Definition

Projective space $\mathbb{P}^n$ is defined as the quotient $\mathbb{A}^{n+1}/\sim$, where $x \sim y$ if $y = \lambda x$ for some $\lambda \neq 0$.

We are interested in finite sets of points in $\mathbb{P}^n \times \mathbb{P}^m$. But! we cannot have points of the form

$$[0 : \ldots : 0] \times [b_0 : \ldots : b_m] \text{ or } [a_0 : \ldots : a_n] \times [0 : \ldots : 0].$$
Defining ideals

Definition
Let $X$ be a subset of $\mathbb{P}^n \times \mathbb{P}^m$. The **Cox ring** of $\mathbb{P}^n \times \mathbb{P}^m$ is $S = k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ and is $\mathbb{Z}^2$-graded, where $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$.
Then
$$I(X) = \{ f \in S \mid f(x) = 0 \text{ for all } x \in X \}$$
is the bihomogeneous **defining ideal** of $X$.

We also have the **irrelevant ideal**
$$B = \langle x_0, \ldots, x_n \rangle \cap \langle y_0, \ldots, y_m \rangle.$$
Cox ring and vanishing ideals

Let \( S = k[x_0, \ldots, x_n, y_0, \ldots, y_m] \).

A finite set of points

\[ X = \{P_1, P_2, \ldots, P_s\} \]

in \( \mathbb{P}^n \times \mathbb{P}^m \) has defining ideal

\[ I(X) = I(P_1) \cap I(P_2) \cap \ldots \cap I(P_s). \]

We call \( S/I(X) \) the Cox ring of \( X \).
Example

In \( \mathbb{P}^2 \times \mathbb{P}^1 \), consider the points \( P_1 = [1 : 0 : 0] \times [1 : 0] \), and \( P_2 = [2 : 1 : 0] \times [1 : 2] \). Then

\[
I(P_1) = \langle x_1, x_2, y_1 \rangle,
\]

\[
I(P_2) = \langle x_1 - 2x_0, x_2, y_1 - 2y_0 \rangle.
\]

For \( X = P_1 \cup P_2 \), then

\[
I(X) = I(P_1) \cap I(P_2)
\]

\[
= \langle x_2, 2y_0y_1 - y_1^2, 2x_0y_1 - x_1y_1, 2x_1y_0 - x_1y_1, 2x_0x_1 - x_1^2 \rangle
\]

<table>
<thead>
<tr>
<th>Degree ( d )</th>
<th>Monomial Basis of ( (S/I)_d )</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(1,0)</td>
<td>( x_0, x_1 )</td>
<td>2</td>
</tr>
<tr>
<td>(0,1)</td>
<td>( y_0, y_1 )</td>
<td>2</td>
</tr>
<tr>
<td>(1,1)</td>
<td>( x_0y_0, x_1y_1 )</td>
<td>2</td>
</tr>
</tbody>
</table>
The **Hilbert function** of $S/I(X)$ is the function $H_{S/I(X)} : \mathbb{N}^2 \rightarrow \mathbb{N}$ defined by

$$H_{S/I(X)}(i,j) = \dim_k(S/I(X))_{i,j}$$

$$= \dim_k S_{i,j} - \dim_k I(X)_{i,j}$$

**Example**

$X = \{ [1 : 0 : 0] \times [1 : 0], [2 : 1 : 0] \times [1 : 2] \}$ as before.

$I(X) = \langle x_2, 2y_0y_1 - y_1^2, 2x_0y_1 - x_1y_1, 2x_1y_0 - x_1y_1, 2x_0x_1 - x_1^2 \rangle$

$$H_{S/I(X)}(i,j) = \begin{cases} 1 & (i,j) = (0,0) \\ 2 & \text{otherwise} \end{cases}, \quad H_{S/I(X)} = \begin{bmatrix} 1 & 2 & \cdots \\ 2 & 2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$
Free resolutions

Definition
A graded free resolution of $S/I(X)$ is an exact sequence of free $S$-modules

$$0 \leftarrow S/I(X) \leftarrow \bigoplus_{d \in \mathbb{N}^2} S(-d)^{\beta_{0,d}} \leftarrow \bigoplus_{d \in \mathbb{N}^2} S(-d)^{\beta_{1,d}} \leftarrow \cdots$$

A free resolution is minimal (MFR) if each free module has the minimal number of generators. The $\beta_{i,d}$ are the Betti numbers of $S/I(X)$. 
Example

$X = \{[1 : 0 : 0] \times [1 : 0], [2 : 1 : 0] \times [1 : 2]\}$ as before. A graded MFR of $S/I(X)$ is given by

\[
S(-1, 0) \oplus S(-2, 0) \oplus S(-1, -1)^2 \oplus S(0, -2) \\
\oplus S(-3, 0) \oplus S(-2, -1)^4 \oplus S(-3, -1)^2 \\
\oplus S(-2, -2)^3 \\
\oplus 0
\]

Theorem (Hilbert’s Syzygy Theorem, 1890)

The minimal free resolution of any module over a polynomial ring has finite length, and this length is bounded by the number of variables.
Virtual Resolutions

Definition

Virtual resolutions (VR) are complexes of free $S$-modules which are not necessarily exact:

$$
0 \leftarrow S/I(X) \overset{\phi_0}{\leftarrow} \bigoplus_{d \in \mathbb{N}^2} S(-d)^{\beta_{0,d}} \overset{\phi_1}{\leftarrow} \bigoplus_{d \in \mathbb{N}^2} S(-d)^{\beta_{1,d}} \overset{\phi_2}{\leftarrow} \cdots
$$

The modules $\text{Ker}(\phi_{i-1})/\text{Im}(\phi_i)$ are allowed to have support in the irrelevant ideal

$$
B = \langle x_0, \ldots, x_n \rangle \cap \langle y_0, \ldots, y_m \rangle.
$$

Note:
(1) Every MFR is a VR;
(2) In $\mathbb{P}^n \times \mathbb{P}^m$, while MFRs have length bounded by $n + m + 2$, VRs can have length bounded by $n + m$
Why study resolutions?

MFRs tell us about the module:

- Hilbert function
- Dimension
- Degree
- Vanishing of cohomology
- Embedded deformation theory
- Smoothness for curves
- Compactness
- Complete intersections
- Intersection theory
- Positivity/ampleness
- and more!

Eisenbud’s *Geometry of Syzygies* book summarizes some of these stories for \( \mathbb{P}^n \).

**BUT!** In products of projective space MFRs are “too long”

- VRs are shorter and still give useful geometric information
- Looking at multiple VRs can show *even more* geometry
Two Approaches Towards Virtual Resolutions

Trimming vs. Intersections
First Approach: Trimming

Let $X$ be a finite set of points in $\mathbb{P}^n \times \mathbb{P}^m$.

**Theorem (Maclagan–Smith 2004)**

The multigraded regularity of $S/I(X)$ is

$$\text{reg}(S/I(X)) = \{d \in \mathbb{Z}^2 \mid H_X(d) = |X|\}.$$

**Example**

$X = \{[1 : 0 : 0] \times [1 : 0], [2 : 1 : 0] \times [1 : 2]\}$ as before.

Hilbert matrix: $H_X = \begin{bmatrix} 1 & 2 & \cdots \\ 2 & 2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$
Trimming: First Example

Definition
Trimming at \( d \): keep the free summands in the MFR of \( X \subseteq \mathbb{P}^n \times \mathbb{P}^m \) generated in degree \( \leq d + (n, m) \).

Example
\( X = \{[1 : 0 : 0] \times [1 : 0], [2 : 1 : 0] \times [1 : 2]\} \) as before.

MFR & VR (trimming at \( (1, 0) + (2, 1) = (3, 1) \)):

\[
\begin{align*}
S(-1, 0) & \oplus S(-3, 0) & \oplus S(-3, -1)^2 & \oplus S(-3, -2) & \mapsto 0 \\
S(-2, 0) & \oplus S(-2, -1)^4 & \oplus S(-2, -2)^3 & \oplus S(-1, -2)^3 & \mapsto 0 \\
S(-1, -1)^2 & \oplus S(-1, -2) & \oplus S(-1, -2) & \oplus S(-1, -2) & \mapsto 0 \\
S & \oplus S & \oplus S & \oplus S & \mapsto 0
\end{align*}
\]

Theorem
(berkesch–Erman–smith 2020)
Trimming the MFR of \( X \) at \( \underline{d} \in \text{Reg}(S/I(X)) \) always yields virtual resolutions.
Conjecture

When the points in $X \subseteq \mathbb{P}^n \times \mathbb{P}^m$ are in sufficiently general position, the Hilbert matrix should have a fixed form; namely, we should have

$$H_X(i,j) = \min \left\{ \binom{i+n}{n} \binom{j+m}{m}, |X| \right\}$$

Example

$X =$ set of 12 random points in $\mathbb{P}^1 \times \mathbb{P}^2$ generated in Macaulay2

$$H_X = \begin{bmatrix}
1 & 3 & 6 & 10 & 12 & 12 & \cdots \\
2 & 6 & 12 & 12 & 12 & 12 & \cdots \\
3 & 9 & 12 & 12 & 12 & 12 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
11 & 12 & 12 & 12 & 12 & 12 & \cdots \\
12 & 12 & 12 & 12 & 12 & 12 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix}$$
A certain difference matrix of $H_X$
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & -8 & \cdots \\
0 & 0 & -6 & 16 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
0 & -3 & 9 & -9 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\n\vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \cdots \\
-1 & 3 & -3 & 1 & \cdots \\
0 & 0 & 0 & 0 & \cdots 
\end{bmatrix}
\]

<table>
<thead>
<tr>
<th>Hom. degree</th>
<th>Degree</th>
<th>Betti number</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 3)</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>(2, 2)</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>(4, 1)</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>(12, 0)</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(2, 3)</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>(4, 2)</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>(12, 1)</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>(4, 3)</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>(12, 2)</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>(12, 3)</td>
<td>1</td>
</tr>
</tbody>
</table>

A virtual resolution of $X$ by trimming at $(11, 0) + (1, 2) = (12, 3)$:

\[
S(-2, -2)^6 \oplus S(-4, -1)^3 \leftarrow S(-4, -2)^9 \oplus S(-12, -2)^3 \leftarrow 0
\]

Here we used a version of the Minimal Resolution Conjecture.
Trimming: Result

Assuming the two conjectures, we have:

**Theorem (B-D-G-S-S 2023+)**

*For* \( X \subseteq \mathbb{P}^1 \times \mathbb{P}^2 \) *in sufficiently general position, when* \(|X| \geq 12\), *doing “trimming” at* \((|X| - 1, 0) \in \text{reg}(S/I(X))\) *will always give us a virtual resolution of length 3 of the form:*

\[
S(-m, -2)^{6-r} \oplus S(-m - 1, -2)^{r} \oplus S(-m', -2)^{9-3r'}
\]

\[
S \leftarrow S(-m', -1)^{3-r'} \oplus S(-m' - 1, -2)^{3r'} \leftarrow S(-n, -2)^{3} \leftarrow 0
\]

\[
S(-m' - 1, -1)^{r'} \oplus S(-n, -1)^{3} \oplus S(-n, 0)
\]

*where* \( n = 6m + r = 3m' + r' \).
Second Approach: Intersection with $\langle x \rangle^a$

**Theorem (Harada–Nowroozi–Van Tuyl 2022)**

Let $X$ be a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $t$ denote the number of unique first coordinates. Then for all $a \geq t - 1$, the MFR of $S/(I(X) \cap \langle x_0, x_1 \rangle^a)$ is a VR of $S/I(X)$ of length two.

Our result:

**Theorem (B-D-G-S-S 2023+)**

Let $X$ be a set of points in $\mathbb{P}^n \times \mathbb{P}^1$. Let $t$ denote the number of first coordinates. For all $a \geq t - 1$, the MFR of $S/(I(X) \cap \langle x_0, \ldots, x_n \rangle^a)$ is a VR of $S/I(X)$ of length $n + 1$. 
Example

Let \( X = \{[1 : 0 : 0] \times [1 : 0], [2 : 1 : 0] \times [1 : 2]\} \subseteq \mathbb{P}^2 \times \mathbb{P}^1 \).

Then \( t = 2 \) and

\[
I(X) = \langle x_1, x_2, y_1 \rangle \cap \langle x_1 - 2x_0, x_2, y_1 - 2y_0 \rangle.
\]

The MFR of \( S/(I(X) \cap \langle x_0, x_1, x_2 \rangle^a) \) has length 3 for all \( a \geq 2 - 1 = 1 \).

This MFR (for \( a = 1 \)) is a VR of \( S/I(X) \).

\[
\begin{array}{cccc}
S(-1, 0) & \oplus & S(-2, -1)^4 & \\
S & \leftarrow & S(-2, 0) & \oplus \\
& \oplus & S(-3, 0) & \\
& & S(-3, -1)^2 & \\
& & & \leftarrow 0 \\
& & & \leftarrow S(-1, -1)^2
\end{array}
\]

Recall the MFR of \( S/I(X) \) is length 4.
Theorem (B-D-G-S-S 2023+)

Let $X$ be a set of points in $\mathbb{P}^n \times \mathbb{P}^m$. Let $t$ denote the number of distinct first coordinates. For all $a \geq t - 1$, the MFR of $S/(I(X) \cap \langle x_0, \ldots, x_n \rangle^a)$ is a VR of $S/I(X)$ of length at most $n + m$.

Tools we used:

▶ Auslander–Buchsbaum
▶ Primary decomposition
▶ Short exact sequences and additivity of the Hilbert Function
Acknowledgements

We would like to thank our project mentor Christine Berkesch and TA Sasha Pevzner for all of their help on this project. This project was partially supported by RTG grant NSF/DMS-1745638. It was supervised as part of the University of Minnesota School of Mathematics Summer 2023 REU program.