$0 \to (S/I_X)(0, -1) \xrightarrow{\cdot L} S/I_X \to S/\langle I_X, L\rangle \to 0.$$

Then

$$H_{S/\langle I_X,L\rangle}(i, j) = H_X(i, j) - H_X(i, j - 1) \text{ for all } (i, j) \in \mathbb{Z}^2_{\geq 0}.$$

By Proposition 7.3, the sequence $\{H_X(0, j)\}_{j\in\mathbb{Z}_{\geq 0}}$ is the Hilbert function of $\pi_2(X)$ in $\mathbb{P}^m$, and by Lemma 2.3 we have $H_X(0, j) = |\pi_2(X)| = t$ for $j \geq t - 1$. Then $H_{S/\langle I_X,L\rangle}(0, t) = H_{S/I_X}(0, t) - H_{S/I_X}(0, t - 1) = t - t = 0$. This implies that $\langle I_X, L\rangle_{0,t} = S_{0,t}$, equivalently, $\langle \mathbf{y}\rangle^t \subseteq \langle I_X, L\rangle$. $\square$

The following lemma is a slightly weaker version of [Eis95, Exercise 3.19].

Lemma 7.9. *Suppose R is a ring containing an infinite field k, and let $I_1, \cdots, I_n$ be ideals of R. If $\langle f_1, \cdots, f_m\rangle \not\subset I_i$ for $i = 1, \cdots, n$, then there is $(a_1, \cdots, a_m) \in k^m$ such that $\sum_j a_i f_i \notin \bigcup_j I_j$.*

Proof. In the case k is infinite, the ideals $\langle f_1, \cdots, f_m\rangle$ and I_i for $i = 1, \cdots, n$ are vector spaces over the infinite field. Suppose the vector space $\langle f_1, \cdots, f_n\rangle \subset \bigcup_j I_j$. A vector space over an infinite field cannot be expressed as a union of finitely many proper subspaces. Then $\langle f_1, \cdots, f_n\rangle \subset I_i$ for some $i = 1, \cdots, n$, a contradiction. Hence there is some element $\sum_j a_i f_i$ of the vector space $\langle f_1, \cdots, f_n\rangle$, a k-linear combination of its generators, such that $\sum_j a_i f_i \notin \bigcup_j I_j$. $\square$

Lemma 7.10. *Let $X \subset \mathbb{P}^n \times \mathbb{P}^m$ be a finite set of distinct points. Let P be a point in $\mathbb{P}^m$ such that $P \notin \pi_2(X)$, and suppose $L_1, \dots, L_m$ are linear forms in the y_i's generating the defining ideal of P. Then there exists a k-linear combination of the L_i's that is a nonzero divisor on S/I_X.*

Proof. We will use $\bar{F}$ to denote the image of F under the quotient map $S \to S/I_X$. Recall that if we regard S/I_X as a module over itself, then the annihilator $\operatorname{ann}(\bar{L}_i)$ is defined as the set of elements $\bar{F} \in S/I_X$ such that $\bar{F}\bar{L}_i = 0$.

We first claim that

$$\bigcap_{i=1}^{m} \operatorname{ann}(\bar{L}_i) = 0 \tag{7.1}$$

To see this, suppose there is $\bar{F} \in S/I_X$ such that $\bar{F}\bar{L}_i = 0$ for all i, we will show that $\bar{F} = 0$ in S/I_X. Note that this is equivalent to saying: If there is $F \in S$ such that $FL_i \in I_X$ for all i, then $F \in I_X$.

So suppose there is such an F, plugging in any point $(P_1, P_2) \in X$, we get $F(P_1, P_2)L_i(B') = 0$ for all i. However, since $P \notin \pi_2(X)$, we know $L_1, \cdots, L_m$ do not vanish simultaneously on any point of X. This means that $L_i(P_2) \neq 0$ for some i, so we must have $F(P_1, P_2) = 0$. This is true for any point in X, so we conclude that $F \in I_X$.

Next, recall that in general, given an R-module M, a prime ideal of R is called an associated prime if it happens to be the annihilator of some element of M, and we denote the set of associated primes as:

$$\operatorname{Ass}(M) := \{\mathfrak{p} \subset R \text{ prime} \mid \mathfrak{p} = \operatorname{ann}(m) \text{ for some } m \in M\}$$

In our case, $R = M = S/I_X$. By [Eis95, Theorem 3.1], $\operatorname{Ass}(S/I_X)$ is a finite set. Moreover, the finite union of all associated primes:

$$\bigcup_{\mathfrak{p} \in \operatorname{Ass}(S/I_X)} \mathfrak{p}$$

is exactly the set of zero divisors of S/I_X together with $0 \in S/I_X$.

Now from Equation (7.1), we see that the ideal $\langle \bar{L}_1, \cdots, \bar{L}_m \rangle$ is not annihilated by any element $\bar{F} \in S/I_X$, i.e. $\langle \bar{L}_1, \cdots, \bar{L}_m \rangle \not\subset \operatorname{ann}(\bar{F})$. In particular, $\langle \bar{L}_1, \cdots, \bar{L}_m \rangle$ is not contained in any associated primes. Therefore, by Lemma 7.9, we know there is a k-linear combination of $\bar{L}_1, \cdots, \bar{L}_m$, say $\bar{L} = c_1\bar{L}_1 + \cdots + c_m\bar{L}_m$, that avoids all of the associated primes, which means that $\bar{L}$ is a non-zero divisor in S/I_X. Taking preimage under the quotient map, we get a k-linear combination $L = c_1L_1 + \cdots + c_mL_m$, which is a non-zero divisor on S/I_X. □

The following theorem is a generalization of [GVT15, Theorem 3.29].

Theorem 7.11. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^m$ be any set of distinct points with associated tuple $\beta_X = (\beta_1, \dots, \beta_{|\pi_2(X)|})$. For all $i \in \mathbb{Z}_{\geq 0}$ if $j \geq |\pi_2(X)| - 1$, then*

$$H_X(i,j) = \beta_1^* + \beta_2^* + \ldots + \beta_{i+1}^*$$

where $\beta_X^* = (\beta_1^*, \dots, \beta_{\beta_1}^*)$ is the conjugate of β_X, and where we make the convention that $\beta_r^* = 0$ if $r > \beta_1$.

Proof. We will prove this statement by induction on $|\pi_2(X)|$. For the base case, consider when $|\pi_2(X)| = 1$. Then we have $X = \{A_1 \times B, A_2 \times B, \ldots, A_\ell \times B\}$, so $\beta_X = (\ell)$, and we have conjugate $\beta_X^* = \underbrace{(1, \ldots, 1)}_{\ell}$. By Lemma 7.7 we know that the Hilbert function of X is:

$$H_X(i,j) = \begin{cases} i+1 & i < \ell - 1 \\ \ell & i \geq \ell - 1 \end{cases}$$

So for all $j \geq |\pi_2(X)| - 1 = 0$, we have:

$$H_X(i,j) = \min\{i+1, \ell\} = \beta_1^* + \beta_2^* + \ldots + \beta_{i+1}^* \text{ for all } i \in \mathbb{Z}_{\geq 0}$$

since $\beta_X^* = \underbrace{(1, \ldots, 1)}_{\ell}$ and we observe the convention that $\beta_r^* = 0$ for $r > \beta_1 = \ell$.

Now suppose the theorem holds when $|\pi_2(X)| < t$. Let $B_1, \ldots, B_t$ denote the unique second coordinates in X. Then for each $1 \leq k \leq t$ let $X_k := \{P \in X \mid \pi_2(P) = B_k\}$. Now we will consider

$$X' = \bigcup_{k=1}^{t-1} X_k$$

So $X = X' \cup X_t$. We then have the short exact sequence:

$$0 \longrightarrow S/(I(X') \cap I(X_t)) \longrightarrow S/I(X') \oplus S/I(X_t) \longrightarrow S/(I(X') + I(X_t)) \longrightarrow 0$$

Note that $I(X_t) = \langle F, L_1, \cdots, L_m \rangle$, where F is a polynomial in the x_i's and $L_1, \cdots, L_m$ are the linear forms in the y_i's generating the defining ideal of the point $B_t \in \mathbb{P}^m$.

Now by Lemma 7.10, we can find an element $L \in \langle L_1, \cdots, L_m \rangle$ of degree $(0,1)$ that is a non-zero divisor on $S/I(X')$, then Lemma 7.8 yields $\langle I(X'), L \rangle_{0,t-1} = S_{0,t-1}$ where $t - 1 = |\pi_2(X')|$. Therefore, we have

$$S_{i,j} \subset \langle I(X'), L \rangle_{i,j} \subset (I(X'), I(X_t))_{i,j} \text{ if } j \geq t - 1$$

Thus from the short exact sequence we get:

$$H_X(i,j) = H_{X'}(i,j) + H_{X_t}(i,j) \text{ for all } (i,j) \in \mathbb{Z}^2_{\geq 0} \text{ with } j \geq t - 1$$

Now since $\beta_X = (\beta_1, \cdots, \beta_t)$, we have $\beta' := \beta_{X'} = (\beta_1, \cdots, \beta_{t-1})$ and $\beta'' := \beta_{X_t} = (\beta_t)$. Since $|\pi_2(X')| = t - 1 < t$, by induction:

$$H_{X'}(i,j) = (\beta')_1^* + \ldots + (\beta')_{i+1}^*$$

for any $j \geq t - 1 > |\pi_2(X')| - 1$. Similarly,

$$H_{X_t}(i,j) = (\beta'')_1^* + \ldots + (\beta'')_{i+1}^*$$

for any $j \geq t - 1 > |\pi_2(X_t)| - 1$. Therefore for any $j \geq t - 1$

$$\begin{aligned} H_X(i,j) &= \big((\beta')_1^* + \ldots + (\beta')_{i+1}^*\big) + \big((\beta'')_1^* + \ldots + (\beta'')_{i+1}^*\big) \\ &= \big((\beta')_1^* + (\beta'')_1^*\big) + \ldots + \big((\beta')_{i+1}^* + (\beta'')_{i+1}^*\big) \\ &= \beta_1^* + \ldots + \beta_{i+1}^* \end{aligned}$$

so we are done. ◻

The following result is a generalization of [HNVT22, Corollary 2.8].

Corollary 7.12. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^m$ be a set of points with associated tuple $\beta^*_X = (\beta^*_1, \beta^*_2, \ldots, \beta^*_{\beta_1})$. Let $L \in S$ correspond to the nonzero divisor of Lemma 1.3 which is of degree $(1,0)$. For all $j \geq |\pi_2(X)| - 1$,*

$$H_{S/(I_X+\langle L\rangle)}(i,j) = \beta^*_{i+1} \text{ where } \beta^*_r = 0 \text{ if } r > \beta_1.$$

Proof. We have the short exact sequence

$$0 \to [S/(I_X : \langle L\rangle)](-1,0) \xrightarrow{\times L} S/I_X \to S/\langle I_X, L\rangle \to 0$$

Since L is a nonzero divisor, $I_X : \langle L\rangle = I_X$. The maps in the short exact sequence have degree (0,0). Thus, since Hilbert functions are additive on short exact sequences, $H_{S/(I_X+\langle L\rangle)}(i,j) = H_{S/I_X}(i,j) - H_{S/I_X}(i-1,j)$. By Theorem 7.11, the right hand side is $\beta^*_1 + \beta^*_2 + \ldots + \beta^*_{i+1} - (\beta^*_1 + \beta^*_2 + \ldots + \beta^*_i) = \beta^*_{i+1}$. □

8. Intersections in $\mathbb{P}^1 \times \mathbb{P}^m$

The following lemma is a generalization of [HNVT22, Lemma 4.3] which proves the analogous statement for $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 8.1. *Let X be any set of points in $\mathbb{P}^1 \times \mathbb{P}^m$ with $\pi_2(X) = \{B_1, \ldots, B_t\}$. Suppose that x_0 is a nonzero divisor on S/I_X. Then for any integer $b \geq |\pi_2(X)| - 1$, the equality*

$$\langle I_X \cap \langle \mathbf{y}\rangle^b, x_0\rangle = \bigcap_{B_k \in \pi_2(X)} \langle x_0, x_1^{\beta_k}, L_{B_{k,1}}, L_{B_{k,2}}, \ldots, L_{B_{k,m}}\rangle \cap \langle \langle \mathbf{y}\rangle^b, x_0\rangle \tag{8.1}$$

is a primary decomposition of $\langle I_X \cap \langle \mathbf{y}\rangle^b, x_0\rangle$ with $L_{B_{k,i}}$ a linear form in the y_i's generating the defining ideals for B_k.

Proof. Recall that by definition of the β_k's, for each $B_k \in \pi_2(X)$ we have a set

$$X_k = \{A_{k,1} \times B_k, A_{k,2} \times B_k, \ldots, A_{k,\beta_k} \times B_k\} \subseteq X$$

so that $X_k = \pi_2^{-1}(B_k) \cap X$ and contains all points in X with second coordinate B_k. So $I_X = \bigcap_{k=1}^t I_{X_k}$. We will prove Equation (8.1) by showing containment in each direction and end the proof by demonstrating that each ideal in the decomposition is primary. For each k we know $I_{X_k} = \langle \prod L_{A_{k,i}}, L_{B_{k,1}}, L_{B_{k,2}}, \ldots, L_{B_{k,m}}\rangle$ with $\deg(L_{B_{k,i}}) = (0,1)$ and $\deg\left(\prod L_{A_{k,i}}\right) = (\beta_k, 0)$.

Recall that for any three ideals I, J, K, we have that $(I \cap J) + K \subseteq (I+K) \cap (J+K)$. By applying this fact, for all $b \geq 0$, we have

$$\begin{aligned} \left\langle I_X \cap \langle \mathbf{y}\rangle^b, x_0\right\rangle &= \left\langle I_{X_1} \cap I_{X_2} \cap \ldots \cap I_{X_t} \cap \langle \mathbf{y}\rangle^b, x_0\right\rangle \\ &\subseteq \langle I_{X_1}, x_0\rangle \cap \ldots \cap \langle I_{X_t}, x_0\rangle \cap \left\langle \langle \mathbf{y}\rangle^b, x_0\right\rangle \end{aligned} \tag{8.2}$$

For any k we have:

$$\langle I_{X_k}, x_0\rangle = \left\langle \prod_{i=1}^{\beta_k} L_{A_{k,i}}, L_{B_{k,1}}, \ldots, L_{B_{k,m}}, x_0 \right\rangle$$

Furthermore, since $\prod_{i=1}^{\beta_k} L_{A_{k,i}}$ is a homogeneous polynomial of degree β_k in only x_0 and x_1 where the coefficient on $x_1^{\beta_k}$ is not zero, we have:

$$\left\langle \prod_{i=1}^{\beta_k} L_{A_{k,i}}, L_{B_{k,1}}, \dots, L_{B_{k,m}}, x_0 \right\rangle = \langle x_0, x_1^{\beta_k}, L_{B_{k,1}}, \dots, L_{B_{k,m}} \rangle. \tag{8.3}$$

Then, by combining Equation (8.2) and Equation (8.3) we have:

$$\langle I_X \cap \langle \mathbf{y} \rangle^b, x_0 \rangle \subseteq \bigcap_{B_k \in \pi_2(X)} \langle x_0, x_1^{\beta_k}, L_{B_{k,1}}, L_{B_{k,2}}, \dots, L_{B_{k,m}} \rangle \cap \langle \langle \mathbf{y} \rangle^b, x_0 \rangle$$

This shows the first containment. Next we will now show that the righthand side of the equality is contained in the lefthand side, for all $b \geq |\pi_2(X)| - 1$. Since both ideals are bihomogeneous, it suffices to show inclusion on the (i,j)-th graded pieces. We will show that for all $(i,j) \in \mathbb{Z}^2_{\geq 0}$ we have:

$$\left[\bigcap_{B_k \in \pi_2(X)} \langle x_0, x_1^{\beta_k}, L_{B_{k,1}}, \dots, L_{B_{k,m}} \rangle \cap \langle \langle \mathbf{y} \rangle^b, x_0 \rangle \right]_{i,j} \subseteq [\langle I_X \cap \langle \mathbf{y} \rangle^b, x_0 \rangle]_{i,j} \tag{8.4}$$

We now have two cases we must consider, when $0 \leq j < b$ and $b \leq j$.

Case 1. Suppose $0 \leq j < b$. Let

$$G \in \left[\bigcap_{B_k \in \pi_2(X)} \langle x_0, x_1^{\beta_k}, L_{B_{k,1}}, \dots, L_{B_{k,m}} \rangle \cap \langle \langle \mathbf{y} \rangle^b, x_0 \rangle \right]_{i,j}.$$

Therefore $G \in \langle \langle \mathbf{y} \rangle^b, x_0 \rangle$. Since $0 \leq j < b$, and we are considering the i,j-th graded component, then $G \in \langle x_0 \rangle$. Therefore $G \in [\langle I_X \cap \langle \mathbf{y} \rangle^b, x_0 \rangle]_{i,j}$.

Case 2. Suppose $b \leq j$. Since $b \leq j$, therefore $[\langle \mathbf{y} \rangle^b]_{i,j} = S_{i,j}$. Since $\langle y_0, \dots, y_m \rangle^b$ is already all of $S_{i,j}$, it follows that $\langle \langle \mathbf{y} \rangle^b, x_0 \rangle]_{i,j} = S_{i,j}$. Furthermore, since intersecting a set with a subset yields the subset, then $[\langle I_X \cap \langle \mathbf{y} \rangle^b, x_0 \rangle]_{i,j} = [\langle I_X, x_0 \rangle]_{i,j}$. So, by substitution into (Equation (8.4)), it suffices to show:

$$\left[\bigcap_{B_k \in \pi_2(X)} \langle x_0, x_1^{\beta_k}, L_{B_{k,1}}, L_{B_{k,2}}, \dots, L_{B_{k,m}} \rangle \right]_{i,j} \subseteq [\langle I_X, x_0 \rangle]_{i,j} \tag{8.5}$$

We now consider each ideal appearing in the intersection of the lefthand side one at a time. Fix a $1 \leq k \leq t$ and consider the following two subcases:

(i). Suppose $\beta_k \leq i$. Then $[\langle x_0, x_1^{\beta_k}, L_{B_{k,1}}, L_{B_{k,2}}, \dots, L_{B_{k,m}} \rangle]_{i,j} = S_{i,j}$ since every monomial of degree (i,j) with $i \geq \beta_k$ is either divisible by x_0 or is a multiple of $x_1^{\beta_k}$.

(ii.) Suppose $i < \beta_k$. Then

$$[\langle x_0, x_1^{\beta_k}, L_{B_{k,1}}, L_{B_{k,2}}, \dots, L_{B_{k,m}} \rangle]_{i,j} = [\langle x_0, L_{B_{k,1}}, L_{B_{k,2}}, \dots, L_{B_{k,m}} \rangle]_{i,j}.$$

since no monomial containing $x_1^{\beta_k}$ will be in the (i,j)-th graded piece of the ideal when $i < \beta_k$.

Taking these two subcases into account, it suffices to show the following:

$$\left[\bigcap_{\substack{B_k\in\pi_2(X)\\ \beta_k>i}} \langle x_0, L_{B_{k,1}}, L_{B_{k,2}}, \ldots, L_{B_{k,m}}\rangle\right]_{i,j} \subseteq [\langle I_X, x_0\rangle]_{i,j} \tag{8.6}$$

Since the vector space on the righthand side of Equation (8.6) is a subspace of the one on the lefthand side, we show equality by proving that both vector spaces have the same dimension.

The ideal on the lefthand side of Equation (8.6) is the ideal of the points

$$Y = \{[0:1]\times B_k \mid \beta_k > i\} \subseteq \mathbb{P}^1\times\mathbb{P}^m$$

Then it suffices to show $H_Y(i,j) = H_{S/\langle I_X,x_0\rangle}(i,j)$. Since x_0 is a non-zero-divisor and since $j \geq b = |\pi_2(X)|-1$, by Corollary 7.12 we have $H_{S/\langle I_X,x_0\rangle}(i,j) = \beta^*_{i+1}$. The number of points in Y is then the number of $\beta_k \in \beta_X$ with $\beta_k \geq i+1$ which is, by definition β^*_{i+1}. So by Lemma 7.6, $H_Y(i,j) = |Y| = \beta^*_{i+1}$. Therefore $H_Y(i,j) = H_{S/\langle I_X,x_0\rangle}(i,j)$ for $j \geq |\pi_2(X)|-1$, and for all $i \in \mathbb{Z}_{\geq 0}$. So Equation (8.6) holds.

Having checked both cases for this containment, we may conclude that the equality holds. Finally, we turn to showing that the decomposition is indeed primary. Recall that a monomial ideal $Q = \langle q_1,\ldots,q_t\rangle$ in a polynomial ring $\Bbbk[z_1,\ldots,z_n]$ is a primary ideal if and only if, after a permutation of the variables, $Q = \langle z_1^{a_1},\ldots,z_r^{a_r}, q_1',\ldots,q_p'\rangle$ and the only variables that divide $q_1',\ldots,q_p'$ are $z_1,\ldots,z_r$ (see [Vil15, Proposition 6.1.7]). Therefore, the ideal $\langle\langle \mathbf{y}\rangle^b, x_0\rangle$ is a primary monomial ideal.

For the ideals $\langle x_0, x_1^{\beta_k}, L_{B_{k,1}}, L_{B_{k,2}},\ldots,L_{B_{k,m}}\rangle$ recall that $L_{B_{k,i}}$ are linear either of the form y_i or $b_iy_j - b_jy_i$ for $B_K = [b_{k,0}:b_{k,1}:\cdots:b_{k,m}]$ (WLOG we assume $b_{k,0}\neq 0$ since at least one coordinate must be nonzero). Then:

$$\begin{aligned}
&S/\langle x_0, x_1^{\beta_k}, y_{i_1},\ldots,y_{i_v}, b_{k,r_1}y_0 - b_{k,0}y_{r_1},\ldots,b_{k,r_w}y_0 - b_{k,0}y_{r_w}\rangle\\
&\qquad = S/\langle x_0, x_1^{\beta_k}, y_{i_1},\ldots,y_{i_v}, \frac{b_{k,r_1}}{b_{k,0}}y_0 - y_{r_1},\ldots,\frac{b_{k,r_w}}{b_{k,0}}y_0 - y_{r_w}\rangle\\
&\qquad\qquad \cong \Bbbk[x_0,x_1,y_0,y_{i_1},\ldots,y_{i_v}]/\langle x_0, x_1^{\beta_k}, y_0, y_{i_1},\ldots,y_{i_v}\rangle
\end{aligned}$$

Since $\langle x_0, x_1^{\beta_k}, y_0, y_{i_1},\ldots,y_{i_v}\rangle$ is primary, so is $\langle x_0, x_1^{a_k}, L_{B_{k,1}}, L_{B_{k,2}},\ldots,L_{B_{k,m}}\rangle$, completing the proof. □

The following theorem is the main result of Section 8, and is a generalization of [HNVT22, Theorem 4.2] which proves the analogous statement for $\mathbb{P}^1\times\mathbb{P}^1$.

Theorem 8.2. *Let X be a set of points in $\mathbb{P}^1\times\mathbb{P}^m$. For all $b \geq |\pi_2(X)|-1$, the minimal free resolution of $S/(I_X\cap\langle\mathbf{y}\rangle^b)$ is a virtual resolution of S/I_X of length $m+1$.*

Proof. We first claim that for all integers b, the minimal free resolution of $S/(I_X\cap\langle\mathbf{y}\rangle^b)$ is a virtual resolution of S/I_X. To see this, note that by Lemma 7.4, the ideal I_X is B-saturated. Moreover, we have the identity

$$I_X\cap\langle\mathbf{y}\rangle^b : B^\infty = I_X \text{ for all integers } b\geq 0$$

We check inclusion on both sides. Because $I_X \cap \langle \mathbf{y}\rangle^b \subseteq I_X$, we get $I_X \cap \langle \mathbf{y}\rangle^b : B^\infty \subseteq I_X : B^\infty$. Then $I_X : B^\infty = I_X$ since I_X is B-saturated. For the reverse inclusion, note that $I_X B^b \subseteq I_X \cap B^b \subseteq I_X \cap \langle \mathbf{y}\rangle^b$. The claim now follows from Lemma 6.3.

Now we want to show that for any $b \geq |\pi_2(X)| - 1$, the minimal free resolution of $S/(I_X \cap \langle \mathbf{y}\rangle^b)$ has length $m+1$. The projective dimension of $S/(I_X \cap \langle \mathbf{y}\rangle^b)$ is by definition the length of its minimal free resolution. By the Auslander-Buchsbaum Formula,

$$\operatorname{pd}(S/(I_X \cap \langle \mathbf{y}\rangle^b) = \operatorname{depth}(S) - \operatorname{depth}(S/I_X \cap \langle \mathbf{y}\rangle^b)$$

. Since $\operatorname{depth}(S) = m+3$, if we can show $\operatorname{depth}(S/I_X \cap \langle \mathbf{y}\rangle^b) = 2$ we are done.

We first find the Krull dimension of $S/(I_X \cap \langle \mathbf{y}\rangle^b)$, since for any S-module M, $\operatorname{depth}(M) \leq \dim(M)$ The topological dimension of an algebraic set Y is the same as the Krull dimension of its coordinate ring $S/I(Y)$. Note that in $\mathbb{A}^{m+3}$, $V(I_X \cap \langle \mathbf{y}\rangle^b) = V(I_X \cap \langle \mathbf{y}\rangle) = V(I_X) \cup V(\langle \mathbf{y}\rangle)$ has topological dimension 2. By the Nullstellensatz $S/(\sqrt{I_X \cap \langle \mathbf{y}\rangle^b})$ is the coordinate ring of this algebraic set, which then has Krull dimension 2. The prime ideals in $S/(\sqrt{I_X \cap \langle \mathbf{y}\rangle^b})$ and $S/(I_X \cap \langle \mathbf{y}\rangle^b)$ are the same, so $S/(I_X \cap \langle \mathbf{y}\rangle^b)$ also has Krull dimension 2. Hence depth of $S/(I_X \cap \langle \mathbf{y}\rangle^b)$ is at most 2.

We now show that for all $b \geq |\pi_2(X)| - 1$, the depth of $S/(I_X \cap \langle \mathbf{y}\rangle^b)$ is at least 2, by finding a regular sequence of length 2. By Remark 7.5 we can assume x_0 is a non-zero divisor of S/I_X. In fact, x_0 is also a non-zero divisor of $S/(I_X \cap \langle \mathbf{y}\rangle^b)$ because if $x_0 F \in (I_X \cap \langle \mathbf{y}\rangle^b)$ then $x_0 F \in I_X$ and $x_0 F \in \langle \mathbf{y}\rangle^b$, and this can only happen if $F \in (I_X \cap \langle \mathbf{y}\rangle^b)$. Because $b \geq |\pi_2(X)| - 1$, Lemma 8.1 gives a primary decomposition of $(I_X \cap \langle \mathbf{y}\rangle^b, x_0)$. The zero divisors of $S/(I_X \cap \langle \mathbf{y}\rangle^b, x_0)$ are the elements in the union of the associated primes, that is elements of the set

$$\left(\bigcup_{B_k \in \pi_2(X)} \langle x_0, x_1, L_{B_{k,1}}, L_{B_{k,2}}, \ldots, L_{B_{k,m}} \rangle \right) \cup \langle \langle \mathbf{y}\rangle, x_0 \rangle \tag{8.7}$$

Let L be any element with $\deg L = (0,1)$ such that L does not vanish at any point in $\pi_2(X)$. Then $L + x_1$ is not in the union (8.7), so $L + x_1$ is a nonzero divisor of $S/(I_X \cap \langle \mathbf{y}\rangle^b, x_0)$. Since it is possible to construct a regular sequence in $S/(I_X \cap \langle \mathbf{y}\rangle^b)$ of length 2, depth is exactly 2. $\square$

9. Intersections in $\mathbb{P}^n \times \mathbb{P}^m$

We recall notation established in Definition 6.4. For a finite set of points $X \subset \mathbb{P}^n \times \mathbb{P}^m$, define $\ell = |\pi_1(X)|$ to be the number of distinct first coordinates with $\pi_1(X) = \{A_1, \ldots, A_\ell\}$. Define $X_k = \pi_1^{-1}(A_k) \cap X$. Then

$$X_k = \{A_k \times B_{k,1}, A_k \times B_{k,2}, \ldots, A_k \times B_{k,\alpha_k}\} \subseteq X$$

where $\alpha_k = |\pi_1^{-1}(A_k)|$.

Lemma 9.1. *For a set of points $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$, there exists a change of coordinates so that every point in X is of the form $[1 : a_1 : \ldots : a_n] \times [1 : b_1 : \ldots : b_m]$.*

Proof. First we will show that there exists linear forms $F(x_0,\dots,x_m) = \sum_{i=0}^{n} c_i x_i$ and $G(y_0,\dots,y_m) = \sum_{j=0}^{m} d_j y_j$ such that for all $P \in X$ with

$$P = [a_0 : a_1 : \dots : a_n] \times [b_0 : \dots : b_m]$$

we have $F(a_0,\dots,a_n) \neq 0$ and $G(b_0,\dots,b_m) \neq 0$.

The existence of such an $F(x_0,\dots,x_m) = \sum_{i=0}^{n} c_i x_i$ follows from the fact that $\Bbbk$ is an infinite field and $|X|$ is finite. Thus it is impossible for every hyperplane through the origin of $\mathbb{A}^{n+1}$ to contain a point $A \in \pi_1(X)$. The result follows similarly for the existence of $G(y_0,\dots,y_m) = \sum_{j=0}^{m} d_j y_j$.

Therefore we may choose appropriate $F(x_0,\dots,x_m) = \sum_{i=0}^{n} c_i x_i$ and $G(y_0,\dots,y_m) = \sum_{j=0}^{m} d_j y_j$. So

$$M = \begin{bmatrix} c_0 & c_1 & \cdots & c_n & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & d_0 & d_1 & \cdots & d_m \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

is in $\mathrm{GL}_{n+m+2}(\Bbbk)$ and $\overline{M} \in \mathrm{PGL}_{n+m+2}(\Bbbk)$ will have the same action on points

$$P = [a_0 : a_1 : \dots : a_n] \times [b_0 : \dots : b_m]$$

as M does on $(a_0,\dots,a_n,b_0,\dots,b_m)$. Therefore, for each $P \in X$, it follows that

$$\overline{M}\left([a_0 : a_1 : \dots : a_n] \times [b_0 : \dots : b_m]\right) = \left[\sum_{i=0}^{n} c_i a_i : a_1 : \dots : a_n\right] \times \left[\sum_{j=0}^{m} d_j b_j : b_1 : \dots : b_m\right]$$

with $\sum_{i=0}^{n} c_i a_i$ and $\sum_{j=0}^{m} d_j b_j \neq 0$. Finally by scaling, each point we obtain the desired result. $\square$

Definition 9.2. For a point $P \subset \mathbb{P}^n \times \mathbb{P}^m$, with

$$P = A \times B = [1 : a_1 : \dots : a_n] \times [1 : b_1 : \dots : b_m]$$

define $L_{A_i} = a_i x_0 - x_i$ and $L_{B_j} = b_j y_0 - y_j$. Furthermore define

$$I_A = \langle L_{A_1},\dots,L_{A_n}\rangle \text{ and } I_B = \langle L_{B_1},\dots,L_{B_m}\rangle$$

Note that $I_A + I_B$ is the defining bihomogeneous ideal for P.

Remark 9.3. Note that for a finite set of points $X \subset \mathbb{P}^n \times \mathbb{P}^m$, then for $1 \le k \le \ell$, we have

$$I_{X_k} = I_{A_k} + \bigcap_{j=1}^{\alpha_k} I_{B_{k,j}}$$

Definition 9.4. For a finite set of points $X \subset \mathbb{P}^n \times \mathbb{P}^m$ and a for $1 \le a \le \ell$, let $J_a = \bigcap_{k=1}^{a} I_{X_k}$.

Remark 9.5. $J_a = I_Y$ where $Y = \bigsqcup_{k=1}^{a} X_k$ for $1 \le a \le \ell$.

Lemma 9.6. *Let X be a finite set of points in $\mathbb{P}^n \times \mathbb{P}^m$. The Krull dimension of $S/(I_X \cap \langle \mathbf{x} \rangle^a)$ for $a \geq |\pi_1(X)| - 1$ is $m+1$.*

Proof. The depth of a module is less than or equal to its Krull dimension, and the topological dimension of an algebraic set coincides with the Krull dimension of its coordinate ring. Then in $\mathbb{A}^{m+n+2}$, the variety

$$V(I_X \cap \langle \mathbf{x} \rangle^a) = V(I_X \cap \langle \mathbf{x} \rangle) = V(I_X) \cup V(\langle \mathbf{x} \rangle)$$

is a union of 2-dimensional hyperplanes with an $m+1$-dimensional hyperplane. Since k is algebraically closed, by the Nullstellensatz, $\dim(S/(I_X \cap \langle \mathbf{x} \rangle^a)) = m+1$. $\square$

Lemma 9.7. *Let X be a set of at least two points in $\mathbb{P}^n \times \mathbb{P}^m$ with $|\pi_1(X)| = \ell$. Then $I_{X_\ell} + J_{\ell-1}$ contains $\langle \mathbf{x} \rangle^a$ for $a \geq |\pi_1(X)| - 1$.*

Proof. We will prove this statement by induction on $|\pi_1(X)|$.

Suppose $|\pi_1(X)| = 2$. Then $I_{X_1} = \langle L_{A_0}, \ldots, L_{A_n}, G_1, \ldots, G_k \rangle$ where L_{A_i} is a linear form in x_0 and x_i, and each G_i is a polynomial in the y variables. $I_{X_2} = \langle L'_{A_0}, \ldots, L'_{A_n}, G'_1, \ldots, G'_{k'} \rangle$ is in the same form. Then $I_{X_1} + I_{X_2} = \langle x_0, \ldots, x_n, G_1, \ldots, G_k, G'_1, \ldots, G'_{k'} \rangle$.

Assume

$$\langle \mathbf{x} \rangle^{\ell-1} \in I_{X_\ell} + J_{\ell-1}$$

for any X' such that $|\pi_1(X')| = \ell$. Then for X with $|\pi_1(X)| = \ell+1$, we can write $X = Y_1 \sqcup Y_2 \sqcup X_{\ell+1}$ where $Y_1 = \bigsqcup_{k=1}^{\ell-1} X_k$ and $Y_2 = X_\ell$ so $|\pi_1(Y_1)| = \ell - 1$ and $|\pi_1(Y_2)| = 1$. By the inductive hypothesis $\langle \mathbf{x} \rangle^{\ell-1} \in I_{X_{\ell+1}} + I_{Y_1}$ and $\langle \mathbf{x} \rangle \in I_{X_{\ell+1}} + I_{Y_2}$.

Then

$$\langle \mathbf{x} \rangle^{\ell-1} \cdot \langle \mathbf{x} \rangle \subset (I_{X_{\ell+1}} + I_{Y_1}) \cdot (I_{X_{\ell+1}} + I_{Y_2}) \subset I_{X_{\ell+1}} + I_{Y_1} \cap I_{Y_2} = I_{X_{\ell+1}} + \bigcap_{k=1}^{\ell} I_{X_k} = I_{X_{\ell+1}} + J_\ell \quad \square$$

Corollary 9.8. *Let $\ell = |\pi_1(X)|$. For $i \geq |\pi_1(X)| - 1$,*

$$H_{S/\left(J_{\ell-1} + \langle I_{X_\ell} \rangle\right)}(i,j) = 0$$

Proof. By Lemma 9.7, the ideal $J_{\ell-1} + \langle I_{X_\ell} \rangle$ contains $\langle \mathbf{x} \rangle^{\ell-1}$. Hence for $i \geq \ell - 1$, we know

$$\dim_{\mathbb{k}} \left[\frac{S}{J_{\ell-1} + \langle I_{X_\ell} \rangle} \right]_{i,j} = 0$$

so the result follows. $\square$

Lemma 9.9. *Let I be an ideal in S and L be a nonzero divisor of degree $(0,1)$ in S/I.*

$$H_{S/(I + \langle L \rangle)}(i,j) = H_{S/I}(i,j) - H_{S/I}(i-1,j).$$

Proof. We have the short exact sequence

$$0 \to [S/(I : \langle L \rangle)](0,-1) \xrightarrow{\times L} S/I_X \to S/\langle I, L \rangle \to 0$$

Since L is a nonzero divisor, $I : \langle L \rangle = I$. The maps in the short exact sequence have degree (0,0). Thus, since Hilbert functions are additive on short exact sequences, $H_{S/(I+\langle L \rangle)}(i,j) = H_{S/I_X}(i,j) - H_{S/I}(i-1,j)$. $\square$

Lemma 9.10. *For a finite set of points $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$, for $i \geq |\pi_1(X)| - 1$, we have*

$$\left[\bigcap_{A_k \in \pi_1(X)} \langle I_{X_k}, y_0\rangle\right]_{i,j} = [\langle I_X, y_0\rangle]_{i,j} \tag{9.1}$$

Proof. Because the vector space on the right hand side of Equation (9.5) is contained in the left hand side, it suffices to show the vector spaces have the same dimension. We will show this by induction on $|\pi_1(X)|$.

For the base case, when $|\pi_1(X)| = 1$, we have

$$\left[\bigcap_{k=1}^{1} \langle I_{X_k}, y_0\rangle\right]_{i,j} \subseteq [\langle I_X, y_0\rangle]_{i,j}.$$

Suppose that $|\pi_1(X)| \geq 1$ and Equation (9.5) holds for all values less than $|\pi_1(X)|$. For the RHS, by Lemma 9.9, it follows that

$$H_{S/\langle I_X, y_0\rangle}(i,j) = H_{S/I_X}(i,j) - H_{S/I_X}(i,j-1).$$

Furthermore, we have $I_X = \bigcap_{k=1}^{\ell} I_{X_k}$. So from the short exact sequence

$$0 \to S/I_X \to S/J_{\ell-1} \oplus S/I_{X_\ell} \to S/(J_{\ell-1} + I_{X_\ell}) \to 0$$

By Corollary 9.8 we know since $i \geq \ell - 1$

$$H_{S/(J_{\ell-1}+I_{X_\ell})}(i,j) = 0.$$

So by additivity of Hilbert functions on short exact sequences we have

$$H_{S/I_X}(i,j) = H_{S/J_{\ell-1}}(i,j) + H_{S/I_{X_\ell}}(i,j).$$

Similarly

$$H_{S/I_X}(i,j-1) = H_{S/J_{\ell-1}}(i,j-1) + H_{S/I_{X_\ell}}(i,j-1).$$

Therefore

$$H_{S/\langle I_X, y_0\rangle}(i,j) = H_{S/J_{\ell-1}}(i,j) + H_{S/I_{X_\ell}}(i,j) - H_{S/J_{\ell-1}}(i,j-1) - H_{S/I_{X_\ell}}(i,j-1). \tag{9.2}$$

For the LHS, by the inductive hypothesis

$$\left[\bigcap_{k=1}^{\ell-1} (I_{X_k} + \langle y_0\rangle)\right]_{i,j} = \left[\bigcap_{k=1}^{\ell-1} I_{X_k} + \langle y_0\rangle\right]_{i,j}.$$

Therefore we have the short exact sequence

$$0 \to \frac{S}{\bigcap_{k=1}^{\ell}\langle I_{X_k}, y_0\rangle} \to \frac{S}{J_{\ell-1} + \langle y_0\rangle} \oplus \frac{S}{I_{X_\ell} + \langle y_0\rangle} \to \frac{S}{J_{\ell-1} + I_{X_\ell} + \langle y_0\rangle} \to 0.$$

Since Hilbert functions are additive on short exact sequences,

$$H_{S/\cap_{k=1}^{\ell}\langle I_{X_k}, y_0\rangle}(i,j) = H_{S/(J_{\ell-1}+\langle y_0\rangle)}(i,j) + H_{S/(I_{X_\ell}+\langle y_0\rangle)}(i,j) - H_{S/(J_{\ell-1}+I_{X_\ell}+\langle y_0\rangle)}(i,j)$$

By Lemma 9.7, since $i \geq |\pi_1(X)| - 1$, thus $\langle \mathbf{x}\rangle^i \subset J_{\ell-1} + I_{X_\ell}$. Therefore $S_{i,j} = [\langle \bigcap_{k=1}^{\ell-1} I_{X_k}, I_{X_\ell}\rangle]_{i,j}$ and

$$H_{S/(J_{\ell-1}+I_{X_\ell}+\langle y_0\rangle)}(i,j) = 0$$

Next, note that y_0 is a nonzero divisor on $S/J_{\ell-1}$ and S/I_{X_ℓ}, so by Lemma 9.9,

$$H_{S/(J_{\ell-1}+\langle y_0\rangle)}(i,j) = H_{S/J_{\ell-1}}(i,j) - H_{S/J_{\ell-1}}(i,j-1)$$

and

$$H_{S/(I_{X_\ell}+\langle y_0\rangle)}(i,j) = H_{S/I_{X_\ell}}(i,j) - H_{S/I_{X_\ell}}(i,j-1)$$

By comparing with Equation (9.2), we see that for $i \geq a$

$$H_{S/\langle I_X, y_0\rangle}(i,j) = H_{S/\cap_{k=1}^{\ell}\langle I_{X_k}, y_0\rangle}(i,j)$$

So Equation (9.1) is an equality. □

Lemma 9.11. *Let R be a commutative ring with unity and suppose I is a primary ideal in R. Then $I[x]$, the ideal of polynomials in $R[x]$ with coefficients in I, is primary in $R[x]$.*

Proof. An ideal is primary if and only if the quotient ring has the property that all zero divisors are nilpotent. Since $R[x]/I[x] \cong (R/I)[x]$, it suffices to show that all zero divisors of $(R/I)[x]$ are nilpotent.

Suppose $f = c_0 + c_1x + \cdots + c_dx^d$ is a zero divisor in $(R/I)[x]$. Then by McCoy's theorem [NHM57], there is $r \in R/I$ such that $rf = 0$. In particular, this shows that all coefficients $c_0, \ldots, c_d$ of f are zero divisors in R/I. Since I is primary, each c_i is nilpotent in R/I, say $c_i^{n_i} = 0$. Then $f^{n_0+n_1+\cdots+n_d} = 0$. Thus, f is nilpotent in $(R/I)[x]$. □

Theorem 9.12 (Primary decomposition)**.** *Let X be a set of point in $\mathbb{P}^n \times \mathbb{P}^m$, with $|\pi_1(X)| = \ell$ and $I_X = \bigcap_{k=1}^{\ell} I_{X_k}$. Suppose y_0 is a nonzero divisor on S/I_X. Then for any integer $a \geq \ell - 1$, the equality*

$$\langle I_X \cap \langle \mathbf{x}\rangle^a, y_0\rangle = \bigcap_{A_k \in \pi_1(X)} \langle I_{X_k}, y_0\rangle \cap \langle\langle \mathbf{x}\rangle^a, y_0\rangle$$

is a primary decomposition of $\langle I_X \cap \langle \mathbf{x}\rangle^a, y_0\rangle$.

Proof. For ideals I_1, I_2, I_3, we have $I_1 \cap I_2 + I_3 \subseteq (I_1 + I_3) \cap (I_2 + I_3)$. It follows that for all $a \geq 0$

$$\langle I_X \cap \langle \mathbf{x}\rangle^a, y_0\rangle \subseteq \bigcap_{A_k \in \pi_1(X)} \langle I_{X_k}, y_0\rangle \cap \langle\langle \mathbf{x}\rangle^a, y_0\rangle \tag{9.3}$$

Now we wish to show that the LHS of Equation (9.3) contains the RHS, for $a \geq |\pi_1(X)| - 1$. Since both sides are bihomogeneous, it suffices to check containment on each bigraded piece, i.e. to show that

$$\left[\bigcap_{A_k \in \pi_1(X)} \langle I_{X_k}, y_0\rangle \cap \langle\langle \mathbf{x}\rangle^a, y_0\rangle\right]_{i,j} \subseteq [\langle I_X \cap \langle \mathbf{x}\rangle^a, y_0\rangle]_{i,j} \tag{9.4}$$

for all $(i,j) \in \mathbb{N}^2$. We consider two cases: (1) $0 \leq i < a$ and (2) $i \geq a$.

(1): Suppose F is in the LHS of Equation (9.4), in particular $F \in \langle\langle \mathbf{x}\rangle^a, y_0\rangle$. Since $i < a$ by assumption, $F \in \langle y_0\rangle$. But then F is contained in the RHS, so Equation (9.4) holds.

(2): Since $i \geq a$, $[\langle \mathbf{x}\rangle^a]_{i,j} = S_{i,j}$. Consequently

$$[\langle\langle \mathbf{x}\rangle^a, y_0\rangle]_{i,j} = S_{i,j}$$

and

$$[\langle I_X \cap \langle \mathbf{x} \rangle^a, y_0 \rangle]_{i,j} = [\langle I_X, y_0 \rangle]_{i,j}$$

Thus it suffices to show that for $i \geq a$:

$$\left[\bigcap_{A_k \in \pi_1(X)} \langle I_{X_k}, y_0 \rangle \right]_{i,j} \subseteq [\langle I_X, y_0 \rangle]_{i,j} . \tag{9.5}$$

This follows from Lemma 9.10, finishing case (2). Thus Equation (9.3) is an equality.

$$\langle I_X \cap \langle \mathbf{x} \rangle^a, y_0 \rangle = \bigcap_{A_k \in \pi_1(X)} \langle I_{X_k}, y_0 \rangle \cap \langle \langle \mathbf{x} \rangle^a, y_0 \rangle \tag{9.6}$$

Finally, we show that the ideals on the RHS of Equation (9.6) are primary. By [Vil15, Proposition 6.1.7] it follows that $\langle \langle \mathbf{x} \rangle^a, y_0 \rangle$ is primary. It remains to show that $\langle I_{X_k}, y_0 \rangle$ is primary for each $1 \leq k \leq \ell$. Recall

$$X_k = \{A_k \times B_{k,1}, \dots, A_k \times B_{k,\alpha_k}\}.$$

Then by Remark 9.3

$$\begin{aligned} \langle I_{X_k}, y_0 \rangle &= I_{A_k} + \bigcap_{j=1}^{\alpha_k} I_{B_{k,j}} + \langle y_0 \rangle \\ &= \left\langle L_{A_{k,1}}, L_{A_{k,2}}, \dots, L_{A_{k,n}}, G_1, \dots, G_s, y_0 \right\rangle \end{aligned}$$

where $L_{A_{k,i}} = a_{k,i}x_0 - x_i$, with $a_{k,i}$ is possibly 0, are the linear forms generating I_{A_k}, and $G_1, \dots, G_s$ are forms in the y variables such that

$$\langle G_1, \dots, G_s \rangle = \bigcap_{j=1}^{\alpha_k} I_{B_{k,j}}.$$

Therefore

$$\begin{aligned} S/\langle I_{X_k}, y_0 \rangle = S/ &\left\langle L_{A_{k,1}}, L_{A_{k,2}}, \dots, L_{A_{k,n}}, G_1, \dots, G_s, y_0 \right\rangle \\ &\cong \Bbbk[x_0, y_0, y_1, \dots, y_m]/ \langle G_1, \dots, G_s, y_0 \rangle . \end{aligned} \tag{9.7}$$

So it suffices to show that the ideal $J = \langle G_1, \dots, G_s, y_0 \rangle$ is primary in $\Bbbk[x_0, y_0, \dots, y_m]$. Let J' denote the ideal generated by the same elements, but now viewed as an ideal of $\Bbbk[y_0, \dots, y_m]$. Since $J = J'[x_0]$, it suffices to show that J' is primary in $\Bbbk[y_0, \dots, y_m]$ by Lemma 9.11.

By Lemma 9.1 we may denote

$$B_{k,j} = [1 : b_{k,j_1} : \dots : b_{k,j_m}],$$

i.e. $b_{k,j_0} = 1$. so by Remark 9.3,

$$I_{B_{k,j}} = \langle b_{k,j_1} y_0 - y_1, \dots, b_{k,j_m} y_0 - y_m \rangle.$$

Furthermore

$$\left\langle \prod_{j=1}^{\alpha_k} I_{B_{k,j}}, y_0 \right\rangle \subseteq \left\langle \bigcap_{j=1}^{\alpha_k} I_{B_{k,j}}, y_0 \right\rangle = J'.$$

Therefore we have

$$\langle y_0, \ldots, y_m\rangle^{\alpha_k} \subseteq J' \subseteq \langle y_0, \ldots, y_m\rangle.$$

So the radical of J' is maximal in $\Bbbk[y_0, \ldots, y_m]$, and thus J' is primary in $\Bbbk[y_0, \ldots, y_m]$. $\square$

Theorem 9.13. *Let X be a set of points in $\mathbb{P}^n \times \mathbb{P}^m$. For all $a \geq |\pi_1(X)| - 1$, the minimal free resolution of $S/(I_X \cap \langle \mathbf{x}\rangle^a)$ is a virtual resolution of S/I_X of length at most $n+m$ and at least $n+1$.*

Proof. We first show the minimal free resolution of $S/(I_X \cap \langle \mathbf{x}\rangle^a)$ is a virtual resolution of S/I_X. By Lemma 7.4, the ideal I_X is B-saturated. Moreover, we have the identity

$$I_X \cap \langle \mathbf{x}\rangle^t \colon B^\infty = I_X \text{ for all integers } t \geq 0.$$

We check the equality above. Since $I_X \cap \langle \mathbf{x}\rangle^t \subseteq I_X$, we have $I_X \cap \langle \mathbf{x}\rangle^t \colon B^\infty \subseteq I_X \colon B^\infty = I_X$, where the last equality holds since I_X is B-saturated. For the reverse inclusion, we have $I_X B^t \subseteq I_X \cap B^t \subseteq I_X \cap \langle \mathbf{x}\rangle^t$. The claim now follows from Lemma 6.3.

Next we find an upper and lower bound on the length of this minimal free resolution. By the Auslander-Buchsbaum Formula,

$$\operatorname{pd}(S/(I_X \cap \langle \mathbf{x}\rangle^a) = \operatorname{depth}(S) - \operatorname{depth}(S/(I_X \cap \langle \mathbf{x}\rangle^a)$$

where projective dimension is by definition the length of the minimal free resolution. The depth of S is $n+m+2$. By Lemma 9.6, the Krull dimension of $S/(I_X \cap \langle \mathbf{x}\rangle^a)$ is $m+1$, so the depth of $S/(I_X \cap \langle \mathbf{x}\rangle^a)$ is at most $m+1$. This implies the projective dimension is at least $n+1$. We now show that the depth of $S/(I_X \cap \langle \mathbf{x}\rangle^a)$, for $a \geq |\pi_1(X)| - 1$, is at least 2. We find a regular sequence of length at least two. By Lemma 9.1, we may take y_0 to be a non-zero divisor of S/I_X. Furthermore, y_0 is a non-zero divisor of $S/(I_X \cap \langle \mathbf{x}\rangle^a)$, because if $y_0 F \in I_X \cap \langle \mathbf{x}\rangle^a$, then $y_0 F \in I_X$ and $y_0 F \in \langle \mathbf{x}\rangle^a$. This may only happen if $F \in I_X \cap \langle \mathbf{x}\rangle^a$. Since $a \geq |\pi_1(X)| - 1$, Theorem 9.12 gives a primary decomposition of $\langle I_X \cap \langle \mathbf{x}\rangle^a, y_0\rangle$:

$$\langle I_X \cap \langle \mathbf{x}\rangle^a, y_0\rangle = \bigcap_{A_k \in \pi_1(X)} \langle I_{X_k}, y_0\rangle \cap \langle \langle \mathbf{x}\rangle^a, y_0\rangle.$$

Recall by Remark 9.3 that

$$\langle I_{X_k}, y_0\rangle = I_{A_k} + \bigcap_{j=1}^{\alpha_k} I_{B_{k,j}} + \langle y_0\rangle$$

The zero divisors of $S/\langle I_X \cap \langle \mathbf{x}\rangle^a, y_0\rangle$ are the elements in the union of the associated primes of $S/\langle I_X \cap \langle \mathbf{x}\rangle^a, y_0\rangle$, i.e.

$$\bigcup_{A_k \in \pi_1(X)} \sqrt{\langle I_{X_k}, y_0\rangle} \cup \sqrt{\langle \langle \mathbf{x}\rangle^a, y_0\rangle} = \bigcup_{A_k \in \pi_1(X)} \langle I_{A_k}, G_1, G_2, \ldots, G_s, y_0\rangle \cup \langle \mathbf{x}, y_0\rangle.$$

where each G_i is a polynomial in only y-variables. Let L be any element of degree $(1,0)$ such that L does not vanish at any point in $\pi_1(X)$. Then $L + y_1$ is not in the union of the associated primes of $S/\langle I_X \cap \langle \mathbf{x}\rangle^a, y_0\rangle$, so $L + y_1$ is a non-zero divisor on $S/\langle I_X \cap \langle \mathbf{x}\rangle^a, y_0\rangle$. This implies that the depth of $S/\langle I_X \cap \langle \mathbf{x}\rangle^a\rangle$ is at least 2. The result follows from the Auslander-Buchsbaum formula. $\square$

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References

[ABLS20] Ayah Almousa, Juliette Bruce, Michael Loper, and Mahrud Sayrafi, *The virtual resolutions package for Macaulay2*, J. Softw. Algebra Geom. **10** (2020), no. 1, 51–60.

[BES20] Christine Berkesch, Daniel Erman, and Gregory G. Smith, *Virtual resolutions for a product of projective spaces*, Algebr. Geom. **7** (2020), no. 4, 460–481.

[BKLY21] Christine Berkesch, Patricia Klein, Michael C. Loper, and Jay Yang, *Homological and combinatorial aspects of virtually Cohen-Macaulay sheaves*, Trans. London Math. Soc. **8** (2021), no. 1, 413–434.

[BP] Caitlyn Booms-Peot, *Hilbert-Burch virtual resolutions for points in* $\mathbb{P}^1 \times \mathbb{P}^1$, arXiv.AC:2304.04953, 21 pages.

[BPC22] Caitlyn Booms-Peot and John Cobb, *Virtual criterion for generalized Eagon-Northcott complexes*, J. Pure Appl. Algebra **226** (2022), no. 12, Paper No. 107138, 8.

[BE] Michael K. Brown and Daniel Erman, *Results on virtual resolutions for toric varieties*, arXiv:2303.14319 [math.AG], 5 pages.

[BS] Michael K. Brown and Mahrud Sayrafi, *A short resolution of the diagonal for smooth projective toric varieties of Picard rank 2*, arXiv:2208.00562 [math.AG], 18 pages.

[Dua20] Eliana and Seceleanu Duarte Alexandra, *Implicitization of tensor product surfaces via virtual projective resolutions*, Math. Comp. **89** (2020), no. 326, 3023–3056.

[Eis95] David Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry*, Springer New York, 1995.

[FH] David Favero and Jesse Huang, *Rouquier dimension is Krull dimension for normal toric varieties*, arXiv.2302.09158 [math.AG], 9 pages.

[GLLM21] Jiyang Gao, Yutong Li, Michael C. Loper, and Amal Mattoo, *Virtual complete intersections in* $\mathbb{P}^1 \times \mathbb{P}^1$, J. Pure Appl. Algebra **225** (2021), no. 1, Paper No. 106473, 15.

[GVT15] Elena Guardo and Adam Van Tuyl, *Arithmetically Cohen-Macaulay sets of points in* $\mathbb{P}^1 \times \mathbb{P}^1$, SpringerBriefs in Mathematics, Springer, Cham, 2015.

[Giu92] S. and Maggioni Giuffrida R. and Ragusa, *On the postulation of* 0*-dimensional subschemes on a smooth quadric*, Pacific J. Math. **155** (1992), no. 2, 251–282.

[Giu96] ———, *Resolutions of generic points lying on a smooth quadric*, Manuscripta Math. **91** (1996), no. 4, 421–444.

[HHL] Andrew Hanlon, Jeff Hicks, and Oleg Lazarev, *Resolutions of toric subvarieties by line bundles and applications*, arxiv:2303.03763 [math.AG], 63 pages.

[HNVT22] Megumi Harada, Maryam Nowroozi, and Adam Van Tuyl, *Virtual resolutions of points in* $\mathbb{P}^1 \times \mathbb{P}^1$, J. Pure Appl. Algebra **226** (2022), no. 12, Paper No. 107140, 18.

[KLM+23] Nathan Kenshur, Feiyang Lin, Sean McNally, Zixuan Xu, and Teresa Yu, *On virtually Cohen-Macaulay simplicial complexes*, J. Algebra **631** (2023), 120–135.

[Lop21] Michael C. Loper, *What makes a complex a virtual resolution?*, Trans. Amer. Math. Soc. Ser. B **8** (2021), 885–898.

[Lor93] Anna Lorenzini, *The minimal resolution conjecture*, J. Algebra **156** (1993), no. 1, 5–35.

[MS04] Diane Maclagan and Gregory G. Smith, *Multigraded Castelnuovo-Mumford regularity*, J. Reine Angew. Math. **571** (2004), 179–212.

[GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry.*

[Mus98] Mircea Mustaţă, *Graded Betti numbers of general finite subsets of points on projective varieties*, 1998, pp. 53–81. Pragmatic 1997 (Catania).

[VT02] Adam Van Tuyl, *The border of the Hilbert function of a set of points in* $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, J. Pure Appl. Algebra **176** (2002), no. 2-3, 223–247.

[Vil15] Rafael H. Villarreal, *Monomial algebras*, 2nd ed., Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2015.

[Yan21] Jay Yang, *Virtual resolutions of monomial ideals on toric varieties*, Proc. Amer. Math. Soc. Ser. B **8** (2021), 100–111.

[NHM57] Neal H. McCoy, *Annihilators in Polynomial Rings*, The American Mathematical Monthly **64** (1957), no. 1, 28–29.

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