CLUSTER MONOMIALS IN GRAPH LP ALGEBRAS

GUILHERME ZEUS DANTAS E MOURA, RAMANUJA CHARYULU TELEKICHERLA KANDALAM, AND DORA WOODRUFF

Contents

1. Introduction

In [FZ02], cluster algebras were introduced by Fomin and Zelevinsky. They are rather ubiquitous throughout math, appearing in Lie theory, triangulations of surfaces, Teichmüller theory, and many other contexts. In [LP16a], Lam and Pylyavskyy define a generalization of cluster algebras, where the exchange polynomials are allowed to have arbitrarily many monomials, rather than being strictly binomial. Then, in [LP16b], Lam and Pylyavskyy define a particularly nice class of LP algebras arising from graphs. LP algebras arising from undirected paths in this way are also cluster algebras, but paths are the only graphs whose associated LP algebras are cluster algebras.

In [LP16b], Lam and Pylyavskyy pose Conjecture 1.1.

Conjecture 1.1 ([LP16b, Conjecture 7.3])**.** Let Γ be a graph, and let $\mathcal{A}_\Gamma$ be its associated LP algebra with coefficient ring R.

(i) The cluster monomials of $\mathcal{A}_\Gamma$ form a basis over R.

(*ii*) (Positivity) Any monomial in the cluster variables can be written as a R-linear combination of cluster monomials with positive coefficients.

The analogues of these conjectures for cluster algebras are well-known statements (some parts are proved, some are still conjectural). In this report, we give a partial positive answer to this conjecture.

1.1. **Organization of the Report.** In Section 2, we give more in-depth background on graph LP algebras. In Section 3, we introduce some quick but useful lemmas which apply to all graphs Γ. In Section 4, we show:

Theorem 1.2. For all graphs, the cluster monomials form a $\mathbb{Z}[A_1 \dots A_n]$-linear spanning set for the associated LP algebra.

In Section 5, we prove formulas for expanding some monomials as positive linear combinations: specifically, we show how to write $Y_I Y_J$ as a linear combination of cluster monomials with positive coefficients when the subgraphs corresponding to I, J satisfy $|I \cap J| \leq 2$. We also conjecture a general method of expanding $Y_I Y_J$ into a positive linear combination of cluster monomials. We also show that when Γ is an undirected graph, in order to prove the general conjecture, it suffices to show that the product of two cluster variables $Y_I Y_J$ is a positive linear combination of cluster monomials. Finally, in Section 6 we show:

Theorem 1.3. Positivity holds for LP algebras given by undirected trees.

Theorem 1.4. Positivity holds for LP algebras given by undirected cycles.

2. Preliminaries

2.1. **Background on LP Algebras.** Laurent phenomenon algebras (LP algebras), introduced by Lam and Pylyavskyy [LP16a], are an extension of cluster algebras. The main idea behind cluster algebras is that the generators of a commutative algebra, called *cluster variables*, can be grouped into sets called *clusters*. A *seed* consists of a cluster and a polynomial associated to each variable in the cluster, called *exchange polynomials*. For cluster algebras, the exchange polynomial is a polynomial in the other variables of the cluster, and is always a binomial. One can then *mutate* the seed to obtain another seed via the following rule:

$$\text{old variable} \times \text{new variable} = \text{exchange binomial}$$

In the more general LP setting, we instead have a mutation rule:

$$\text{old variable} \times \text{new variable} = \text{exchange Laurent polynomial}$$

This exchange Laurent polynomial no longer needs to be binomial, and it can be a Laurent polynomial in the cluster variables (so, we may divide by a monomial in the cluster variables). The motivation for working in this general setting is that many desirable properties that hold for cluster algebras also hold, or seem to hold, for LP algebras. For instance, the Laurent phenomenon (from which LP algebras get their name), a remarkable property of cluster algebras, extends to all LP algebras. As another example, it appears that for finite type LP algebras, the cluster monomials form linear bases. Moreover, it appears that for finite type algebras, all monomials can be written as a linear combination of cluster monomials with *positive coefficients*. These statements generalize well known properties, some proved and some still conjectural, about cluster algebras.

Our project is interested in proving these kinds of statements for particularly nice classes of LP algebras, which arise from graphs in a manner defined by Lam and Pylyavskyy [LP16b]. We summarize their construction in the next subsection.

2.2. **Graph LP Algebras.** First, we recall some graph theory terminology:

Definition 2.1 (Induced subgraphs)**.** Let Γ be a directed graph with vertex set $[n]$. Given a subset $I \subset [n]$, the ***subgraph induced by*** I is the maximal subgraph whose vertices are I, and whose edges are *all* of the edges in Γ between vertices in I.

Unless otherwise stated, all subgraphs in this report are induced subgraphs. So, by abuse of notation, we will sometimes refer to 'the' subgraph on I or refer to a subgraph and its vertex set interchangeably.

Definition 2.2 (Strongly connected subset)**.** Let Γ be a directed graph with vertex set $[n]$. A subset $I \subset [n]$ is said to be ***strongly connected*** if for all $v, w \in I$, there is a directed path from v to w with edges in I.

Now, we can define graph LP algebras following [LP16b]:

Definition 2.3 (Graph LP Algebra)**.** Let Γ be a strongly connected, directed graph with vertex set $[n]$ with edge set E and let R be the ring $\mathbb{Z}[A_1, A_2, ..., A_n]$. For each $i \in [n]$, let E_i denote the set of edges in E that are directed from i to some other vertex in $[n]$. Let t be the seed with cluster variables $\{X_1, X_2, ..., X_n\}$ and exchange polynomials $F_i = A_i + \sum_{(i,j)\in E_i} X_j$. The ***graph LP algebra*** A_Γ asscociated to Γ is the LP algebra generated by the seed t.

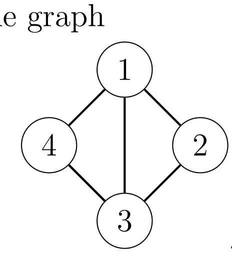
Throughout this report, whenever we refer to an undirected Γ, we really mean a bidirected Γ, in which each edge has both possible orientations. Similarly, in our diagrams, every undirected edge is really a bidirected edge.

In [LP16b], Lam and Pylyavskyy show that graph LP algebras have an especially nice structure; in particular, they give a characterization of the cluster variables and clusters. To describe this structure, we define nested collections of subsets and acyclic functions on subsets.

Definition 2.4 (Nested collections)**.** Let Γ be a directed graph with vertex set $[n]$. Let $\mathcal{I}$ be the set of strongly connected subsets of $[n]$. We say that $\mathcal{S} \subset \mathcal{I}$ is ***nested*** if

(i) for every pair $I, J \in S$,
$$I \subset J \quad \text{or} \quad J \subset I \quad \text{or} \quad I \cap J = \varnothing$$
(ii) for any $\mathcal{T} \subset \mathcal{S}$ such that $I \cap J = \varnothing$ for all $I, J \in \mathcal{T}$, each $I \in \mathcal{T}$ is a strongly connected component of the subgraph induced by $\bigcup_{J \in \mathcal{T}} J$.

Example 2.5. Suppose that Γ is the graph



.

The set $\{\{1, 2, 4\}, \{4\}\}$ is a nested set, since $\{4\} \subset \{1, 2, 4\}$. However, $\{\{1, 2\}, \{4\}\}$ is *not* a nested set since $\{1, 2\}$ and $\{4\}$ are not the connected components of the subgraph induced by $\{1, 2\} \cup \{4\}$.

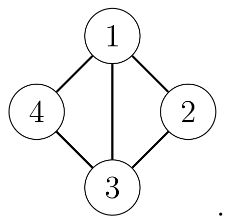
For the purposes of this paper, we consider certain types of multigraphs that are defined in terms of Γ. We describe these multigraphs in the definition below.

Definition 2.6 (Multifunctions)**.** Let Γ be a directed graph with edge set E and vertex set $[n]$. Let I be a multiset with support on V. A ***multifunction*** f of Γ on I is a directed multigraph with vertex set V and edge multiset E' such that

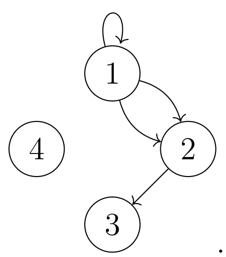
(i) for a vertex $v \in V$, the outdegree of v in f is its multiplicity in I, and
(ii) each edge in E' is either a loop or an edge in E. or a loop

When it is clear what the underlying graph Γ is, we will just say that f is a multifunction on I.

Example 2.7. Let Γ be the graph below.



Let I be the multiset $\{1, 1, 1, 2\}$. The directed multigraph below is a multifunction on I.



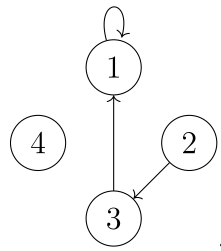
Note that when I is a set, any multifunction of Γ on I corresponds to a function $f : I \to V$. For a function $f : I \to V$ we denote by f both the multigraph corresponding to f and the function f. The following definition formalizes this notion.

Definition 2.8. Suppose that Γ is a graph with vertex set V and edge set E. For $I \subseteq V$, let $f : I \to V$ be a function such that for each $i \in I$, either the edge $(i, f(i))$ is in E or $i = f(i)$. The multifunction corresponding to f is the graph with vertex set V and edge set E' where E' is the set of edges of the form $(i, f(i))$.

Example 2.9. Let Γ be the same graph as Examples 2.5 and 2.7. Let I be the set $\{1, 2, 3\}$ and define $f : I \to [4]$ by

$$f(i) = \begin{cases} 1 & \text{if i = 1} \\ 3 & \text{if i = 2} \\ 1 & \text{if i = 3} \end{cases}.$$

The multifunction corresponding to f is



.

We shall say a multifunction f is acyclic if the only cycles in f are loops. We will say that f is acyclic over a subset of its vertices I if the induced subgraph on I is acyclic as a multifunction on I. We are now able to define acyclic functions over G.

Definition 2.10 (Acyclic functions)**.** Let I $\subseteq V$ and let $f : I \to V$ be a function for which there exists a corresponding multifunction m of I. Then, we say that f is an acyclic function over Γ on I if m is acyclic.

Observe that the function from Example 2.9 is an acyclic function.

Definition 2.11 (Weight of a function)**.** Let $I \subseteq [n]$, for a function $f : I \to [n]$, we denote the weight of f by $w(f)$. We define $w(f)$ by

$$w(f) = \prod_{i \in I} \tilde{x}_{(i,f(i))}, \qquad \tilde{x}_{(i,f(i))} = \begin{cases} x_{(f(i)} & \text{if } f(i) \neq i, \\ A_i & \text{if } f(i) = i. \end{cases}$$

This definition can be extended to multifunctions.

Definition 2.12 (Weight of a Multifunction)**.** Let Γ be a directed graph with edge set E and vertex set V and let I be a multiset with support on $J \subseteq V$. Let f be a multifunction on I with edge set E'. The weight of f is defined by

$$w(f) = \prod_{(i,j) \in E'} \tilde{x}_{(i,j)}, \qquad \tilde{x}_{(i,j)} = \begin{cases} x_j & \text{if } j \neq i, \\ A_i & \text{if } j = i. \end{cases}$$

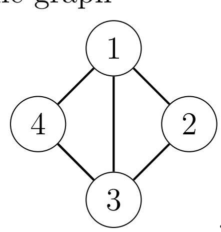
We shall now define certain Laurent polynomials which shall later be used to define the cluster variables of Graph LP algebras explicitly.

Definition 2.13. Let Γ be a graph with vertex set V and edge set E and let $I \subseteq V$. Let F denote the set of acyclic functions on I We define the Laurent polynomial Y_I by

$$Y_I = \frac{\sum_{f \in F} w(f)}{\prod_{i \in I} X_i}.$$

We will often write the polynomial $Y_{\{s_1, s_2, \ldots, s_k\}}$ as $Y_{s_1 s_2 \ldots s_k}$.

Example 2.14. Suppose that Γ is the graph



.

Then, the Laurent polynomial Y_{13} is

$$\frac{A_1A_3+A_1X_1+A_1x_2+A_1x_3+A_3x_2+A_3x_3+A_3x_4+x_2x_1+x_2^2++2x_2x_3+x_3^2+x_4x_1+x_4x_2+x_4x_3}{x_1x_3}.$$

In [LP16b], Lam and Pylyavskyy prove the following description of clusters and cluster variables:

Theorem 2.15. The cluster algebra associated to a graph Γ has:

(i) Cluster variables of the form

$$\{X_1, X_2 \dots X_n\} \cup \{Y_I | I \text{ is strongly connected}\} \tag{1}$$

(ii) Clusters of the form

$$\{X_{i_1}, X_{i_2} \dots X_{i_k}\} \cup \{Y_S | S \in \mathcal{S}\} \tag{2}$$

where $\mathcal{S}$ is a maximal nested collection on $\Gamma \setminus \{i_1, i_2 \dots i_k\}$

3. Introductory Lemmas

Lemma 3.1 allows us to express a single cluster variable as a cluster monomial.

Lemma 3.1 ([LP16b, Lemma 4.2])**.** Let $I_1, I_2 \dots I_k \in \mathcal{I}$ be the strongly connected components of I. Then,

$$Y_I = \prod_{j=1}^{k} Y_{I_j}.$$

Proof. Any combination of acyclic functions on the I_js yields an acyclic function on I by taking their disjoint union, as there are no additional cycles between the I_js. Conversely, any acyclic function on I yields a family of acyclic functions on the I_js by taking restrictions. ☐

Lemma 3.2 allows us to express X_iY_j as a positive linear combination of cluster monomials:

Lemma 3.2 ([LP16b, Lemma 4.7])**.** Let $S \subset \Gamma$, and let $i \in \Gamma$ be a vertex. Furthermore, let $P_I^{i,j} = \sum_{p:i \to_I j} Y_{I \setminus p}$, where p is a path from i to j with all intermediary vertices are in I (but i, j are not necessarily in I). Then,

$$X_iY_{S\cup i} = \sum_{j \in S\cup i} P_S^{i,j} X_j + \sum_{j \in S \cup i} P_S^{i,j} A_j.$$

Proof. For every acyclic function counted in the numerator of $Y_{S\cup i}$, follow the outputs of i until either the function leaves $S \cup i$ or ends in a loop. ☐

Lemmas 3.1 and 3.2 guarantee that, in order to show Conjecture 1.1(ii) for a graph LP algebra $\mathcal{A}_\Gamma$, it suffices to show that any product of Y variables can be written as a sum of cluster monomials.

3.1. The preimages method.

Lemma 3.3. Let $\mathcal{S}, \mathcal{R}$ be sets of tuples of multifunctions. If $\phi\colon \mathcal{S} \to \mathcal{R}$ preserves weights, that is, $w(s) = w(\phi(s))$ for all $s \in \mathcal{S}$, then

$$\sum_{s\in\mathcal{S}} w(s) = \sum_{r\in\mathcal{R}} \left|\phi^{-1}(r)\right| w(r).$$

Proof. Apply the weight-preserving property and double-count pairs $(s, r) \in \mathcal{S}\times\mathcal{R}$ such that $f(s) = r$ to obtain

$$\sum_{s\in\mathcal{S}} w(s) = \sum_{s\in\mathcal{S}} w(\phi(s)) = \sum_{r\in\mathcal{R}} \left|\phi^{-1}(r)\right| w(r). \qquad \square$$

Lemma 3.4 (Preimages Lemma)**.** Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{R}$ be sets of tuples of multifunctions. If $\phi_1\colon \mathcal{S}_1 \to \mathcal{R}$ and $\phi_2\colon \mathcal{S}_2 \to \mathcal{R}$ preserve weights, and $|\phi_1^{-1}(r)| = |\phi_2^{-1}(r)|$ for all $r \in \mathcal{R}$, then

$$\sum_{s\in\mathcal{S}_1} w(s) = \sum_{s\in\mathcal{S}_2} w(s).$$

Proof. Apply Lemma 3.3 on ϕ_1 and on ϕ_2 to obtain

$$\sum_{s\in\mathcal{S}_1} w(s) = \sum_{r\in\mathcal{R}} \left|\phi_1^{-1}(r)\right| w(r) = \sum_{r\in\mathcal{R}} \left|\phi_2^{-1}(r)\right| w(r) = \sum_{s\in\mathcal{S}_2} w(s). \qquad \square$$

4. Integer Coefficients

In the following paragraphs, we introduce technical notation that we will use in this section.

A Y***-monomial*** is a monomial of the form $\prod_{J\in\mathcal{J}} Y_J$, where $\mathcal{J}$ is a multiset of sets of vertices. For example, $Y_1Y_2Y_3^2Y_{123}$ is a Y-monomial. We define a partial order on Y-monomials. Let $Y_{\mathcal{J}_1} = \prod_{J\in\mathcal{J}_1} Y_J$ and $Y_{\mathcal{J}_2} = \prod_{J\in\mathcal{J}_2} Y_J$ be Y-monomials. Let the cardinality vector of $\mathcal{J}$ be the tuple of the cardinalities of the elements of $\mathcal{J}$, sorted increasingly. For example, the cardinality vector of $Y_1Y_{34}^2$ is $(1, 2, 2)$. Since $Y_\varnothing = 1$, assume $\mathcal{J}_1$ has the same number of elements as $\mathcal{J}_2$, by adding $\varnothing$ to one of them. We say that $Y_{\mathcal{J}_1}$ is larger than $Y_{\mathcal{J}_2}$ if $\sum_{J\in\mathcal{J}_1} |J| > \sum_{J\in\mathcal{J}_2} |J|$, or if $\sum_{J\in\mathcal{J}_1} |J| = \sum_{J\in\mathcal{J}_2} |J|$ and the cardinality vector of $\mathcal{J}_1$ is lexicographically larger than $\mathcal{J}_2$. For example, $Y_{1234}Y_{3456}$ is larger than $Y_{16}Y_{34}^2$, since $|1234| + |3456| = 8 > 6 = |16| + |34| + |34|$; and $Y_{1234}Y_{3456}$ is larger than $Y_{34}123456$, since $|1234| + |3456| = 8 = |34| + |123456|$ and $(4, 4)$ is lexicographically larger than $(2, 6)$.

The important properties of this order are that a Y-monomial with more elements among its indices than another is larger, that Y_IY_J is larger $Y_{I\cup J}Y_{I\cap J}$, and that $Y_{\mathcal{J}_1}$ being larger than $Y_{\mathcal{J}_2}$ implies that $Y_{\mathcal{J}}Y_{\mathcal{J}_1}$ is larger than $Y_{\mathcal{J}}Y_{\mathcal{J}_2}$.

Given a set S of vertices, let $\mathcal{Y}_S$ denote the set of acyclic functions on S. Given a multiset S of vertices, let $\mathcal{W}_S$ denote the set of multifunctions on S. Given a tuple $(f_1, ..., f_n)$ of multifunctions, the weight of the tuple is the product $\prod_{i\in[n]} w(f_i)$. Given a set $\mathcal{S}$ of tuples of multifunctions, its weight sum is the sum of the weights of its elements. Given sets $\mathcal{S}_1, \mathcal{S}_2$ of tuples of multifunctions, we say that $\mathcal{S}_1 \sim \mathcal{S}_2$ if they have the same weight sums.

The goal of this section is to prove Theorem 4.1.

Theorem 4.1. A Y-monomial can be written as a linear combination, with integer coefficients, of cluster Y-monomials.

Theorem 4.1 follows from Lemma 4.2 by applying induction on the order of Y-monomials.

Lemma 4.2. A Y-monomial can be written as a linear combination, with integer coefficients, of cluster monomials or smaller Y-monomials.

Let $\mathcal{J}$ be a multiset of sets of vertices, with cardinality k. Let $U = \bigsqcup_{J\in\mathcal{J}} J$ be the multiunion of these sets of vertices. Let $\mathcal{L} = \{L_1, L_2, \ldots, L_k\}$, where L_i denotes the set of vertices that appear at least i times in U.

Proposition 4.3. Let $\mathcal{J}$ be a multiset of sets of vertices, with cardinality k. Let $U = \bigsqcup_{J\in\mathcal{J}} J$ be the multiunion of these sets of vertices. Let $\mathcal{L} = \{L_1, L_2, \ldots, L_k\}$, where L_i denotes the set of vertices that appear at least i times in U. Then,

$$\bigtimes_{J\in\mathcal{J}} \mathcal{W}_J \quad \sim \quad \bigtimes_{L\in\mathcal{L}} \mathcal{W}_L.$$

Proof. Note that $U = \bigsqcup_{L\in\mathcal{L}} L = \bigsqcup_{J\in\mathcal{J}} J$. Consider the functions

$$\phi_J \colon \bigtimes_{J\in\mathcal{J}} \mathcal{W}_J \to \mathcal{W}_U$$

and

$$\phi_L \colon \bigtimes_{L\in\mathcal{L}} \mathcal{W}_L \to \mathcal{W}_U$$

where both functions send a tuple of functions to the multifunction obtaining by taking the union (as a multiset) of the edges of each function. Note that, for all $r \in \mathcal{W}_{I\sqcup J}$,

$$|\phi_J^{-1}(r)| = \prod_{v\in V} \#\,(\text{colorings of the } t \text{ edges from } v \text{ into } t \text{ distinct colors}) = |\phi_L^{-1}(r)|.$$

Hence, applying the Preimages Lemma on ϕ_J and ϕ_L, we obtain that

$$\bigtimes_{J\in\mathcal{J}} \mathcal{W}_J \quad \sim \quad \bigtimes_{L\in\mathcal{L}} \mathcal{W}_L.$$ □

Proposition 4.4. Let S be a set of vertices. Then,

$$\mathcal{W}_S \quad \sim \quad \bigsqcup_{F\in\mathcal{F}_S} \mathcal{Y}_{S\setminus F},$$

where F ranges over all families of vertex-disjoint cycles in the restriction of Γ to S.

Proof. We prove that these sets have the same weight sum by constructing a weight-preserving bijection ϕ between them. Let $w \in \mathcal{W}_S$ be a function on S. Let F denote the family of cycles in w. Since the outdegree in w of every vertex $v \in S$ is 1, the cycles in w are vertex-disjoint. Let $\phi(w)$ be the function obtained by removing the edges of F from the function w. Note that $\phi(w) \in \mathcal{Y}_{S\setminus F}$. Moreover, since F consists of a union of disjoint cycles, each of which has weight 1; therefore, ϕ is weight-preserving. Finally, this map is invertible, since ϕ^{-1} can be defined by adding to an acyclic function $y \in \mathcal{Y}_{S\setminus F}$ the edges in the cycles of F. □

Corollary 4.5. Let $\mathcal{J}$ be a multiset of sets of vertices, with cardinality k. Let $U = \bigsqcup_{J\in\mathcal{J}} J$ be the multiunion of these sets of vertices. Let $\mathcal{L} = \{L_1, L_2, \ldots, L_k\}$, where L_i denotes the set of vertices that appear at least i times in U. Then,

$$\bigtimes_{J\in\mathcal{J}} \bigsqcup_{F\in\mathcal{F}_J} \mathcal{Y}_{J\setminus F} \quad \sim \quad \bigtimes_{L\in\mathcal{L}} \bigsqcup_{F\in\mathcal{F}_L} Y_{L\setminus F}.$$

Moreover,

$$\prod_{J\in\mathcal{J}}\sum_{F\in\mathcal{F}_J} Y_{J\setminus F} = \prod_{L\in\mathcal{L}}\sum_{F\in\mathcal{F}_L} Y_{L\setminus F}.$$

Note that the largest Y-monomial on the left-hand expression is $\prod_{J\in\mathcal{J}} Y_J$, the largest Y-monomial on the right-hand expression is $\prod_{L\in\mathcal{L}} Y_L$, and all other expressions are smaller than these. Therefore, Lemma 4.2 follows.

5. Multiplying Monomials with Small Intersection

5.1. **The Disjoint Case.** Let I, J be sets of vertices. Given a multiset S of vertices, let $\mathcal{Z}_S$ denote the set of multifunctions on S which are with no cycles in I and no cycles in J.

Lemma 5.1. Let I, J be disjoint sets of vertices. Then,

$$\mathcal{Y}_I \times \mathcal{Y}_J \quad \sim \quad \mathcal{Z}_{I\cup J}.$$

Proof. We prove these sets have the same weight sum by constructing a weight preserving bijection from $\mathcal{Y}_I \times \mathcal{Y}_J$ to $\mathcal{Z}_{I\cup J}$. Define $\psi : \mathcal{Y}_I \times \mathcal{Y}_{\mathcal{J}} \to \mathcal{Z}_{I\cup J}$ such that

$$\psi(f_I, f_J) = g_{(f_I,f_J)}$$

where $g_{(f_I,f_J)}$ satisfies

$$g_{(f_I,f_J)}(i) = \begin{cases} f_I(i) & \text{if } i \in I \\ f_J(i) & \text{if } i \in J \end{cases}.$$

Note that $g_{(f_I,f_J)}$ contains no cycles in I or J since that would contradict the fact that f_I and f_J are acyclic. Thus, ψ is a well defined map from $\mathcal{Y}_I \times \mathcal{Y}_{\mathcal{J}}$ to $\mathcal{Z}_{I\cup J}$ since I and J are disjoint. Observe that ψ is weight preserving by Lemma 3.1. Finally, we note that ψ is invertible since ψ^{-1} can be constructed by mapping $g \in \mathcal{Z}_{I\cup J}$ to a tuple $(f_I, f_J) \in \mathcal{Y}_I \times \mathcal{Y}_{\mathcal{J}}$ satisfying

$$g_{(f_I,f_J)}(i) = \begin{cases} f_I(i) & \text{if } i \in I \\ f_J(i) & \text{if } i \in J \end{cases}.$$

Proposition 5.2 is very similar to Proposition 4.4, as well as their proofs.

Proposition 5.2. Let S be a set of vertices. Then,

$$\mathcal{Z}_S \quad \sim \quad \bigsqcup_F \mathcal{Y}_{S\setminus F},$$

where F ranges over all families of vertex-disjoint cycles in the restriction of Γ to S. that are not entirely in I nor entirely in J.

Proof. The proof is similar to the proof of Lemma 5.1. Let $w \in \mathcal{Z}_S$ be a function on S with no cycles in I and no cycles in J. Let F denote the family of cycles in w, which consists of cycles not entirely in I nor entirely in J. Since the outdegree in w of every vertex $v \in S$ is 1, the cycles in w are vertex-disjoint. Let $\phi(w)$ be the function obtained by removing the edges of F from the function w. Note that $\phi(w) \in \mathcal{Y}_{S\setminus F}$. Moreover, since F consists of a union of disjoint cycles, each of which has weight 1; therefore, ϕ is weight-preserving. Finally, this map is invertible, since ϕ^{-1} can be defined by adding to an acyclic function $y \in \mathcal{Y}_{S\setminus F}$ the edges in the cycles of F. $\square$

Theorem 5.3. Let I, J be disjoint sets of vertices. Let $\mathcal{F}$ denote the collection of families of vertex-disjoint cycles in the restriction of Γ to $I \cup J$ that are not entirely in I nor entirely in J. Then,

$$\mathcal{Y}_I \times \mathcal{Y}_J \quad \sim \quad \bigsqcup_{F \in \mathcal{F}} \mathcal{Y}_{I \cup J \setminus F}.$$

Moreover,

$$Y_I Y_J \;=\; \sum_{F \in \mathcal{F}} Y_{I \cup J \setminus F},$$

Proof. The first part of the theorem follows from Proposition 5.2 and Lemma 5.1. The second part follows from the fact that the weight sum of $\mathcal{Y}_I \times \mathcal{Y}_J$ is $Y_I Y_J$ and $\sum_{F \in \mathcal{F}} Y_{I \cup J \setminus F}$ is the weight sum of $\sqcup_{F \in \mathcal{F}} \mathcal{Y}_{I \cup J \setminus F}$. ☐

Remark. If we let ϕ be the function in 5.2 (taking S to be $I \cup J$) and let ψ be the function in 5.1, the map $\phi \circ \psi$ is a weight preserving bijection from $\mathcal{Y}_I \times \mathcal{Y}_J$ to $\sqcup_{F \in \mathcal{F}} \mathcal{Y}_{I \cup J \setminus F}$.

5.2. **The Intersection 1 Case.** In this subsection, we prove Theorem 5.4 using Lemma 3.4.

Theorem 5.4. Let I and J be sets of vertices and suppose that $I \cap J = \{v\}$. Define $\mathcal{C}_I$ and $\mathcal{C}_J$ as the sets of cycles in I containing v and in J containing v respectively. Also define $\mathcal{F}$ as the collection of families of vertex disjoint cycles in $I \cup J$ not contained entirely in I or J, and similarly define $\mathcal{F}'$ as the subset of $\mathcal{F}$ containing only families of cycles in $I \cup J \setminus C_1 \setminus C_2$. Then,

$$Y_I Y_J = \sum_{F \in \mathcal{F}} Y_{I \cup J \setminus F} + \sum_{C_I \mathcal{C}_I} \sum_{C_J \in \mathcal{C}_J} \sum_{F' \in \mathcal{F}'} Y_{I \cup J \setminus C_I \setminus C_J \setminus F'}.$$

Proof. Let $\mathcal{S}_1$ be the set $\mathcal{Y}_I \times \mathcal{Y}_J$, let $\mathcal{S}_2$ be the set $\mathcal{Y}_v \times \mathcal{Z}_{I \cup J}$ and let $\mathcal{S}_3$ be the set of functions mapping $I \sqcup J$ to $[n]$ which have exactly one cycle in J containing v, exactly one cycle in I containing v, and no other cycles contained in I or J, v having outdegree 2 and all other vertices having outdegree 1. We have the following observations

- (i) Observe that $\mathcal{S}_2$ and $\mathcal{S}_3$ are disjoint.
- (ii) The weight sum of $\mathcal{S}_1$ is $Y_I Y_J$
- (iii) The weight sum of $\mathcal{S}_2$ is $\sum_{F \in \mathcal{F}} Y_{I \cup J \setminus F}$
- (iv) The weight sum of $\mathcal{S}_3$ is $\sum_{C_I \in \mathcal{C}_I} \sum_{C_J \in \mathcal{C}_J} \sum_{F' \in \mathcal{F}'} Y_{I \cup J \setminus C_I \setminus C_J \setminus F'}$

Finally, we let $\mathcal{R}$ denote the set of multifunctions from $I \cup J$ to $[n]$ for which v has outdegree 2 and all other vertices in $I \cup J$ have outdegree 1.

We now define $\phi_1 : \mathcal{S}_1 \to \mathcal{R}$ by (f_I, f_J) maps to $g = (I \cup J, E)$ where E the multiset of edges containing the edge (i, j) with multiplicity equal to the number of times $f_I(i) = j$ or $f_J = (i)$. Define ϕ_2 similarly. Finally define ϕ_3 by inclusion. Note that ϕ_1, ϕ_2, and ϕ_3 are all weight preserving. We will now show that $|\phi_1^{-1}(g)| = |\phi_2^{-1}(g)| + |\phi_3^{-1}(g)|$. First we note that if g has a cycle contained entirely in $I \setminus v$ or $J \setminus v$, then $|\phi_1^{-1}(g)| = |\phi_2^{-1}(g)| = |\phi_3^{-1}(g)| = 0$. We will now assume that g has no cycles contained entirely in I or J. Similarly, note that if g has 2 cycles in I or two cycles in J, then $|\phi_1^{-1}(g)| = |\phi_2^{-1}(g)| = |\phi_3^{-1}(g)| = 0$. This leaves us with four cases where at least one of $|\phi_1^{-1}(g)|$, $|\phi_2^{-1}(g)|$, or $|\phi_3^{-1}(g)|$ is non-zero. These cases are

Case 1: When g has one cycle in I and one cycle in J, we have $|\phi_1^{-1}(g)| = 1$, $|\phi_2^{-1}(g)| = 0$, and $|\phi_3^{-1}(g)| = 1$.

Case 2: When g has one cycle in I and no cycles in J, we have $|\phi_1^{-1}| = 1$, $|\phi_2^{-1}(g)| = 1$, and $|\phi_3^{-1}(g)| = 0$.

Case 3: When g has no cycles in I and no cycles in J, we have $|\phi_1^{-1}(g)| = 1$, $|\phi_2^{-1}(g)| = 1$, and $|\phi_3^{-1}(g)| = 0$.

Case 4: When g has no cycles in I nor in J, we have $|\phi_1^{-1}(g)| = 2$, $|\phi_2^{-1}(g)| = 2$, and $|\phi_3^{-1}(g)| = 0$.

Thus, for all $g \in \mathcal{R}$, we have $|\phi_1^{-1}(g)| = |\phi_2^{-1}(g)| + |\phi_3^{-1}(g)|$. Since each ϕ_i is weight preserving, by Lemma 3.4 we have that the weight sum of $\mathcal{S}_1$ is equal to the weight o f$\mathcal{S}_2 \cup \mathcal{S}_2$. Therefore,

$$Y_I Y_J = \sum_{F \in \mathcal{F}} Y_{I \cup J \setminus F} + \sum_{C_I \mathcal{C}_I} \sum_{C_J \in \mathcal{C}_J} \sum_{F' \in \mathcal{F}'} Y_{I \cup J \setminus C_I \setminus C_J \setminus F'}. \qquad \square$$

In the case where $|I \cap J| = 2$, we can apply a similar method to find a formula for $Y_I Y_J$ that has all positive coefficients; however, this proof is much longer and more computational than the previous two cases, so for brevity we omit it from this report. As $|I \cap J|$ gets larger, in principle this method would still work, but the computations become much more complicated quickly.

6. Proving Positivity for Special Graphs

The goal of this section is to prove Theorem 1.3 and 1.4:

Theorem 6.1. When Γ is an undirected tree or an undirected cycle, the LP algebra associated to Γ satisfies [LP16b, Conjecture 7.3(2)]: every monomial can be expressed as a linear combination of cluster monomials with *positive* coefficients.

First, we provide a general overview of the strategy and introduce some helpful vocabulary.

6.1. Multiplying Monomials in the Undirected Graph case.

Let Γ be an undirected graph.

Recall the definition of the order on the Y-monomials from Section 4. The important properties of this order are that a Y-monomial with more elements among its indices than another is larger, that $Y_I Y_J$ is larger $Y_{I\cup J} Y_{I \cap J}$, and that $Y_{\mathcal{J}_1}$ being larger than $Y_{\mathcal{J}_2}$ implies that $Y_{\mathcal{J}} Y_{\mathcal{J}_1}$ is larger than $Y_{\mathcal{J}} Y_{\mathcal{J}_2}$.

Lemma 6.2. Let Γ be an undirected graph. Assume that any non-cluster monomial $Y_I Y_J$ can be written as a sum of smaller Y-monomials. Then, any Y-monomial can be written as a sum of cluster monomials.

Proof. We prove it by induction on the order of the Y-monomial. Consider an Y-monomial $\prod_{K \in \mathcal{K}} Y_K$. Assume, by induction hypothesis, that any smaller Y-monomial can be written as a sum of cluster monomials.

If $\prod_{K \in \mathcal{K}} Y_K$ is a cluster monomial, we are done. Otherwise, we claim that, because Γ is undirected, there exists distinct sets $I, J \in \mathcal{J}$ such that $\{I, J\}$ is non-nested. To see why, suppose that $\mathcal{J}$ is non-nested. Then, either there must be a pair I, J with $I \cap J \neq \emptyset$ and $I \not\subset J, J \not\subset I$, or there is some collection $I_1 \dots I_k \in \mathcal{J}$ which are pairwise disjoint but not the strongly connected components of their intersection. In the first case, $\{I, J\}$ is non-nested. In the second case, since Γ is strongly connected, there must be some path from I_i to I_j for some $I, I \in \{I_1 \dots I_k\}$. But, since Γ is undirected, this path is also a path from J to I. Therefore, the only strongly connected component of $I \cup J$ is $I \cup J$, and so the pair $\{I, J\}$ is not nested.

By the assumption, we can write the non-cluster monomial $Y_I Y_J$ as a sum of smaller Y-monomials. Hence, we can write $\prod_{K\in\mathcal{K}} Y_K = Y_I Y_J \prod_{K\in\mathcal{K}\setminus\{I,J\}} Y_K$ as a sum of smaller Y-monomials, for which we can apply the induction hypothesis and finally write $\prod_{K\in\mathcal{K}} Y_K$ as a sum of cluster monomials. □

Therefore, for the rest of this section we will focus on multiplying two cluster variables $Y_I Y_J$.

The general strategy is to use the preimages method. We use the following notation for the rest of this section:

(i) We let $\mathcal{S}_0$ denote the set of pairs of functions (f_I, f_J) where f_I is an acyclic function on I and f_J is an acyclic function on J.

(ii) Similarly, $\mathcal{S}_1$ will denote the set of pairs of functions $(f_{I\cup J}, f_{I\cap J})$ where $f_{I\cup J}$ is an acyclic function on $I \cup J$ and $f_{I\cap J}$ is an acyclic function on $I \cap J$.

(iii) We let $\mathcal{R}$ denote the set of multifunctions on $I \cup J$ such that every vertex in $I \cap J$ has outdegree 2, and every vertex in $(I \cup J) \setminus (I \cap J)$ has outdegree 1. (Intuitively, all the multifunctions we could possibly obtain by gluing together pairs of functions in $\mathcal{S}_0$ or $\mathcal{S}_1$).

(iv) There are natural maps $\phi_0 : \mathcal{S}_0 \to \mathcal{R}$ and $\phi_1 : \mathcal{S}_1 \to \mathcal{R}$ taking multiunions of edges.

Remark. We can think of $r \in \mathcal{R}$ either as a multifunction, where each vertex in $I \cap J$ has two outputs, or as a directed (not necessarily simple) graph where each vertex in $I \cap J$ has outdegree 2. Thus, by abuse of notation when we say 'cycles in r,' we mean cycles that appear in r when we think of r as a graph.

We can see that ϕ_0 and ϕ_1 are weight-preserving maps. So, in order to apply the preimages method, we will want to compare $|\phi_0^{-1}(r)|$ and $|\phi_1^{-1}(r)|$ for a given $r \in \mathcal{R}$. We will not generally have $|\phi_0^{-1}(r)| = |\phi_1^{-1}(r)|$; the idea is that we always will have $|\phi_0^{-1}(r)| \geq |\phi_1^{-1}(r)|$, and in cases where the inequality is strict, we will add in suitable 'correction terms.' We do this by analyzing the possible collections of cycles that can appear in r; in principal, this method applies to every undirected graph, but this step is relatively simple for trees and cycles, and can be much more complicated for general graphs.

In order to investigate these preimage sizes, we will use the convenient language of colorings:

6.2. **Colorings.** Fix some $r \in \mathcal{R}$. The tools in this subsection in principal apply

Definition 6.3. A $\mathcal{S}_0$-*valid* coloring of r is a labeling of the edges of r with either I or J such that:

(i) Every edge originating from a vertex in $I \setminus J$ is colored with I.

(ii) Every edge originating from a vertex in $J \setminus I$ is colored with J.

(iii) For every vertex $v \in I \cap J$, the to edges originating from v are colored differently.

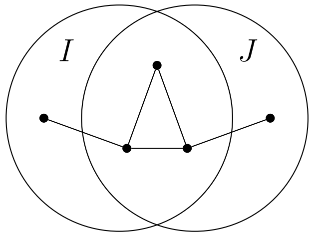
Similarly, a $\mathcal{S}_1$-*valid* coloring of r is a labeling of the edges of r with either $I \cap J$ or $I \cup J$ such that:

(i) Every edge originating from a vertex in $(I \cup J) \setminus (I \cap J)$ is colored with $I \cup J$.

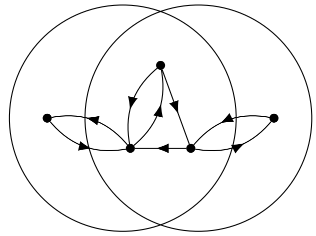
(ii) For every vertex $v \in I \cap J$, the to edges originating from v are colored differently.

We call a coloring *acyclic* if there is no monochromatic cycle.

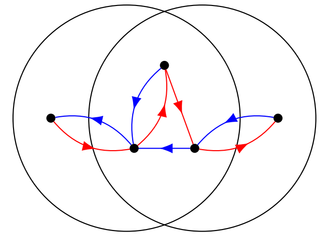
Example 6.4. In the example below, I-colored edges are represented with red and J-colored edges are represented with blue. Refer to Figure 1.



(A) Graph Γ.



(B) A multifunction r on $I \sqcup J$.



(C) An acyclic $\mathcal{S}_0$-valid coloring of $I \sqcup J$.

FIGURE 1. Example of $\mathcal{S}_0$-valid coloring. I-colored edges are represented with red and J-colored edges are represented with blue.

Notice that the number of acyclic, $\mathcal{S}_i$-valid colorings of r is just exactly $|\phi_i^{-1}(r)|$.

In order to give a $\mathcal{S}_i$-valid acyclic coloring on some r, it essentially suffices to color the 'connected components' of cycles independently. Below we rigorize this notion:

Definition 6.5. Say that two cycles are *adjacent* if they have at least one vertex in common. We can draw an auxiliary graph G, where cycles are represented by vertices and edges are given by cycle adjacencies. We say that *connected families* of cycles correspond to the connected components of this graph G.

Notice that, by definition, connected families of cycles are pairwise vertex disjoint.

Lemma 6.6. Let $\mathcal{C}_1, \mathcal{C}_2 \dots \mathcal{C}_k$ be the connected families of cycles of some r, and say that $|(I \cap J) \setminus \bigsqcup_i \mathcal{C}_i| = k$. Then,

$$\#(\mathcal{S}_i\text{-valid acyclic colorings of } r) = 2^k \prod_j \#(\mathcal{S}_i\text{-valid acyclic colorings of } \mathcal{C}_j)$$

Proof. Choose a coloring of each $\mathcal{C}_i$; since these cycles are vertex disjoint, these colorings can be chosen independently. Then, choose colorings for the remaining vertices; there are no more cycles left in r once we have already colored all of the connected families, so any valid coloring will be acyclic. $\square$

Lemma 6.7. Let $\mathcal{C}$ be a connected family of cycles such that $\mathcal{C} \subset I$ (or $\mathcal{C} \subset J$). Then,

$$\#(\mathcal{S}_0\text{-valid acyclic colorings of } \mathcal{C}) = \#(\mathcal{S}_0\text{-valid acyclic colorings of } \mathcal{C})$$

Proof. There is a bijection between acyclic $\mathcal{S}_0$-colorings and acyclic $\mathcal{S}_1$-colorings of $\mathcal{C}$ by sending edges colored with I (respectively, J if $\mathcal{C} \subset J$) to edges colored with $I \cup J$ and edges colored with J (respectively I) to edges colored with $I \cap J$. $\square$

In particular, if for some $r \in \mathcal{R}$, each connected families of cycles is contained either in I or in J, then Lemmas 6.7 and 6.6 imply that $|\phi_0^{-1}(r)| = |\phi_1^{-1}(r)|$.

6.3. **Trees.** In this subsection, we apply the ideas of the previous section to the case where Γ is an undirected (really, bidirected) tree. So, we want to be able to multiply two cluster variables $Y_I Y_J$. The only cycles in Γ are thus the 2-cycles associated to every edge of Γ.

Definition 6.8. Let $v_1, v_2 \dots v_k$ be the vertices of a path in $I \cup J$, where $v_1 \in I \setminus J$, $v_k \in J \setminus I$, and $v_i \in I \cap J$ for $1 < i < k$. Then, an *anchored chain* is given by all of the two-cycles $(v_1, v_2) \dots (v_{k-1}, v_k)$.

Intuitively, an anchored chain is a 'path of cycles' that travels from I to J.

Lemma 6.9. Let I, J be strongly connected induced subgraphs in Γ with nontrivial intersection. We can compute the product of their corresponding cluster variables as follows:

Let $\mathcal{P}$ be the set of all families of disjoint anchored chains. Then

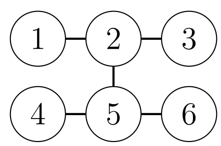
$$Y_I Y_J = \sum_{P \in \mathcal{P}} Y_{I \cup J \setminus \mathcal{P}} Y_{I \cap J \setminus \mathcal{P}} \tag{3}$$

Example 6.10. This lemma gives a second proof of the formula by Lam and Pylyavskyy [LP16b] for multiplying two monomials in the case where Γ is an undirected path. Let I and J be overlapping segments with vertices $1, 2 \dots k$ and $l \dots m$: then their formula gives

$$Y_I Y_J = Y_{I \cup J} Y_{I \cap J} + Y_{1,2 \dots l-2} Y_{k+2 \dots m} \tag{4}$$

In this case, the only anchored chain between I and J is the chain on the vertices $l-1, l \dots k, k+1$. Thus, the second term corresponds to removing this chain from $I \cup J$ and $I \cap J$, and the first term corresponds to the empty family of chains.

Example 6.11. Let Γ be the following tree:



Let $I = \{1, 2, 4, 5\}$ and let $J = \{2, 3, 5, 6\}$. Then the possible collections of disjoint anchored chains between I and J are given by:

(i) $\{(12), (23)\}$
(ii) $\{(45), (56)\}$
(iii) $\{(12), (23)\}$ and $\{(45), (56)\}$ (as they are disjoint)
(iv) $\{(12), (25), (56)\}$

By the lemma, we compute

$$Y_{1245} Y_{2356} = Y_{123456} Y_{25} + Y_{456} + Y_{123} + Y_{43} + Y_{16} + 1. \tag{5}$$

Now, we prove Lemma 6.9:

Proof. For a given family P of disjoint anchored chains in $\mathcal{P}$, let $\mathcal{S}_P$ denote the set of pairs of functions $(f_{I \cup J}, f_{I \cap J})$ such that:

(i) $f_{I \cup J}$ is a function on $I \cup J$ such that for every path $v_1 \to v_2 \cdots \to v_k \in P$, we have $f_{I \cup J}(v_i) = v_{i+1}$ for $i < k$ and $f_{I \cup J}(v_k) = v_{k-1}$. Furthermore, $f_{I \cup J}$ is acyclic on $I \cup J \setminus P$.
(ii) $f_{I \cap J}$ is a function on $I \cap J$ such that for every path $v_1 \to \cdots \to v_k \in P$, we have $f_{I \cap J}(v_i) = v_{i-1}$ for $1 < i < k$. Furthermore, $f_{I \cap J}$ is acyclic on $I \cap J \setminus P$.

Let $\phi_P : \mathcal{S}_P \to \mathcal{R}$ be the usual multiunion function, which is weight preserving. Furthermore, notice that

$$\sum_{(f_{I \cup J}, f_{I \cap J}) \in P} w(f_{I \cup J}) w(f_{I \cap J}) \tag{6}$$

is the same as the numerator of $Y_{I \cup J \setminus P} Y_{I \cap J \setminus P}$.

We will show that for all $r \in \mathcal{R}$:

$$|\phi_0^{-1}(r)| = |\phi_1^{-1}(r)| + \sum_{P \in \mathcal{P}} |\phi_P^{-1}(r)| \tag{7}$$

By the preimages method, the lemma will then follow.

As outlined in the overview previously, we start by classifying possible arrangements of cycles for some $r \in \mathcal{R}$. We have the following observations:

(i) Since Γ is an undirected (i.e. bidirected) tree, the only cycles in Γ are the two-cycles corresponding to every edge.

(ii) Since I and J are strongly connected and Γ is a tree, there can be no edges in $I \cup J$ that are not contained in either I or J. (If there were such an edge between $v \in I$ and $w \in J$, then there would be a cycle in Γ containing v and w).

(iii) The only possibly connected families of cycles which are not entirely contained in either I or J are of the following form: choose a path $v_1, v_2 \dots v_k$ between I and J as specified in the statement of Lemma 6.9, and take the two-cycles (v_1, v_2), (v_2, v_3), $\dots$, (v_{k-1}, v_k).

To justify the last observation: by definition, the maximum outdegree of a vertex in $r \in \mathcal{R}$ is 2. Therefore, each two-cycle C can be adjacent to at most two other two-cycles, because each vertex of C can be in at most one other two-cycle. Furthermore, any cycles in $I \setminus J$ or $J \setminus I$ cannot be adjacent to any other two-cycles, since vertices not in $I \cap J$ have outdegree one. Therefore, any connected family of cycles not in I and not in J must have: one cycle in I with exactly one vertex in $I \cap J$, one cycle in J with exactly one vertex in $I \cap J$, and all other cycles in the intersection forming a path between them.

Now, let's look at the possible colorings of an anchored chain p on $v_1, v_2 \dots v_k$, where $v_1 \in I$ and $v_k \in J$.

Claim 6.12. The number of acyclic $\mathcal{S}_0$-valid colorings of p is 1. The number of acyclic $\mathcal{S}_1$-valid colorings of p is 0.

To show this claim, first, we count the number of acyclic $\mathcal{S}_0$-valid colorings. We must color the edge $v_1 \to v_2$ with I, since it originates from a vertex in $I \setminus J$. To avoid a monochromatic I-cycle between v_1, v_2, we must then color $v_2 \to v_1$ with J. Now, because the edges originating from $v_2 \in I \cap J$ must have two different colors and $v_2 \to v_1$ is colored with J, $v_2 \to v_3$ must be colored with I. Inductively, all edges $v_i \to v_{i+1}$ must be colored with I, and all edges $v_i \to v_{i-1}$ must be colored with J. When we color the last cycle (v_{k-1}, v_k), we must color $v_k \to v_{k-1}$ with J, since $v_k \in J \setminus I$, and this condition is satisfied by coloring $v_i \to v_{i+1}$ with I and $v_i \to v_{i-1}$ with J. All of our choices were forced, so there is 1 coloring of p which is acyclic and $\mathcal{S}_0$-valid.

Now, we count the number of acyclic $\mathcal{S}_1$-valid colorings. Since $v_1 \notin I \cap J$, we must color $v_1 \to v_2$ with $I \cup J$. To avoid a monochromatic $I \cup J$-cycle between v_1, v_2, we must color $v_2 \to v_1$ with $I \cap J$. By analogous reasoning to before, we must color $v_i \to v_{i+1}$ with $I \cup J$ and $v_i \to v_{i-1}$ with $I \cap J$. However, the last edge we color, $v_k \to v_{k-1}$, cannot be colored with $I \cap J$ since $v_k \notin I \cap J$. Therefore, there are *no* colorings of p which are acyclic and $\mathcal{S}_1$-valid.

Now, fix $r \in \mathcal{R}$. If every connected family of cycles is either in I or in J, then by Lemma 6.7 and Lemma 6.6, we have

$$|\phi_0^{-1}(r)| = |\phi_1^{-1}(r)|$$

Furthermore, for all other $\mathcal{S}_P$, by definition the image of any $\phi_P(f_{I\cup J}, f_{I\cap J}) \in \mathcal{R}$ contains some connected family of cycles not in I and not in J. Thus, we have $|\phi_P^{-1}(r)| = 0$. In this case,

$$|\phi_0^{-1}(r)| = |\phi_1^{-1}(r)| + \sum_{P\in\mathcal{P}} |\phi_P^{-1}(r)|$$

With similar reasoning, we do casework on which families of anchored chains can occur in r. Such families of anchored chains are given exactly by disjoint families of paths from I to J in $\mathcal{P}$. By the same reasoning as the above paragraph, we have

$$|\phi_P^{-1}(r)| = |\phi_0^{-1}(r)|$$

Furthermore, we claim that $|\phi_{P'}^{-1}(r)| = 0$ for $P' \neq P$. This is because for any paths $p \in P$ and $p' \in P'$, if $p \neq p'$ we have $p \not\subset p'$ and $p' \not\subset p$. Therefore, for any $P' \neq P$, we have that either:

- (i) There is some $p' \in P'$ which is not in P. In this case, $\phi_{P'}((f_{I\cup J}, f_{I\cap J}))$ has cycles which are not in r (so that $|\phi_{P'}^{-1}(r)| = 0$
- (ii) Every $p' \in P'$ is also found in P, but there is some $p \in P$ not in P' (otherwise, $P' = P$). In this case, every path in P' is disjoint from p. But then, $|\phi_{P'}^{-1}(r)| = 0$, because of the fact that the number of ways to give an acyclic $\mathcal{S}_1$-valid coloring of an anchored chain is 0.

Therefore, we have that $|\phi_0^{-1}(r)| = |\phi_1^{-1}(r)| + \sum_{P\in\mathcal{P}} |\phi_P^{-1}(r)|$ for all $r \in \mathcal{R}$, as desired. □

Remark. A key observation that makes this argument for trees work, somewhat hidden in this proof, is the fact that for any two distinct paths p_1, p_2, $p_1 \not\subset p_2$ and $p_2 \not\subset p_1$. This observation allows us to show that $|\phi_{P'}^{-1}(r)| = 0$ when the family of anchored chains appearing in r is $P \neq P'$, and in general this is the difficult step to generalize to all graphs.

Now, we can prove Theorem 1.3:

Proof. By Lemma 6.2, it is sufficient to expand the product of any two cluster variables $Y_I Y_J$ as a linear combination of cluster monomials with positive coefficients. We induct on $|I \cap J|$. When $I \cap J = \emptyset$, we showed in the previous section how to expand $Y_I Y_J$, proving the base case. Now, by Lemma 6.9, we can expand

$$Y_I Y_J = Y_{I\cup J} Y_{I\cap J} + \sum_{P\in\mathcal{P}} Y_{I\cup J\setminus P} Y_{I\cap J\setminus P}$$

The term $Y_{I\cup J} Y_{I\cap J}$ is already a cluster monomial, but the terms $Y_{I\cup J\setminus P} Y_{I\cap J\setminus P}$ need not be. Furthermore, $I \cup J \setminus P$ and $I \cap J \setminus P$ are no longer strongly connected. However, by Lemma 3.1, we can expand each monomial $Y_{I\cup J\setminus P} Y_{I\cap J\setminus P}$ into $Y_{A_1} Y_{A_2} \dots Y_{A_k} Y_{B_1} Y_{B_2} \dots Y_{B_{k'}}$, where A_i's are the connected components of $I \cup J \setminus P$ and the B_i's are the connected components of $I \cap J \setminus P$. Furthermore, since each P contains vertices in $I \cap J$, we have for all A_i, B_j that $|A_i \cap B_j| < |I \cap J|$. Therefore, by induction on the size of the intersection, we may express any $Y_{A_i} Y_{B_j}$ as a positive linear combination of cluster monomials. Finally, applying Lemma 6.2 one more time shows that we can expand $Y_{A_1} \dots Y_{B_{k'}} = Y_{I\cup J\setminus P} Y_{I\cap J\setminus P}$ as a positive linear combination of cluster monomials, as desired. □

6.4. **Cycles.** Finally, we apply similar methods to prove that positivity holds for LP algebras arising from undirected cycles. In this case, for every edge of Γ, there is a directed cycle of length 2, and there are also two directed cycles that each contain all of the vertices of Γ.

Remark. Studying LP algebras arising from undirected cycles is motivated by an observation in [LP16b]. In [LP16b, Corollary 6.2], Lam and Pylyavskyy show that LP algebras arising from undirected paths can be identified with Type A cluster algebras, proving positivity for paths. They do so by giving a correspondence between cluster monomials and triangulations of polygons, and showing that a Ptolemy-like formula for expanding $Y_I Y_J$ into cluster monomials holds.

However, although LP algebras coming from undirected cycles have the same cluster complexes as Type B cluster algebras, they cannot be identified with Type B cluster algebras since their exchange polynomials are no longer binomial.

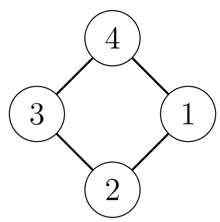
Lemma 6.13. Suppose I, J are connected segments of an undirected cycle Γ such that $I \cap J$ consists of two disjoint segments, K_1, K_2. Let K_1' be K_1 together with the vertex to the left and the vertex to the right of K_1, and let K_2' be defined similarly. First, suppose that $K_1' \cap K_2' \neq \emptyset$. Then, we have:

$$Y_I Y_J = Y_{I\cup J} Y_{I\cap J} + Y_{(I\cup J)\setminus K_1'} Y_{K_2} + Y_{(I\cup J)\setminus K_2'} Y_{K_1} + 2Y_{I\cap J} \tag{8}$$

If $K_1' \cap K_2' = \emptyset$, we get the same equation with one more term:

$$Y_I Y_J = Y_{I\cup J} Y_{I\cap J} + Y_{(I\cup J)\setminus K_1'} Y_{K_2} + Y_{(I\cup J)\setminus K_2'} Y_{K_1} + Y_{(I\cup J)\setminus K_1'\setminus K_2'} + 2Y_{I\cap J} \tag{9}$$

Example 6.14. Suppose Γ is a 4-cycle as below. Let $I = \{1, 2, 3\}$ and $J = \{1, 4, 3\}$. Then, $K_1 = \{1\}, K_2 = \{3\}, K_1' = \{1, 2, 4\}$ and $K_2' = \{2, 3, 4\}$. In this case, $K_1' \cap K_2' \neq \emptyset$.



By Lemma 6.13, we obtain:

$$Y_I Y_J = Y_{1234} Y_{13} + Y_2 + Y_4 + 2Y_{13} \tag{10}$$

Proof. We first assume that $K_1' \cap K_2' \neq \emptyset$. Similarly to the case for trees, there are three kinds of connected families of cycles which are not contained in I and not contained in J which can appear in any $r \in \mathcal{R}$:

- (i) One of the two cycles C_1, C_2 containing every vertex of Γ (as there are two possible orientations of this cycle).
- (ii) The anchored chain P_1 whose vertices are the vertices of K_1'.
- (iii) The anchored chain P_2 whose vertices are the vertices of K_2'.

Because of the fact that vertices not in $I \cap J$ have outdegree one in r, we cannot have both P_i and C_j appearing in r or both C_1, C_2 appearing in r; furthermore, since $K_1' \cap K_2' \neq \emptyset$, we cannot have both P_1 and P_2 appearing in r. So, at most one of the above three families of cycles occurs in r.

Similarly to the proof for trees, we will let $\mathcal{S}_{C_i}$ consist of pairs $(f_{I\cup J}, f_{I\cap J})$ of functions on $I\cup J$, $I\cap J$ respectively, where $f_{I\cup J}$ is just exactly the cyclic function given by C_i and $f_{I\cap J}$ is acyclic. Notice that

$$w(f_{I\cup J})w(f_{I\cap J}) = w(f_{I\cap J})$$

so that $\sum_{(f_{I\cup J}, f_{I\cap J})\in\mathcal{S}_{C_i}} w((f_{I\cup J}, f_{I\cap J}))$ is the numerator of $Y_{I\cap J}$. We define $\mathcal{S}_{P_i}$ in the same way as we did for trees, and ϕ_{P_i}, ϕ_{C_i} are, as usual, the weight-preserving functions taking multiunions of edges. By the preimages method, it will suffice to show that for all $r\in\mathcal{R}$:

$$|\phi_0^{-1}(r)| = |\phi_1^{-1}(r)| + |\phi_{P_1}^{-1}(r)| + |\phi_{P_2}^{-1}(r)| + |\phi_{C_1}^{-1}(r)| + |\phi_{C_2}^{-1}(r)| \tag{11}$$

We do this by splitting into cases on which connected families of cycles appear in r. If all connected families of cycles in r are contained either in I or in J, then we have

$$|\phi_0^{-1}(r)| = |\phi_1^{-1}(r)|, |\phi_{P_i}^{-1}(r)| = |\phi_{C_i}^{-1}(r)| = 0$$

and we are done. If C_1 (or symmetrically, C_2) appears in r, then we have

$$|\phi_0^{-1}(r)| - 1 = |\phi_1^{-1}(r)|$$

This is because every $\mathcal{S}_0$-valid coloring of C_1 is acyclic, as C_1 has vertices both in $J\setminus I$ and $I\setminus J$, hence has edges colored both with I and with J. However, there is one $\mathcal{S}_1$-valid coloring of C_1 which is not acyclic, namely, when every edge is colored with $I\cup J$. Then, Lemma 6.6 implies that $|\phi_0^{-1}(r)| - 1 = |\phi_1^{-1}(r)|$. Furthermore, we have:

$$|\phi_{C_1}^{-1}(r)| = 1, |\phi_{P_i}^{-1}(r)| = |\phi_{C_2}^{-1}(r)| = 0 \tag{12}$$

Finally, if P_1 (or symmetrically, P_2) appears in r, then by the same reasoning as the case with trees, we have

$$|\phi_1^{-1}(r)| = |\phi_{C_i}^{-1}(r)| = |\phi_{P_2}^{-1}(r)| = 0, |\phi_0^{-1}(r)| = |\phi_{P_1}^{-1}(r)| \tag{13}$$

Thus, Equation 11 holds in all cases, and we are done.

When $K_1'\cap K_2' = \emptyset$, we could also have both anchored chains P_1 and P_2 appearing in r. So, we add in another term corresponding to excising both chains at once, and the argument is exactly the same. □

Now, we can prove that positivity holds for LP algebras arising from undirected cycles:

Proof. By Lemma 6.2, it suffices to show that products of two cluster variables $Y_I Y_J$ can be expanded as a positive linear combination of cluster monomials. If I, J are disjoint, we know how to expand this product since we can always multiply $Y_I Y_J$ when $I\cap J = \emptyset$. If $I\cap J$ has one connected component, we know how to do this by the formula for paths in [LP16b, Theorem 6.1]. So, it suffices to look at cases where $I\cap J$ has two connected components. We use the same argument as in the case for trees: inducting on the size of $I\cap J$, we can rewrite $Y_I Y_J$ as a sum of monomials of the form $Y_{A_1}Y_{A_2}\dots Y_{A_k}Y_{B_1}\dots Y_{B_{k'}}$, where $|A_i\cap B_j| < |I\cap J|$. Then, by Lemma 6.2, we can further expand each $Y_{A_1}\dots Y_{B_{k'}}$ as a positive linear combination of cluster monomials, as desired. □

6.5. **Future Work.** In the future, we hope to extend these arguments to show positivity for all graphs Γ, or to show positivity for undirected Γ. If not, it may also be interesting to look at other special classes of graphs; possible directions to explore are planar graphs, graphs with small maximal degree, or edges of polytopes.

Acknowledgements

This project was partially supported by RTG grant NSF/DMS-1745638. It was supervised as part of the University of Minnesota School of Mathematics Summer 2023 REU program. Guilherme Zeus Dantas e Moura was supported by Haverford College's KINSC Summer Scholar funding. We would like to thank our mentor Pasha Pylyavskyy and our TA Robbie Angarone for their guidance and support.

References

[FZ02] Sergey Fomin and Andrei Zelevinsky. "Cluster Algebras I: Foundations". In: *Journal of the American Mathematical Society* 15.2 (Apr. 1, 2002), pp. 497–529. ISSN: 0894-0347, 1088-6834. DOI: 10.1090/S0894-0347-01-00385-X. arXiv: math/0104151.

[LP16a] Thomas Lam and Pavlo Pylyavskyy. "Laurent Phenomenon Algebras". In: *Cambridge Journal of Mathematics* 4.1 (2016), pp. 121–162. ISSN: 2168-0949. DOI: 10.4310/CJM.2016.v4.n1.a2. arXiv: 1206.2611 [math].

[LP16b] Thomas Lam and Pavlo Pylyavskyy. "Linear Laurent Phenomenon Algebras". In: *International Mathematics Research Notices* 2016.10 (2016), pp. 3163–3203. ISSN: 1073-7928. DOI: 10.1093/imrn/rnv237. arXiv: 1206.2612 [math].

Department of Mathematics and Statistics, Haverford College, Haverford PA, 19041, USA
Email address: zeusdanmou@gmail.com, gdantasemo@haverford.edu

School of Mathematics, University of Minnessota, Minneapolis MN, 55455, USA
Email address: telek002@umn.edu

Department of Mathematics, Harvard University, Cambridge MA, 02138, USA
Email address: dorawoodruff@college.harvard.edu