Simplicial Complexes and Jeu de Taquin Theory

2023 Twin Cities REU in Combinatorics & Algebra

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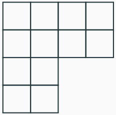
Table of Contents

Background

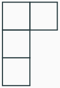
Young Diagram

Straight diagram

$\lambda = (4, 4, 2, 2)$

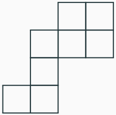


$\mu = (2, 1, 1)$



Skew diagram

$\lambda/\mu = (4, 4, 2, 2)/(2, 1, 1)$



Young Tableaux

A **standard tableau** is a filling of the boxes of a diagram with $[n]$ such that:

- each number is used once;
- numbers increase from left to right;
- numbers increase from top to bottom.

1	2	5	6
3	4		

1	3	4
2		

	2	3
1		

Reading Word

The **reading word** $w_r(T)$ of T is obtained by reading the rows of T, from the last row to the first row.

Example

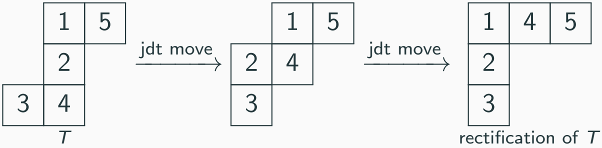
$$w_r\left(\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}\right) = 6341257.$$

The reading word of a standard tableau is a permutation.

Jeu de Taquin (JDT)

Jeu de taquin is an algorithm that turns a skew tableau into a straight tableau.

Each **jeu de taquin move** slides a removed box out of the diagram.



Robinson-Schensted Correspondence

The Robinson-Schensted correspondence (RS) is a bijection:

permuations w of $[n]$	$\xleftrightarrow{RS}$	straight standard tableaux (P, Q) of a same shape λ of size n

Notation P: insertion tableau; Q: recording tableau.

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Notation P: insertion tableau; Q: recording tableau.

Example

$$w = 12534 \quad \xleftrightarrow{RS} \quad P(w) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array}, \; Q(w) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}$$

Dual Equivalence

Definition (Haiman (1992))

Permutations w, z are **dual equivalent** if $Q(w) = Q(z)$.
Tableaux S, T of same shape are **dual equivalent** if $w_{\mathrm{r}}(S) \sim w_{\mathrm{r}}(T)$.

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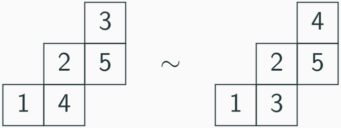
Example

$$14253 \quad \sim \quad 13254$$



Dual equivalence classes are indexed by their recording tableaux Q.

Fact Jeu de taquin moves preserve dual equivalence classes.

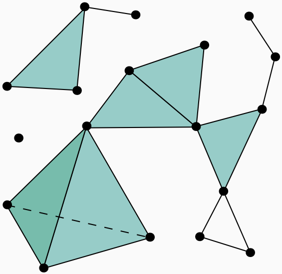
Simplicial Complexes

Definition

An **abstract simplicial complex** Δ is a family of sets, called **faces**, closed by inclusion, that is

$$\text{if } F \subset G \in \Delta, \quad \text{then } F \in \Delta.$$

Facets are maximal elements of Δ.

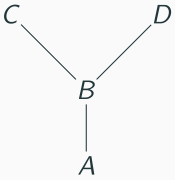


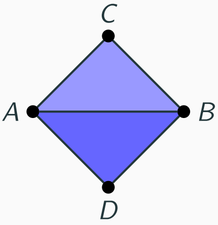
Order Complexes

Let P be a poset.

A **chain** is a totally ordered subset of elements $x_1 < x_2 < \cdots < x_k$.

The **order complex** $\Delta(P)$ is the simplicial complex whose faces are chains of P.





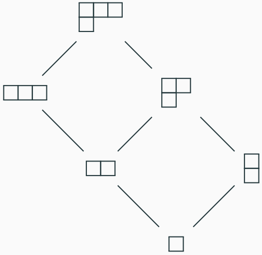
0-dim: A, B, C, D.

1-dim: $A < B$, $B < C$, $B < D$, $A < C$, $A < D$.

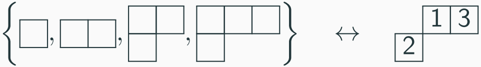
2-dim: $A < B < C$, $A < B < D$.

Young's Lattice is the poset whose elements are Young diagrams, where the ordering is given by inclusion.

We consider finite closed intervals $[\mu, \lambda]$ of Young's lattice.



A maximal chain in $[\mu, \lambda]$ is related to a standard tableau of shape λ/μ.



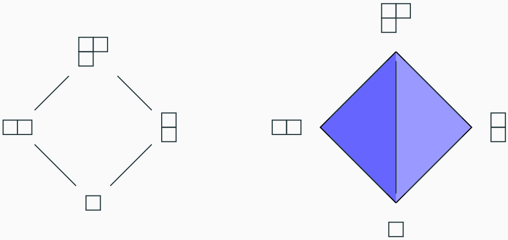
Motivating Theorem

Theorem (Björner and Brenti (2005, Theorem 2.7.7))

The order complex $\Delta([\mu, \lambda])$ is piecewise-linear homeomorphic to a ball.

Example

Consider the order complex $\Delta([\square, \ydiagram{2,1}])$:



Dual Equivalence Complexes

Definition

Let Q be a straight standard tableau. The **dual equivalence complex** $\Delta(Q)$ is the complex with facets corresponding to tableaux T in a dual equivalence class indexed by Q.

Remark Up to isomorphism, $\Delta(Q)$ does not depend on the choice of dual equivalence class.

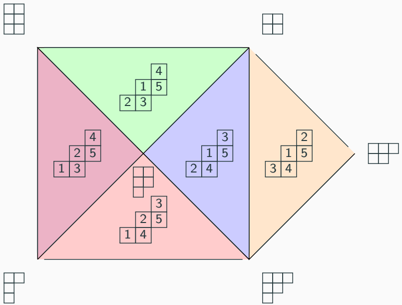
Dual Equivalence Complex Example

The dual equivalence class

$$\left\{ \begin{array}{|c|c|c|} \hline & & 3 \\ & 1 & 5 \\ 2 & 4 & \\ \hline \end{array}, \begin{array}{ccc} & & 2 \\ & 1 & 5 \\ 3 & 4 & \end{array}, \begin{array}{ccc} & & 4 \\ & 1 & 5 \\ 2 & 3 & \end{array}, \begin{array}{ccc} & & 4 \\ & 2 & 5 \\ 1 & 3 & \end{array}, \begin{array}{ccc} & & 3 \\ & 2 & 5 \\ 1 & 4 & \end{array} \right\}.$$

is indexed by $Q = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array}$.

Note that $\Delta(Q)$ has vertices $\ydiagram{2,1}, \ydiagram{3,3}, \ydiagram{3,2}$ that are in every face. Ignoring them, $\Delta(Q)$ is:



Starting Conjecture

Conjecture

Let Q be a straight tandard tableau. The simplicial complex $\Delta(Q)$ is homeomorphic to a ball.

Proof Techniques

Shellability:

- Examine the combinatorial structure of these simplicial complexes, and show that they are **shellable**.
- Any simplicial complex which is pure, subthin, and shellable is homeomorphic to a ball.
- This is how Björner's proof for Young's lattice goes.

Simplicial isomorphisms:

- We know that subcomplexes $\Delta([\mu, \lambda])$ of the order complex are homeomorphic to balls.
- Find a simplicial isomorphism to $\Delta([\mu, \lambda])$?
- *Jeu de taquin moves* preserve dual equivalence classes. Maybe it induces simplicial isomorphisms?

Progress

Small Cases for the Conjecture

Proposition (D., L., W. (2023))

Let Q be a standard tableau with at most 6 boxes. The simplicial complex $\Delta(Q)$ is homeomorphic to a ball.

Counterexamples ☹

For the choices of

$$Q = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 7 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array} \quad \text{or} \quad Q = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}.$$

$\Delta(Q)$ is not Cohen-Macaulay, hence cannot be homeomorphic to a ball. ☹

More Counterexamples, with 8 boxes

$$\begin{matrix}1&2&3&7&8\\4&6\\5\end{matrix}\quad\begin{matrix}1&2&3&4&8\\5&7\\6\end{matrix}\quad\begin{matrix}1&3&5&7\\2&4&8\\6\end{matrix}\quad\begin{matrix}1&2&5&7\\3&4&8\\6\end{matrix}\quad\begin{matrix}1&3&5&6\\2&4&8\\7\end{matrix}\quad\begin{matrix}1&2&5&6\\3&4&8\\7\end{matrix}\quad\begin{matrix}1&3&5&7\\2&6\\4&8\end{matrix}$$

$$\begin{matrix}1&3&4&7\\2&6\\5&8\end{matrix}\quad\begin{matrix}1&3&4&8\\2&7\\5\\6\end{matrix}\quad\begin{matrix}1&2&4&8\\3&7\\5\\6\end{matrix}\quad\begin{matrix}1&2&3&8\\4&7\\5\\6\end{matrix}\quad\begin{matrix}1&4&5&8\\2&6\\3\\7\end{matrix}\quad\begin{matrix}1&2&3&8\\4&6\\5\\7\end{matrix}\quad\begin{matrix}1&4&5&7\\2&6\\3\\8\end{matrix}$$

$$\begin{matrix}1&2&3&7\\4&6\\5\\8\end{matrix}\quad\begin{matrix}1&4&5&6\\2&7\\3\\8\end{matrix}\quad\begin{matrix}1&3&5&6\\2&7\\4\\8\end{matrix}\quad\begin{matrix}1&2&5&6\\3&7\\4\\8\end{matrix}\quad\begin{matrix}1&2&5\\3&6&8\\4\\7\end{matrix}\quad\begin{matrix}1&2&4\\3&6&8\\5\\7\end{matrix}\quad\begin{matrix}1&3&7\\2&4\\5&8\\6\end{matrix}$$

$$\begin{matrix}1&2&7\\3&4\\5&8\\6\end{matrix}\quad\begin{matrix}1&3&6\\2&4\\5&8\\7\end{matrix}\quad\begin{matrix}1&2&6\\3&4\\5&8\\7\end{matrix}\quad\begin{matrix}1&5&6\\2&7\\3\\4\\8\end{matrix}\quad\begin{matrix}1&4&5\\2&6\\3\\7\\8\end{matrix}$$

Evacuation and Transposition on Counterexamples

Note that

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 7 \\ \hline 4 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 7 \\ \cline{1-1}\end{array}$$

are transposes of each other, and are self-evacuating.

Evacuation

Definition

Let Q be a straight standard tableau with n boxes.
The **evacuation** of Q, $\epsilon(Q)$, is obtained by:

- Replacing each entry j with $n+1-j$.
- Rotating the tableau $180°$.
- Rectifying the resulting standard skew tableau.

Example

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array} \xrightarrow{\text{replace}} \begin{array}{|c|c|c|}\hline 5 & 4 & 2 \\ \hline 3 & 1 & \\ \cline{1-2}\end{array} \xrightarrow{\text{rotate}} \begin{array}{|c|c|c|}\cline{2-3} \multicolumn{1}{c|}{} & 1 & 3 \\ \hline 2 & 4 & 5 \\ \hline\end{array} \xrightarrow{\text{rectify}} \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \cline{1-2}\end{array} = \epsilon(T)$$

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Theorem (D., L., W. (2023))

Let Q be a straight standard tableau. Then,

$$\Delta(Q) \cong \Delta(\epsilon(Q)) \cong \Delta(Q^\top).$$

Framing the updated goal

For which recording tableaux Q,
is $\Delta(Q)$ homeomorphic to a ball?

Framing the updated goal

For which recording tableaux Q,
is $\Delta(Q)$ isomorphic to something that is homeomorphic to a ball?

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$$\Delta([\mu, \lambda])$$

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$$\Delta([\varnothing, \lambda])$$

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For which recording tableaux Q,
is $\Delta(Q)$ isomorphic to something that is homeomorphic to a ball?

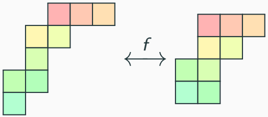
What are the things that are homeomorphic to a ball?

$\Delta([\varnothing, \lambda])$: facets are straight standard tableaux of shape λ.

A Key Technical Lemma

Bijection of Boxes Lemma

If a bijection f of the boxes of two diagrams that bijects facets (a.k.a. standard tableaux) of two subcomplexes, then f induces a simplicial isomorphism between the complexes.



Dual Reading Tableaux

Lemma (Haiman (1992))

Any straight standard tableaux of shape λ *are dual equivalent.*

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$\Delta([\varnothing, \lambda])$ has facets in a dual equivalence class. What's the Q?

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Formally, what's the recording tableau of the reading word of a straight standard tableau of shape λ?

Dual Reading Tableaux

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Any straight standard tableaux of shape λ are dual equivalent.

$\Delta([\varnothing, \lambda])$ has facets in a dual equivalence class. What's the Q?

Formally, what's the recording tableau of the reading word of a straight standard tableau of shape λ? The *dual reading tableaux* of shape λ.

1	3	4	8	13	14
2	6	7	12		
5	10	11			
9					

Proposition (D., L., W. (2023))

If Q is a dual reading tableau, then $\Delta(Q)$ is homeomorphic to a ball.

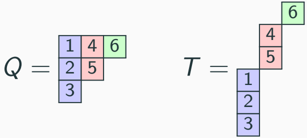
Superstandard Tableaux

Theorem (D., L., W. (2023))

If Q is a column superstandard tableau, $\Delta(Q)$ is homeomorphic to a ball.

1	5	8	11	13	14
2	6	9	12		
3	7	10			
4					

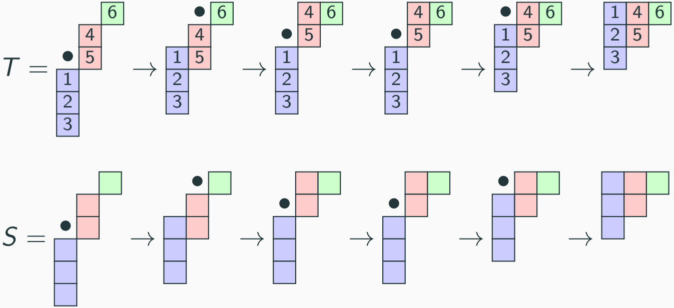
Since $\Delta(Q) \cong \Delta(Q^\top)$, it also holds for a row superstandard tableau.



Fact Q is the recording tableaux of $w_r(T) = 321\,54\,6$.

That is, $\Delta(Q)$ is (isomorphic to) the complex with facets corresponding to the tableaux dual equivalent to T.

Idea for Superstandard Tableaux: Linear Slides (2/2)



Idea 1 Linear slides move the same boxes in S as in T.

Idea 2 Apply "Bijection of Boxes Lemma".

Hence, $\Delta(Q) \cong \Delta([\varnothing, \lambda])$, which is homeomorphic to a ball.

Rectangular Tableaux

Theorem (D., L., W. (2023))
If Q has a rectangular shape λ, then $\Delta(Q) \cong \Delta([\varnothing, \lambda])$, which is homeomorphic to a ball.

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Lemma

$$\overset{P}{\begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline e & f & g & h \\ \hline i & j & k & l \\ \hline\end{array}}, \overset{Q}{\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 9 \\ \hline 4 & 6 & 8 & 11 \\ \hline 5 & 7 & 10 & 12 \\ \hline\end{array}} \overset{RS}{\longleftrightarrow} \overset{w}{[i, j, k, e, a, f, b, g, l, c, h, d]}$$

The permutation w is obtained by reading P in the order defined by

$$Q^{\text{flip}} = \begin{array}{|c|c|c|c|}\hline 5 & 7 & 10 & 12 \\ \hline 4 & 6 & 8 & 11 \\ \hline 1 & 2 & 3 & 9 \\ \hline\end{array}.$$

With "Bijection of Boxes Lemma," we have $\Delta([\varnothing, \lambda]) \cong \Delta(Q)$.

(Some) Hooks

Theorem (D., L., W. (2023))
If Q is an alternating hook-shaped tableau, then $\Delta(Q)$ is homeomorphic to a ball.

1	2	3	4	5	7	9	11	13	14	15	16	17	18
6													
8													
10													
12													

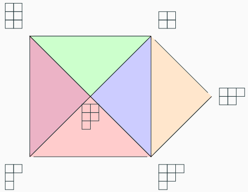
1	2	3	4	5	7	9	11
6							
8							
10							
12							
13							
14							
15							

Why? We don't use "Bijection of Boxes Lemma."

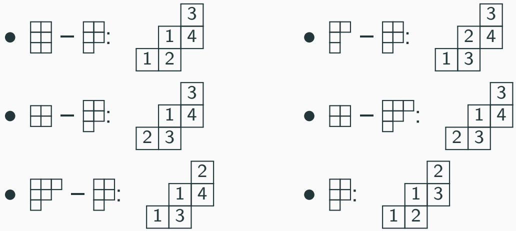
K-jdt and Interior Faces

K-**jeu de taquin** is an analogue of jdt that operates on increasing tableaux.

Interior faces of $\Delta(Q)$ are indexed by increasing tableaux.

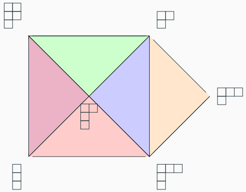


Interior faces:

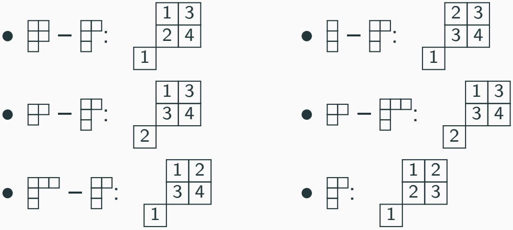


Theorem (D., L., W. (2023))

If a jeu de taquin move at a box induces a simplicial isomorphism, K-jdt "at that box" is the induced map on the interior faces.



Interior faces:



Future Directions

- Do all hook tableaux index shellable complexes?
- Do all sequences of jeu de taquin slides induce simplicial isomorphisms on dual equivalence complexes?
- Provide a more complete classification of which tableaux index complexes homeomorphic to balls.
- Does jdt being a simplicial isomorphism have to do with Q being a **unique rectification target**?

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