(K)NOT DETECTING BOUNDARY SLOPES VIA INTERSECTIONS IN THE CHARACTER VARIETY ARISING FROM EPIMORPHISMS

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Abstract. We describe intersection points in the character varieties of a family of hyperbolic two-bridge knot groups that have epimorphisms onto the trefoil knot. Using the technique of Farey recursion, we show that these intersection points correspond to algebraic non-integral representations. We also determine the boundary slopes detected by these intersection points.

Contents

1. Introduction

As young children, our first words were "no", "bird", "car", "Farey", and "recursion". As we grew older we started expressing more complex thoughts through phrases like "essential surfaces" and "holonomy representations". Eventually we graduated to complete sentences, such as "this proof is left as an exercise to the reader". In this paper, we fulfill our childhood dream of studying detected essential surfaces via intersections in the character variety arising from epimorphisms. We hope you enjoy the journey.

The $\mathrm{SL}_2(\mathbb{C})$ character variety has long been an important tool in the study of 3-manifolds. In [CS83], the authors give a general approach based on Bass-Serre theory ([Ser80]) to construct essential surfaces in the knot complement of hyperbolic knots using their $\mathrm{SL}_2(\mathbb{C})$

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character varieties. These essential surfaces arise from non-trivial actions of the knot group on SL_2-trees, and are said to be *detected by* SL_2-trees. One particular example of this approach is [CS83], where the authors used ideal points in $SL_2(\mathbb{C})$ character varieties to construct SL_2-trees and detect essential surfaces. Another method of constructing SL_2-trees is via algebraic non-integral (ANI) representations of hyperbolic knot groups. [SZ01] compared these two methods, and proved that any essential surface detected by ideal points would also be detected by ANI-representations.

The $SL_2(\mathbb{C})$ character variety of a hyperbolic knot group contains multiple irreducible components, including the canonical component that contains the character of a holonomy representation. The intersection points between these components often correspond to ANI-representations, and therefore detect essential surfaces. [Chu17] studied a family of hyperbolic two-bridge knots whose character varieties are known to contain two distinct irreducible components corresponding to irreducible characters, and proved that their intersection points always detect a Seifert surface.

In this paper, we take on the study of essential surfaces detected by ANI-representations using a slightly different approach. We study a specific family $\mathcal{K}(n,k)$ of hyperbolic two-bridge knots whose knot groups are all known to have epimorphisms onto the knot group of the trefoil knot by [ORS08]. In particular, we define $\mathcal{K}(n,k)$ to denote the two-bridge knot with normal form $q/p = [3, 2, \dots, 3, 2, 3k]$ with n-many 2's. The $SL_2(\mathbb{C})$ character varieties of these knots all contain an irreducible component corresponding to the character variety of the trefoil knot, which intersects the canonical component at finitely many points. Our first main result states that all such intersection points correspond to ANI-representations of the knot group:

Theorem 1.1. *For every two-bridge knot $K_r = \mathcal{K}(n,k)$, there exists an epimorphism $\Gamma_r \to \Gamma_{1/3}$. Moreover, for every $(x_0, y_0) \in \mathbb{C}^2$ that is an intersection point of $X_0(\Gamma_r)$ and the irreducible component $x^2 - y - 1$ of $X(\Gamma_r)$, and for any $SL_2(\mathbb{C})$-representation ρ of Γ_r corresponding to (x_0, y_0),*
(1) There exists a number field F such that the image of ρ is in $SL_2(F)$;
(2) There exists a prime ideal $\mathcal{P}$ of $\mathcal{O}_F$ such that ρ is an ANI-representation of Γ_r with respect to the discrete valuation $v_{\mathcal{P}}$.

Although the essential surface detected by these intersection points is not necessarily unique, [SZ01, Corollary 3] proves that the boundary slope of the essential surface is unique. This boundary slope is said to be *detected by* an SL_2-tree. Our second main result determines this detected boundary slope corresponding to the intersection points in Theorem 1.1 for the knots $\mathcal{K}(1,k)$.

Theorem 1.2. *Under the setting of Theorem 1.1, if K_r is of the form $\mathcal{K}(1,k)$, then the boundary slope of K_r detected by (x_0, y_0) is $6k+6$.*

Remark 1.3. One question currently unsolved by this paper is whether intersection points between $X_0(\Gamma_r)$ and x^2-y-1 always exist. However, it follows from our proof of Theorem 1.1 and Theorem 1.2 that their conclusions hold not only for the intersection points between $X_0(\Gamma_r)$ and $x^2 - y - 1$, but also for the intersection points between $x^2 - y - 1$ and any other irreducible component of $X(\Gamma_r)$ that correspond to irreducible characters. Therefore,

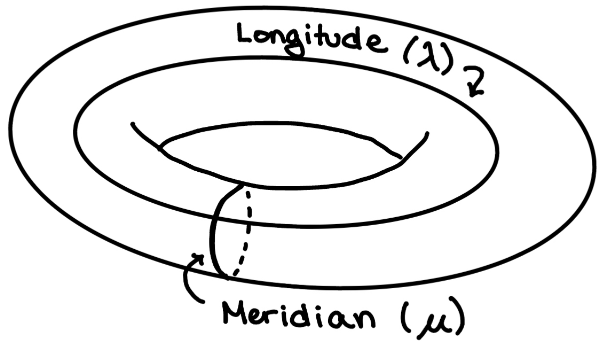

FIGURE 2.2.1. The longitude and meridian of $M(K)$

there always exists some intersection point in the character variety of K_r that detects the boundary slope in Theorem 1.2.

This paper is outlined as follows: in Section 2 we provide the necessary background and the setup of our problem. In Section 3, we characterize the vanishing polynomials for intersection points in the character variety of a given knot via Farey recursion. In Section 4 we calculate the boundary slopes for $\mathcal{K}(n,k)$ and in Section 5 we calculate explicitly the boundary slope detected by the nontrivial action on a tree.

2. Background

2.1. Two-bridge knots and their character varieties.

Definition 2.1. A *knot* K is an embedding of S^1 into S^3. Its *knot complement* is the 3-manifold $M(K) = S^3 \backslash N(K)$, where $N(K)$ is an open tubular neighborhood of K in S^3. Note that $\partial M(K)$ is homeomorphic to T^2. The *knot group* of K is defined as $\Gamma_K = \pi_1(M(K))$.

Definition 2.2. For every knot K, we define two canonical generators of $\pi_1(\partial M(K)) \cong \mathbb{Z}^2$, called the *longitude* and *meridian* of $M(K)$, as follows: the longitude λ is the homotopy class of a loop that goes around the torus longitudinally, and the meridian μ is the homotopy class of a loop that goes through the hole of the torus. See Figure 2.2.1.

Definition 2.3. A *slope* of K is a rational number $a/b \in \mathbb{Q} \cup \{\infty\}$, where $\mu^a\lambda^b \in \pi_1(\partial M(K))$. A *boundary slope* of K is a slope $a/b \in \mathbb{Q} \cup \{\infty\}$ such that there exists an essential surface S in $M(K)$ whose boundary ∂S is a non-empty set of parallel simple essential loops in $\partial M(K)$ of the form $\mu^a\lambda^b$. For a definition of essential surface, see [Sha01] pg 10. Through a slight abuse of notation, we use *boundary slope* to denote the rational number as well as the corresponding loop.

In this paper, we are particularly interested in a family of knots known as *two-bridge knots*.

Definition 2.4. A *two-bridge knot* is a non-trivial knot with a diagram having two local maxima.

It is known from [Sch56] that every two-bridge knot can be associated to a reduced fraction q/p with $q < p$ and p odd. Conversely, every rational number $r \in \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ uniquely determines either a two-bridge knot or a two-bridge link. For a given 2-bridge knot, we find q/p by computing the continued fraction $q/p = [a_1, \ldots, a_k]$ where each a_i denotes the number of half-twists in a box of the plat presentation (see Figure 2.4.2). For example, see Figure 2.4.3 which has $q/p = 1/3$ with continued fraction notation [3].

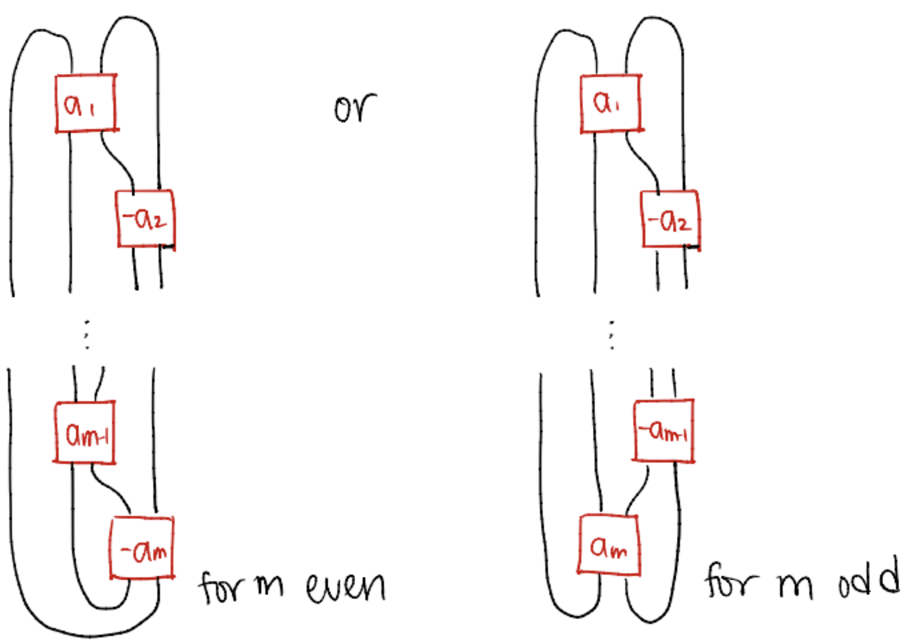

FIGURE 2.4.2. Plat Presentations for 2-Bridge Knots

See [Kaw96, Section 2.1] for further details of this correspondence.

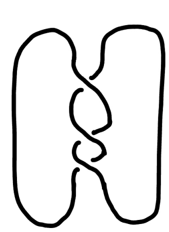

FIGURE 2.4.3. Plat Presentation for the Trefoil Knot $\Gamma_{1/3}$

Remark 2.5. In this paper we use the continued fraction notation defined

$$[a_1, a_2, a_3, \dots, a_m] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_m}}}}$$

where all a_i's are allowed to be arbitrary integers.

Remark 2.6. Different sources have different conventions with regard to whether they denote a two-bridge knot $K = (p, q)$ by $\frac{q}{p}$ or $\frac{p}{q}$. For instance, [Kaw96] uses $\frac{p}{q}$ while [ORS08] and [Che20] use $\frac{q}{p}$. In this paper, we will always use $\frac{q}{p}$ in line with [ORS08] and [Che20].

Lemma 2.7. [Kaw96, Theorem 2.1.3] *Two reduced fractions $\frac{q}{p}$ and $\frac{q'}{p'}$ (where $p, p' > 0$ are odd) represent the same two-bridge knot if and only if $p = p'$ and $qq' \equiv \pm 1 \pmod{p}$ or $q \equiv -q' \pmod{p}$.*

As a consequence of the previous lemma, every two-bridge knot K can be represented by a reduced fraction $r = \frac{q}{p}$ such that $0 < r < 1$ and q, p are both odd (though this reduced fraction is not unique). This fraction is called the *two-bridge normal form* of K, which we denote by either $r = \frac{q}{p}$ or (p, q). The knot group of K is denoted by either Γ_K or $\Gamma_{q/p}$.

For every two-bridge knot, [May74, Proposition 1] gives a canonical presentation of its knot group, as well as canonical representatives for its meridian and longitude:

Theorem 2.8. [May74, Proposition 1] *Given a two-bridge knot $K = (p, q)$, $\Gamma_{q/p}$ has the following presentation:*

$$\Gamma_{q/p} = \langle a, b \mid wa = bw \rangle$$

where $w = a^{\epsilon_1} b^{\epsilon_2} a^{\epsilon_3} \dots b^{\epsilon_{p-1}}$ with $\epsilon_i = (-1)^{\lfloor \frac{iq}{p} \rfloor}$, and a, b are conjugate elements in $\Gamma_{q/p}$.

Moreover, in this presentation, $\mu = a$ is a representative of the meridian of $M(K)$, and $\lambda = w^* w a^{\epsilon}$ is a representative of its longitude, where w^* denotes the word w written backwards, and ϵ is chosen so that the sum of the exponents in λ is zero.

Definition 2.9. For any finitely generated group Γ, we denote the set of its representations in $\mathrm{SL}_2(\mathbb{C})$ by $R(\Gamma)$, and the set of all characters of elements in $R(\Gamma)$ by $X(\Gamma)$. Once we fix a set of generators for Γ, both $R(\Gamma)$ and $X(\Gamma)$ can be given the structure of an affine algebraic variety; for the general construction, see [Sha01, Sections 4.1 and 4.4]. We call $R(\Gamma)$ and $X(\Gamma)$ the *representation variety* and *character variety* of Γ, respectively.

In this paper, since we are only concerned with two-bridge knot groups, we will use a particular parametrization of $X(\Gamma_{q/p})$ that comes from the presentation of $\Gamma_{q/p}$ in Theorem 2.8. We first record a lemma about the $\mathrm{SL}_2(\mathbb{C})$-representations of $\Gamma_{q/p}$, which comes from [Chu17, Pg. 3] and [Sha01, Proposition 1.1.1]:

Lemma 2.10. *Let $\Gamma_{q/p}$ be a two-bridge knot group. Then*
(1) $\rho \in R(\Gamma_{q/p})$ is reducible if and only if there exists $A \in \mathrm{SL}_2(\mathbb{C})$ such that $A\rho(\gamma)A^{-1}$ is upper triangular for all $\gamma \in \Gamma_{q/p}$;

(2) If $\rho_1, \rho_2 \in R(\Gamma_{q/p})$ are both irreducible, then they are equivalent if and only if their characters are the same.

Lemma 2.11. [MPvL11, Proposition 2.1] *If we write $\Gamma_{q/p} = \langle a, b \mid wa = bw \rangle$ as in Theorem 2.8, then every $\rho \in R(\Gamma_{q/p})$ is equivalent (by conjugation) to a representation $\rho' \in R(\Gamma_{q/p})$ such that:*

$$\rho'(a) = \begin{bmatrix} \alpha & 1 \\ 0 & 1/\alpha \end{bmatrix} \text{ and } \rho'(b) = \begin{bmatrix} \alpha & 0 \\ t & 1/\alpha \end{bmatrix}$$

for some $\alpha \in \mathbb{C}^*$ and $t \in \mathbb{C}$. Conversely, every pair $(\alpha, t) \in \mathbb{C}^2$ with $\alpha \neq 0$ determines a representation $\rho \in R(\Gamma_{q/p})$ by the above two equations.

Remark 2.12. Following from Lemma 2.11 we may always assume $\rho \in R(\Gamma_{q/p})$ has the given form.

For every pair $(\alpha, t) \in \mathbb{C}^2$ with $\alpha \neq 0$, let $\rho = \rho_{\alpha,t}$ denote the representation of $\Gamma_{q/p}$ given in the above lemma, and define $A = \rho(a)$, $B = \rho(b)$, $W = \rho(w)$.

Theorem 2.13. [Ril84, Theorem 1] *For every two-bridge knot q/p with knot group $\Gamma_{q/p} = \langle a, b \mid wa = bw \rangle$, and $\rho \in R(\Gamma_{q/p})$, the matrix $WA - BW$ always has the form*

$$\begin{bmatrix} 0 & f(\alpha, t) \\ -f(\alpha, t)t & 0 \end{bmatrix}$$

where f is a rational function in α and t. If we denote the numerator of $f(\alpha, t)$ by $p(\alpha, t)$, then p defines an algebraic curve in $\mathbb{C}^2$ whose points are in one-to-one correspondence with elements of $R(\Gamma_{q/p})$.

The following lemma is a special case of [Sha01, Proposition 4.4.2] adapted to our setting. It will imply that, up to a change of variables, the polynomial p also determines the character variety $X(\Gamma_{q/p})$:

Lemma 2.14. *Every character χ of $\Gamma_{q/p}$ is uniquely determined by $\chi(a)$ and $\chi(ab^{-1})$.*

For each $\rho = \rho_{\alpha,t} \in R(\Gamma_{q/p})$, we have

$$\chi_\rho(a) = \text{tr}(A) = \alpha + 1/\alpha \quad \text{and} \quad \chi_\rho(ab^{-1}) = \text{tr}(AB^{-1}) = 2 - t,$$

so if we define $x = \alpha + 1/\alpha$, $y = 2 - t$, and substitute these two variables into $p(\alpha, t)$, then by Lemma 2.10 and Lemma 2.14, the roots of the resulting polynomial $p(x, y)$ are in one-to-one correspondence with the elements of $X(\Gamma_{q/p})$.

Definition 2.15. We define $p(x, y)$ to be the algebraic variety in $\mathbb{C}^2$ identified with $X(\Gamma_{q/p})$. Furthermore, we identify a pair $(x, y) \in \mathbb{C}^2$ satisfying $p(x, y) = 0$ with the character $\chi_\rho \in X(\Gamma_{q/p})$ determined by $\chi_\rho(a) = x$ and $\chi_\rho(ab^{-1}) = y$.

Remark 2.16. By Lemma 2.10 (1), a character $\chi \in X(\Gamma_{q/p})$ is reducible if and only if $\rho(a)$ and $\rho(b)$ are upper-triangular, where $\rho = \rho_{\alpha,t} \in R(\Gamma_{q/p})$ is any representation with character χ. This happens if and only if $t = 0$, which is equivalent to $y = 2$. This implies that $X(\Gamma_{q/p})$ always has an irreducible component given by $y - 2 = 0$, which corresponds to all the reducible characters of $\Gamma_{q/p}$. The other irreducible components of $X(\Gamma_{q/p})$ correspond to irreducible characters.

Example 2.17. We calculate the character variety of $\Gamma_{1/3}$, the knot group of the trefoil knot. The canonical presentation of $\Gamma_{1/3}$ is $\langle a, b \mid wa = bw\rangle$, where $w = ab$. We calculate

$$WA - BW = \begin{bmatrix} 0 & (\alpha^4 + \alpha^2 t - \alpha^2 + 1)/\alpha^2 \\ (-\alpha^4 t - \alpha^2 t^2 + \alpha^2 t - t)/\alpha^2 & 0 \end{bmatrix}$$

so we have $p(\alpha, t) = \alpha^4 + \alpha^2 t - \alpha^2 + 1$, and substituting $x = \alpha + 1/\alpha$, $y = 2 - t$ gives $p(x, y) = (y-2)(x^2 - y - 1)$, the defining polynomial of $X(\Gamma_{1/3})$. The irreducible component corresponding to irreducible characters is then given by $x^2 - y - 1 = 0$. $\diamond$

In the case that $K = (p, q)$ is a hyperbolic knot, there exists a unique discrete faithful $\mathrm{PSL}_2(\mathbb{C})$-representation ρ_0 of Γ_K that corresponds to the holonomy representation of the hyperbolic structure of $M(K)$. Moreover, all lifts of ρ_0 into $\mathrm{SL}_2(\mathbb{C})$-representations are contained in one particular irreducible component of $X(\Gamma_K)$ (see [Sha01, Sections 1.6 and 4.5] for details).

Definition 2.18. Let $K = (p, q)$ be a two-bridge hyperbolic knot. The unique irreducible component of $X(\Gamma_K)$ that contains all lifts of ρ_0 into $\mathrm{SL}_2(\mathbb{C})$-representations is called the *canonical component* of $X(\Gamma_K)$ and is denoted by $X_0(\Gamma_K)$.

2.2. **Knot group epimorphisms.** As mentioned in the introduction, we would like to study those two-bridge knots whose character variety have multiple irreducible components. The following theorem ([HS10, Theorem 2.3(3)]) says that every two-bridge knot whose knot group has an epimorphism (i.e. a surjective homomorphism) onto another two-bridge knot group will always satisfy this property.

Theorem 2.19. *Let (p, q) and (p', q') be two two-bridge knots. Every epimorphism $\Gamma_{q/p} \twoheadrightarrow \Gamma_{q'/p'}$ induces an injective, algebraic, and Zariski-closed map $X(\Gamma_{q'/p'}) \to X(\Gamma_{q/p})$; in particular, every irreducible component of $X(\Gamma_{q'/p'})$ will appear as an irreducible component of $X(\Gamma_{q/p})$.*

For every fixed two-bridge knot $r = q/p$, [ORS08, Proposition 5.1] gives a way to systematically generate an infinite family of two-bridge knots whose knot groups all have epimorphisms onto Γ_r:

Theorem 2.20. *Let $r = q/p \in \mathbb{Q}$ be a two-bridge knot, and let $r = [a_1, \dots, a_m]$ be the continued fraction expansion of r. We define*

$$a = [a_1, a_2, \dots, a_m], \quad -a = [-a_1, -a_2, \dots, -a_m],$$
$$a^{-1} = [a_m, a_{m-1}, \dots, a_1], \quad -a^{-1} = [-a_m, -a_{m-1}, \dots, -a_1].$$

Then for any $r' \in \mathbb{Q}$ that has odd denominator and can be written as a continued fraction of the form

$$r' = 2c + [\epsilon_1 a, 2c_1, \epsilon_2 a^{-1}, 2c_2, \epsilon_3 a, \dots, 2c_{n-1}, \epsilon_n a^{(-1)^{n-1}}]$$

where $n \geq 2$, $c \in \mathbb{Z}$, $(c_1, \dots, c_{n-1}) \in \mathbb{Z}^{n-1}$, and $(\epsilon_1, \dots, \epsilon_n) \in \{1, -1\}^n$, there exists an epimorphism $\Gamma'_r \to \Gamma_r$.

Example 2.21. Consider the rational number $\frac{5}{27}$. This number can be written as

$$\frac{5}{27} = [3, 0, 3, -2, 3]$$

in continued fraction notation. Since the continued fraction for the trefoil knot is [3], there exists a surjection $\Gamma_{5/27} \to \Gamma_{1/3}$. ◇

The main object of study of this paper is hyperbolic two-bridge knot groups Γ_r that have epimorphisms onto $\Gamma_{1/3}$, the knot group of the trefoil knot. By Theorem 2.19, for every such knot group Γ_r, the character variety $X(\Gamma_r)$ will have an irreducible component defined by $x^2-y-1=0$. Consequently, for every intersection point (x_0, y_0) of $X_0(\Gamma_r)$ and $x^2-y-1=0$, x_0 will be a root of the polynomial $\tilde{p}(x) := p(x, x^2-1)/((x^2-1)-2) \in \mathbb{Z}[x]$, obtained by taking the defining polynomial $p(x, y)$ of $X(\Gamma_r)$, dividing by $y-2$ (the factor corresponding to the reducible characters), and then plugging in $y = x^2 - 1$.

Example 2.22. We compute the polynomial $\tilde{p}(x)$ for the two-bridge knot $5/27 = [3, 0, 3, -2, 3]$. By Theorem 2.20, there exists an epimorphism $\Gamma_{5/27} \to \Gamma_{1/3}$. The word w in the presentation of $\Gamma_{5/27}$ is

$$w = ababab^{-1}a^{-1}b^{-1}a^{-1}b^{-1}abababa^{-1}b^{-1}a^{-1}b^{-1}a^{-1}babab.$$

From this, one can compute the matrix $WA - BW$, and obtain the polynomial $p(x, y)$:

$$\begin{aligned}
p(x,y)/(x^2-y-1) &= x^{20}y^2 - 4x^{20}y - 10x^{18}y^3 + 4x^{20} + 36x^{18}y^2 + 45x^{16}y^4 - 24x^{18}y \\
&\quad - 144x^{16}y^3 - 120x^{14}y^5 - 16x^{18} + 36x^{16}y^2 + 336x^{14}y^4 + 210x^{12}y^6 \\
&\quad + 144x^{16}y + 96x^{14}y^3 - 504x^{12}y^5 - 252x^{10}y^7 - 561x^{14}y^2 - 504x^{12}y^4 \\
&\quad + 504x^{10}y^6 + 210x^8y^8 - 60x^{14}y + 1239x^{12}y^3 + 1008x^{10}y^5 - 336x^8y^7 \\
&\quad - 120x^6y^9 + 60x^{14} + 414x^{12}y^2 - 1701x^{10}y^4 - 1176x^8y^6 + 144x^6y^8 \\
&\quad + 45x^4y^{10} - 393x^{12}y - 1224x^{10}y^3 + 1491x^8y^5 + 864x^6y^7 - 36x^4y^9 \\
&\quad - 10x^2y^{11} - 30x^{12} + 1074x^{10}y^2 + 2010x^8y^4 - 819x^6y^6 - 396x^4y^8 \\
&\quad + 4x^2y^{10} + y^{12} + 268x^{10}y - 1575x^8y^3 - 1980x^6y^5 + 261x^4y^7 \\
&\quad + 104x^2y^9 - 80x^{10} - 883x^8y^2 + 1320x^6y^4 + 1170x^4y^6 - 39x^2y^8 \\
&\quad - 12y^{10} + 367x^8y + 1452x^6y^3 - 615x^4y^5 - 384x^2y^7 + y^9 + 38x^8 - 648x^6y^2 \\
&\quad - 1288x^4y^4 + 138x^2y^6 + 54y^8 - 213x^6y + 542x^4y^3 + 592x^2y^5 - 9y^7 \\
&\quad + 42x^6 + 411x^4y^2 - 208x^2y^4 - 111y^6 - 120x^4y - 335x^2y^3 + 27y^5 - 10x^4 \\
&\quad + 107x^2y^2 + 99y^4 + 35x^2y - 29y^3 - 4x^2 - 27y^2 + 6y + 1
\end{aligned}$$

and substituting $y = x^2 - 1$ yields $\tilde{p}(x) = -4x^2 + 9$. Consequently, the x-coordinates of the two intersection points of $X_0(\Gamma_{5/27})$ and $x^2 - y - 1 = 0$ are $\pm 3/2$. The y-coordinates can be computed using $y = x^2 - 1$, so the intersections points themselves are $(\pm 3/2, 5/4)$. ◇

In the above example, the leading term coefficient of $\tilde{p}(x)$ is not equal to ± 1, which implies that the roots are non-integral algebraic numbers. As we will prove in Section 3, this is a general fact that holds for a large family of two-bridge knots having epimorphisms onto the trefoil knot. To define this family, we introduce some new notation and an important lemma.

Definition 2.23. We use the notation in [CEK+21] to give a matrix form of continued fractions: for any $r = q/p = [a_1, a_2, \cdots, a_n] \in \hat{\mathbb{Q}}$ where $a_1, ..., a_n \in \mathbb{Z}$, we have

$$\begin{bmatrix} q \\ p \end{bmatrix} = \pm 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Lemma 2.24. *For every continued fraction expansion* $q/p = [\pm 3, 2c_1, \pm 3, 2c_2, \cdots, \pm 3, 2c_n, \pm 3]$ *with* $c_i \in \{-1, 0, 1\}$ *for all* i*, the denominator* p *is odd if and only if* n *is even.*

Proof. We use the matrix notation for continued fractions, which will be defined in Definition 2.23. We first compute

$$\begin{bmatrix} 0 & 1 \\ 1 & 2c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & \pm 3 \end{bmatrix} = \begin{bmatrix} 1 & \pm 3 \\ 2c & 1 \pm 6c \end{bmatrix} = \begin{bmatrix} \text{odd} & \text{odd} \\ \text{even} & \text{odd} \end{bmatrix}.$$

We claim that the matrix form of q/p is $\begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix}$ if n is odd, and is $\begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{odd} \end{bmatrix}$ if n is even. This follows from the fact that $\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$ has the desired form and by induction with the relation

$$\begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{odd} \end{bmatrix} \begin{bmatrix} \text{odd} & \text{odd} \\ \text{even} & \text{odd} \end{bmatrix} = \begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix},$$

$$\begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix} \begin{bmatrix} \text{odd} & \text{odd} \\ \text{even} & \text{odd} \end{bmatrix} = \begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{odd} \end{bmatrix}.$$

□

Remark 2.25. This lemma implies that q/p corresponds to a two-bridge knot if and only if n in the above notation is even. When p is even, we have a two-bridge link.

We now define the family of knots that is the main object of study of this paper.

Definition 2.26. Let $r = [3, 2, 3, 2, ..., 3, 2, 3, 0, 3, 0, ..., 3, 0, 3]$, with 2 appearing n times and 0 appearing $k-1$ times. Whenever $n+k$ is odd, we define $\mathcal{K}(n, k)$ to be the two-bridge knot represented by r, and define $\tilde{p}(\mathcal{K}(n, k)) \in \mathbb{Z}[x]$ by

$$\tilde{p}(\mathcal{K}(n, k))(x) := p(x, x^2 - 1)/(x^2 - 3)$$

where $p(x, y)$ is the defining polynomial for $X(\Gamma_r)$ as in Definition 2.15.

Remark 2.27. It follows from Lemma 2.24 that $r = \mathcal{K}(n, k)$ is always a knot when $n+k$ is odd, and it follows from Theorem 2.20 that every $\mathcal{K}(n, k)$ has a knot group epimorphism onto $\Gamma_{1/3}$. Moreover, for every intersection point (x_0, y_0) of $X_0(\Gamma_r)$ and $x^2 - y - 1 = 0$, we always have $\tilde{p}(\mathcal{K}(n, k))(x_0) = 0$, i.e. $\tilde{p}$ is a vanishing polynomial for x_0.

2.3. SL_2**-trees and detected essential surfaces.** Given a field $\mathfrak{K}$ and a discrete valuation v on $\mathfrak{K}$, Bass-Serre Theory ([Ser80]) gives a canonical way to construct a tree T_v on which $\mathrm{SL}_2(\mathfrak{K})$ acts simplicially and without inversions, known as the *Bruhat-Tits tree* for $\mathrm{SL}_2(\mathfrak{K})$. In this subsection, we first give a brief summary of the construction of this tree, and then relate it to the study of two-bridge knots.

Definition 2.28. For a field $\mathfrak{K}$, let $V = \mathfrak{K}^2$. Denote the valuation ring of v in $\mathfrak{K}$ by R_v.
(1) A *lattice* in V is a finitely generated R_v-submodule Λ of V that spans V (viewed as a $\mathfrak{K}$-vector space);
(2) Two lattices Λ_1, Λ_2 in V are *homothety equivalent* if there is some $\alpha \in \mathfrak{K}$ such that $\Lambda_1 = \alpha\Lambda_2$;
(3) Given two lattices Λ_1, Λ_2 in V, we say that Λ_1 is *snugly embedded* in Λ_2 if $\Lambda_1 \subset \Lambda_2$ and $\Lambda_2/\Lambda_1 \cong \mathbb{Z}/\beta\mathbb{Z}$ for $\beta \in R_v$.

Lemma 2.29 ([Sha01, Lemma 3.6.8])**.** *For any two lattices Λ_1 and Λ_2 in V, there is a unique lattice Λ_1' homothety equivalent to Λ_1 such that Λ_1' is snugly embedded in Λ_2.*

Theorem 2.30 ([Sha01, Theorem 3.6.14])**.** *Let $T^{(0)}$ be the set of homothety equivalence classes of lattices in V. Define a graph $T_{\mathfrak{K},v}$ as follows:*
(1) The vertex set of $T_{\mathfrak{K},v}$ is $T^{(0)}$;
(2) For any two homothety classes $s_1, s_2 \in T^{(0)}$, there is an edge between them if there exist representatives Λ_1 and Λ_2 of s_1 and s_2, respectively, such that Λ_1 is snugly embedded in Λ_2, and that for any $A \in \mathrm{GL}_2(\mathfrak{K})$ with $A(\Lambda_1) = \Lambda_2$, we have $v(\det(A)) = 1$.

Then $T_{\mathfrak{K},v}$ is a tree, called the Bruhat-Tits tree for $\mathrm{SL}_2(\mathfrak{K})$ *(with respect to the discrete valuation v).*

Any $A \in \mathrm{GL}_2(\mathfrak{K})$ will map a homothety class of lattices to another homothety class, and therefore $\mathrm{GL}_2(\mathfrak{K})$ acts on the vertex set $T^{(0)}$. The following theorem, which comes from [Sha01, Section 3.7], extends this to a simplicial action of $\mathrm{GL}_2(\mathfrak{K})$ on $T_{\mathfrak{K},v}$:

Theorem 2.31. *The natural action of* $\mathrm{GL}_2(\mathfrak{K})$ *on $T^{(0)}$ extends to a simplicial action (takes vertices to vertices, action on edges is linear) on $T_{\mathfrak{K},v}$, whose restriction to* $\mathrm{SL}_2(\mathfrak{K})$ *is an action on $T_{\mathfrak{K},v}$ without inversions. Furthermore, for this* $\mathrm{SL}_2(\mathfrak{K})$*-action, the stabilizers of the vertices of $T_{\mathfrak{K},v}$ are conjugates of the subgroup* $\mathrm{SL}_2(R_v)$*.*

A group action on a tree T is called *non-trivial* if no vertex of T is fixed by the entire group. As a consequence of the previous theorem, we have:

Corollary 2.32. *For any group representation $\rho : \Gamma \to \mathrm{SL}_2(\mathfrak{K})$, if there exists $\gamma \in \Gamma$ such that $v(\mathrm{tr}(\rho(\gamma))) < 0$, then the induced action of $\rho(\Gamma)$ on $T_{\mathfrak{K},v}$ is non-trivial.*

In general, for any 3-manifold M equipped with a $\pi_1(M)$-action on a tree T that is simplicial, without inversions, and non-trivial, [Sha01] gives a way to associate an essential surface in M to this $\pi_1(M)$-action on T. If the tree T is the SL_2-tree of a field $\mathfrak{K}$, then such an associated essential surface S in M is said to be *detected by an* SL_2*-tree*, and ∂S is called a *boundary slope of K detected by an* SL_2*-tree.*

One of the main purposes of introducing Bruhat-Tits trees into the study of knots is that, for any knot K and any representation $\rho : \Gamma_K \to \mathrm{SL}_2(F)$ where F is a number field with a discrete valuation v, one gets an induced action of Γ_K on $T_{F,v}$, and if ρ satisfies the condition in Corollary 2.32, then it detects an essential surface in $M(K)$. The following definition gives a large family of discrete valuations on F:

Definition 2.33. Let F be a number field, and let $\mathcal{O}_F$ denote the ring of integers of F. Let $\mathcal{P}$ be a prime ideal of $\mathcal{O}_F$. Define a discrete valuation $v_{\mathcal{P}}$ on F as follows:

(1) For any $x \in \mathcal{O}_F$, let $v_{\mathcal{P}}(x) = \max\{n \in \mathbb{Z}_{\geq 0} : x \in \mathcal{P}^n\}$;
(2) For $x \in F - \mathcal{O}_F$, write $x = a/b$ where $a, b \in \mathcal{O}_F$, and define $v_{\mathcal{P}}(x) = v_{\mathcal{P}}(a) - v_{\mathcal{P}}(b)$.
The discrete valuation $v_{\mathcal{P}}$ is called the $\mathcal{P}$-adic valuation on F.

For any knot K and representation $\rho : \Gamma_K \to \mathrm{SL}_2(F)$ where F is a number field, suppose that there exists some $\gamma \in \Gamma_K$ such that $\mathrm{tr}(\rho(\gamma))$ is not an algebraic integer; then there must exist some prime ideal $\mathcal{P}$ of $\mathcal{O}_F$ such that $v_{\mathcal{P}}(\mathrm{tr}(\rho(\gamma))) < 0$. (See Lemma 3.30 for a proof.) It then follows from Corollary 2.32 that there always exists an essential surface in $M(K)$ detected by an $\mathrm{SL}_2(F)$-tree. This motivates the following definition:

Definition 2.34. Let $\rho : \Gamma_K \to \mathrm{SL}_2(F)$ be a representation of Γ_K, where F is a number field with a $\mathcal{P}$-adic valuation $v_{\mathcal{P}}$. We call ρ an ANI (algebraic non-integral) representation of Γ_K (with respect to $v_{\mathcal{P}}$) if there exists some $\gamma \in \Gamma_K$ such that $v_{\mathcal{P}}(\mathrm{tr}(\rho(\gamma))) < 0$.

In general, the essential surfaces detected by an SL_2-tree associated to an ANI-representation of Γ_K are not unique. However, it turns out that the boundary slope detected by an ANI-representation is unique. In fact, since we know that two-bridge knot complements are small (i.e. they do not contain closed essential surfaces; see [HT85, Theorem 1(a)] for a proof), [SZ01, Corollary 3] implies the following theorem:

Theorem 2.35. *Let $\rho : \Gamma_K \to \mathrm{SL}_2(F)$ be an ANI-representation of Γ_K with respect to a $\mathcal{P}$-adic valuation $v_{\mathcal{P}}$. Then there exists a unique boundary slope γ of K such that $v_{\mathcal{P}}(\mathrm{tr}(\rho(\gamma))) \geq 0$, and γ is the unique boundary slope of K detected by an SL_2-tree.*

The rest of this paper is divided into three main sections. In Section 3, we use the technique of Farey recursion to show that for every $\mathcal{K}(n,k)$, all coefficients of $\tilde{p}(\mathcal{K}(n,k))$ but the constant term are even; this will lead to a proof of Theorem 1.1. In Section 4, we use the techniques in [HT85] to determine all the boundary slopes of $\mathcal{K}(n,k)$. Finally, in Section 5, we prove Theorem 1.2.

3. Vanishing polynomials for intersection points

In [Che20], the author uses the close connection between continued fractions and the modular tessellation of the hyperbolic plane (also called the Farey graph; see Figure 3.3.4) to describe a recursive method for finding the character varieties of two-bridge knot groups. In Section 3.1, we introduce the technique of Farey recursion; then, in Section 3.2, we use this technique to systematically study $\tilde{p}(\mathcal{K}(n,k))$ for all n and k. This allows us to obtain an explicit formula for $\tilde{p}(\mathcal{K}(n,k))$ (Theorem 3.27), which then leads to the proof of Theorem 1.1.

3.1. **Farey recursion.**

Definition 3.1. We call a pair of reduced fractions $(q/p, s/r) \in \hat{\mathbb{Q}}^2$ a *Farey pair* if $qr - ps = \pm 1$. For the Farey pair $(q/p, s/r)$, we define the *Farey sum* to be

$$\frac{q}{p} \oplus \frac{s}{r} = \frac{q+s}{p+r}.$$

Remark 3.2. By convention we write $1 = 1/1$, $0 = 0/1$, and $\infty = \pm 1/0$. We also make the convention that for any negative $r \in \mathbb{Q}$, whenever r appears in a Farey sum, we always write it as $r = q/p$ with $p > 0$ for all $r \neq \infty$.

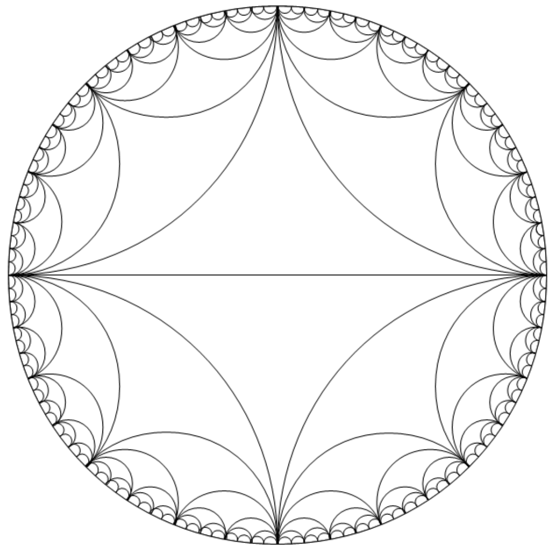

FIGURE 3.3.4. Farey Graph

Remark 3.3. If $(\alpha, \gamma) = (q/p, s/r)$ is a Farey pair, then it is straightforward to show that $q+s$ and $p+r$ are coprime, and that $(\alpha, \alpha \oplus \gamma)$ and $(\gamma, \alpha \oplus \gamma)$ are also Farey pairs.

Lemma 3.4. *Suppose that $(a/b, c/d)$ is a Farey pair which is not $(1/0, 0/1)$. If $a > c$, then $b \geq d$.*

Proof. Suppose for contradiction $a > c$ and $b < d$. Thus $c+1 \leq a$ and $b+1 \leq d$. Therefore $ad - bc \geq (c+1)(b+1) - bc = b+c+1 > 1$. So if $a > c$, then $b \geq d$. □

Definition 3.5. Let $(q/p, s/r)$ be a Farey pair; for any $k \in \mathbb{Z}$, define

$$\frac{q}{p} \oplus^k \frac{s}{r} = \frac{q+ks}{p+kr}$$

Lemma 3.6. *For any reduced fraction q/p there is a pair of reduced fractions α, γ such that $q/p = \gamma \oplus^2 \alpha$.*

Proof. First, we claim there exist $a/b, c/d \in \hat{\mathbb{Q}}$ satisfying $ad - bc = 1$ such that

$$\frac{q}{p} = \frac{a+c}{b+d}.$$

Since $q = a+c$ and $p = b+d$, by substitution it is sufficient to find a, b such that $a(p-b) - b(q-a) = ap - qb = 1$. Since p, q are coprime, choose $0 \leq a < q$ such that $ap \equiv 1 \pmod{q}$. Therefore, $b = \frac{ap-1}{q}$, and using $q = a+c$ and $p = b+d$ we can find c and d.

By Lemma 3.4 if $a > c$, then $b \geq d$. Choose

$$\alpha = \frac{c}{d},\ \gamma = \frac{a-c}{b-d}$$

Then $\alpha \oplus \gamma = a/b$, so $\alpha \oplus \alpha \oplus \gamma = q/p$ □

Example 3.7. Consider the two-bridge knot 5/27. Note that

$$\frac{5}{27} = \frac{3+2}{16+11} = \frac{3}{16} \oplus \frac{2}{11}$$

with 3/16, 2/11 a Farey pair since $3(11) - 2(16) = -1$. As in Lemma 3.6, we then have

$$\frac{5}{27} = \frac{1+2(2)}{5+2(11)} = \frac{1}{5} \oplus^2 \frac{2}{11}.$$

◇

Definition 3.8. Let R be a commutative ring. A function $\mathcal{F} : \hat{\mathbb{Q}} \to R$ is called a *Farey recursive function* if for any Farey pair $(\alpha, \gamma) \in \hat{\mathbb{Q}}^2$, we have

$$\mathcal{F}(\gamma \oplus^2 \alpha) = -\mathcal{F}(\gamma) + \mathcal{F}(\alpha)\mathcal{F}(\gamma \oplus \alpha)$$

Remark 3.9. Note that if $\mathcal{F}$ is a Farey recursive function, then for a Farey pair (α, γ), we also have

$$\mathcal{F}(\gamma \oplus^{-2} \alpha) = -\mathcal{F}(\gamma) + \mathcal{F}(\alpha)\mathcal{F}(\gamma \oplus^{-1} \alpha)$$

We demonstrate this matrix decomposition in Example 3.11. The following lemma establishes the relationship between Farey sums and continued fractions with integer terms.

Lemma 3.10. *Suppose that $r \in \hat{\mathbb{Q}}$ can be written as a continued fraction $r = [a_1, ..., a_m]$, where $a_i \in \mathbb{Z}$ for all $1 \le i \le m$. If we set $r_j = [a_1, ..., a_j]$ for $1 \le j \le m$, and define $r_{-1} = 1/0$, $r_0 = 0/1$, then*
(1) (r_{j-1}, r_j) is a Farey pair for all $0 \le j \le m$;
(2) We have $r_j = r_{j-2} \oplus^{(\eta_{j-2}\eta_{j-1})a_j} r_{j-1}$ for all $1 \le j \le m$, where $\eta_j \in \{1, -1\}$ is the unique constant such that if we write $r_j = q_j/p_j$ under the convention of Remark 3.2, then

$$\begin{bmatrix} q_j \\ p_j \end{bmatrix} = \eta_j \cdot \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Proof. We prove this by induction on j. When $j = 0$, $(r_{-1}, r_0) = (1/0, 0/1)$ is clearly a Farey pair. Suppose that (r_{j-2}, r_{j-1}) is already a Farey pair; by Remark 3.3, to show that (r_{j-1}, r_j) is a Farey pair, it suffices to show that the recursive formula for r_j in (2) holds. To see this, note that

$$\begin{aligned} &\begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_{j-2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + a_j \cdot \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_{j-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_{j-2} \end{bmatrix} \cdot \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & a_{j-1} \end{bmatrix} \begin{bmatrix} 0 \\ a_j \end{bmatrix} \right) \\ &= \begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & a_{j-2} \end{bmatrix} \cdot \left(\begin{bmatrix} 0 & 1 \\ 1 & a_{j-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & a_j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \end{aligned}$$

When $\eta_{j-2} = \eta_{j-1} = \pm 1$, the left hand side is equal to $\pm \begin{bmatrix} q_{j-2} + a_j q_{j-1} \\ p_{j-2} + a_j p_{j-1} \end{bmatrix}$, which by our convention in Remark 3.2 corresponds to the fraction $r_{j-2} \oplus^{a_j} r_{j-1}$. When $\eta_{j-2} = 1$, $\eta_{j-1} = -1$

or when $\eta_{j-2} = -1$, $\eta_{j-1} = 1$, the left hand side is equal to $\pm \begin{bmatrix} q_{j-2} - a_j q_{j-1} \\ p_{j-2} - a_j p_{j-1} \end{bmatrix}$, which corresponds to the fraction $r_{j-2} \oplus^{-a_j} r_{j-1}$. Since the right hand side in the above equation is equal to $\eta_j \cdot \begin{bmatrix} q_j \\ p_j \end{bmatrix} = \pm \begin{bmatrix} q_j \\ p_j \end{bmatrix}$, we have

$$r_j = \frac{\pm q_j}{\pm p_j} = \frac{\pm(q_{j-2} + (\eta_{j-2}\eta_{j-1})a_j q_{j-1})}{\pm(p_{j-2} + (\eta_{j-2}\eta_{j-1})a_j p_{j-1})} = r_{j-2} \oplus^{(\eta_{j-2}\eta_{j-1})a_j} r_{j-1}.$$

Here by $\frac{\pm a}{\pm b}$ we mean $\frac{a}{b}$ or $\frac{-a}{-b}$. □

Example 3.11. Consider the continued fraction

$$[3, 0, 3, -2, 3] = 5/27.$$

Additionally, consider the partial sums

$$r_4 = [3, 0, 3, -2] = 2/11 \text{ and } r_3 = [3, 0, 3] = 1/6.$$

We can check that $5(11) - 27(2) = 55 - 54 = 1$, and $2(6) - 1(11) = 1$, so $[3, 0, 3, -2, 3]$ and $[3, 0, 3, -2]$ are a Farey pair, as are $[3, 0, 3, -2]$ and $[3, 0, 3]$. Now we compute

$$\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -11 \end{bmatrix}.$$

So $\eta_4 = -1$. Furthermore,

$$\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

So $\eta_3 = 1$. Therefore $[3, 0, 3] \oplus^{(-1)(1)(3)} [3, 0, 3, -2]$. So

$$[3, 0, 3] \oplus^{(-1)(1)(3)} [3, 0, 3, -2] = \frac{1 - 3(2)}{6 - 3(33)} = \frac{5}{27}.$$

Note that this matches the results of Lemma 3.10. ◇

3.2. **Vanishing polynomials.** In this section we use the general setup in Section 3.1 to give an explicit description (Theorem 3.27) of the vanishing polynomial $\tilde{p}(\mathcal{K}(n, k))$ defined in Definition 2.26, which would then allow us to prove Theorem 1.1.

We begin by introducing the particular Farey recursive function that will be used throughout this section. The following lemma is a restatement of [CEK+21, Theorem 4.2] adapted for our purposes (akash: i reworded this from "adapted from our specific context" lmk if its bad just change it back):

Lemma 3.12. *There exists a unique Farey recursive function $\mathcal{T} : \hat{\mathbb{Q}} \to \mathbb{Z}[w, z]$ such that*

$$\mathcal{T}(0) = w, \quad \mathcal{T}(1/0) = 0, \quad \mathcal{T}(1) = z$$

Definition 3.13. Define $f_1, f_2 : \hat{\mathbb{Q}} \to \{0, 1\}$ by

$$f_1(q/p) = pq + 1 \pmod 2, \quad f_2(q/p) = q \pmod 2$$

and define $\mathcal{T}_0 : \hat{\mathbb{Q}} \to \mathbb{Z}[w, z]$ by $\mathcal{T}_0(\alpha) = \mathcal{T}(\alpha)/f(\alpha)$, where $f(\alpha) := w^{f_1(\alpha)} z^{f_2(\alpha)}$.

The following lemma is a restatement of [Che20, Lemma 5.5] and [Che20, Theorem 7.3].

Lemma 3.14. *The function $\mathcal{T}_0$ satisfies the following:*

(1) *For all $\alpha \in \hat{\mathbb{Q}}$, we have $\mathcal{T}_0(\alpha) \in \mathbb{Z}[w^2, z^2]$, so we can write $\mathcal{T}_0(\alpha) \in \mathbb{Z}[W, Z]$ using the change of variables $W = w^2$, $Z = z^2$;*
(2) *A point $\chi = (W, Z) \in \mathbb{C}^2$ is an irreducible character of Γ_α if and only if $WZ \neq 0$ and χ satisfies the polynomial $\mathcal{T}_0(\alpha)$.*

The following lemma is a restatement of a result from [Che20, Section 7.2], which allows us to recover our defining polynomial $p(x, y)$ of $X(\Gamma_r)$ (see Definition 2.15) from $\mathcal{T}_0(r)(W, Z)$.

Lemma 3.15. *For any two-bridge normal form $r \in \mathbb{Q}$, we have $p(x, y) = \pm\mathcal{T}_0(r)(2 + y - x^2, 2 - y)$, where $\mathcal{T}_0(r)$ is written in the variables W and Z, and $p(x, y)$ is defined as in Definition 2.15.*

Proof. In the paper [MPvL11, Proposition 2.2], the authors define character varieties of two bridge knots using the parameters r, v where $v = \alpha^2 + \frac{1}{\alpha^2}$, and $r = 2-t$. Therefore $v = x^2-2$, and $y = r$ in the notation of Lemma 2.14. In [Che20, Section 7.2], the author notes that $v = 2 - W - Z$ and $r = 2 - Z$. So by substitution, it follows that $W = 2 + y - x^2$ and $Z = 2 - y$. □

Corollary 3.16. *Let r be the two-bridge normal form for $\mathcal{K}(n, k)$. Then*

$$\frac{\mathcal{T}(r)}{z(w^2 - 1)}(1, z)$$

is an element of $\mathbb{Z}[z^2]$, and substituting $z^2 = 3 - x^2$ yields the polynomial $\tilde{p}(\mathcal{K}(n, k))$ defined in Definition 2.26.

Proof. We have $\mathcal{T}_0(r) = \mathcal{T}(r)/z$ by the definition of $\mathcal{T}_0$ and the fact that the numerator and denominator in a two-bridge normal form are both odd. Let $p(x, y)$ be the defining polynomial for $X(\Gamma_r)$. Since Γ_r surjects onto $\Gamma_{1/3}$ we know p always has the factor $x^2 - y - 1$ and $w^2 = W = 2 + y - x^2$, we know by Lemma 3.15 that $\mathcal{T}_0(r)$ must have the factor $w^2 - 1$. The fact that $\frac{\mathcal{T}(r)}{z(w^2-1)}(1, z) \in \mathbb{Z}[z^2]$ follows from Lemma 3.14.

By Definition 2.26, to obtain $\tilde{p}(x)$ from $p(x, y)/y - 2$, the relation we need to plug in is $y = x^2 - 1$; in terms of the variables $W = 2 + y - x^2$ and $Z = 2 - y$, this then becomes $W - 1 = 0$ and $Z = 3 - x^2$. Since $W = w^2$, $Z = z^2$, in terms of w and z, these relations then become $w^2 = 1$ and $z^2 = 3 - x^2$. Since $\frac{\mathcal{T}(r)}{z(w^2-1)} \in \mathbb{Z}[w^2, z^2]$ by Lemma 3.14, the substitution $w^2 = 1$ can be replaced by $w = 1$; it then follows from Lemma 3.15 that these substitutions yield $\tilde{p}(\mathcal{K}(n, k))$. □

Example 3.17. We have

$$\frac{1}{3} = \frac{1}{0} \oplus^3 \frac{0}{1}$$

Therefore

$$\begin{aligned}\mathcal{T}\left(\frac{1}{3}\right) &= -\mathcal{T}\left(\frac{1}{0}\oplus\frac{0}{1}\right)+\mathcal{T}\left(\frac{0}{1}\right)\mathcal{T}\left(\frac{1}{0}\oplus^2\frac{0}{1}\right)\\ &= -z+w\left(-\mathcal{T}\left(\frac{1}{0}\right)+\mathcal{T}\left(\frac{0}{1}\right)\mathcal{T}\left(\frac{1}{1}\right)\right)\\ &= -z+w(0+wz)\\ &= (w^2-1)z\end{aligned}$$

and $f(1/3)=z$, so $\mathcal{T}_0(1/3)=w^2-1=W-1$. It then follows from Lemma 3.15 that the defining polynomial for $X(\Gamma_{1/3})$ is $\mathcal{T}_0(1/3)(2+y-x^2,2-y)=(2+y-x^2)-1=y-x^2+1$, which corresponds to the calculation in Example 2.17. ◇

Example 3.18. Using Lemma 3.6, we implemented a function in SageMath that calculates $\mathcal{T}(r)$ recursively. For example, we obtain

$$\begin{aligned}\mathcal{T}(5/27) &= w^{22}z^5-17w^{20}z^5+124w^{18}z^5-507w^{16}z^5+1275w^{14}z^5-3w^{14}z^3\\ &\quad -2040w^{12}z^5+31w^{12}z^3+2083w^{10}z^5-123w^{10}z^3-1331w^8z^5+234w^8z^3\\ &\quad +508w^6z^5-219w^6z^3-105w^4z^5+2w^6z+95w^4z^3+9w^2z^5-8w^4z\\ &\quad -15w^2z^3+7w^2z-z\end{aligned}$$

and therefore $\frac{\mathcal{T}(5/27)}{z(w^2-1)}(1,z)=4z^2-3$. By Corollary 3.16, we then have $\tilde{p}(x)=4(3-x^2)-3=9-4x^2$ is a vanishing polynomial for any $x_0\in\mathbb{C}$ such that (x_0,x_0^2-1) is an intersection point of $X_0(\Gamma_{5/27})$ and $x^2-y-1=0$. This matches the explicit calculation done in Example 2.22. (Note that although $(27,5)$ is not a knot in the family $\mathcal{K}(n,k)$, the conclusion of Corollary 3.16 still applies, since its proof only relies on Lemma 3.15, which holds for every two-bridge normal form $r\in\mathbb{Q}$.) ◇

Definition 3.19. For the rest of this section, we fix the following notations. Let $r\in\mathbb{Q}$ be a continued fraction of the form $r=[\pm3,\pm2,...,\pm3,\pm2]$, (e.g. $[3,-2,3,2,-3]$) and let m be the length of this continued fraction, which is always even. For $1\le j\le m$, let a_j be the j-th entry in r, and let r_j denote the continued fraction consisting of the first j terms of r. For every $k\in\mathbb{N}$ and $2\le j\le m$, using the notation in Lemma 3.10, we define

$$P_{k,j}=\mathcal{T}(r_{j-1}\oplus^{(\eta_{j-1}\eta_j)k}r_j)\in\mathbb{Z}[w,z]$$

$$F_{k,j}=\frac{P_{3k,j}}{z(w^2-1)}(1,z)\in\mathbb{Z}[z]$$

And we make the convention that $P_k:=P_{k,m}$, $F_k:=F_{k,m}$.

Remark 3.20. It follows immediately from Corollary 3.16 and Lemma 3.10 that when all terms in the continued fraction expansion of r are positive, we have $F_{k,2n}=\tilde{p}(\mathcal{K}(n,k))$ up to the change of variables $z^2=3-x^2$.

The goal of the rest of this section is to obtain an explicit formula for F_k when r is of the form $[3,2,3,2,...,3,2]$, which would then allow us to describe $\tilde{p}(\mathcal{K}(n,k))$ and prove Theorem 1.1.

We first have the following recursive formula for P_k and F_k:

Lemma 3.21. *For all $k \geq 1$, we have*

$$F_k = \frac{\mathcal{T}(r_m)^2 - 1}{w^2 - 1}(1, z) \cdot \frac{P_{3k-2}(1, z)}{z} - \mathcal{T}(r_m)(1, z)F_{k-1} \tag{3.1}$$

Proof. Since $\mathcal{T}$ is a Farey recursive function, we have

$$\mathcal{T}(r_{m-1} \oplus^{\pm k} r_m) = \mathcal{T}(r_m)\mathcal{T}(r_{m-1} \oplus^{\pm(k-1)} r_m) - \mathcal{T}(r_{m-1} \oplus^{\pm(k-2)} r_m).$$

Therefore $P_k = \mathcal{T}(r_m)P_{k-1} - P_{k-2}$ for all $k \geq 2$. This implies that

$$\begin{aligned} P_{3k} &= \mathcal{T}(r_m)P_{3k-1} - P_{3k-2} = \mathcal{T}(r_m)(\mathcal{T}(r_m)P_{3k-2} - P_{3k-3}) - P_{3k-2} \\ &= (\mathcal{T}(r_m)^2 - 1)P_{3k-2} - \mathcal{T}(r_m)P_{3k-3}. \end{aligned} \tag{3.2}$$

The result follows from dividing both sides of the above equation by $z(w^2 - 1)$ and letting $w = 1$. ☐

Lemmas 3.22, 3.23, 3.25 obtain general formulae for each term in Equation (3.1) (other than F_k and F_{k-1}); Corollary 3.24 and Corollary 3.26 then obtain explicit formulae in the special case that $r = [3, 2, 3, 2, ..., 3, 2]$.

Lemma 3.22. *The polynomial $\mathcal{T}(r_j)(1, z)$ has the following properties:*

1. *If j is odd, then $\mathcal{T}(r_j)(1, z) = \mathcal{T}_0(r_j)(1, z) = 0$;*
2. *If j is even, then the numerator of r_j is always even, and $\mathcal{T}(r_j)(1, z) = \mathcal{T}_0(r_j)(1, z) = (-1)^{j/2}$.*

Proof. (1) When j is odd, by Theorem 2.20, there always exists an epimorphism $\Gamma_{r_j} \to \Gamma_{1/3}$. Then by [Che20, Corollary 7.6], every factor of $\mathcal{T}(1/3)$ divides $\mathcal{T}(r_j)$. By Example 3.17, $\mathcal{T}(1/3)(1, w) = 0$.
(2) We prove this by induction on j. When $j = 2$, we have $r_j = \pm 2/7$ or $\pm 2/5$, and these cases can be verified directly by consulting the list of $\mathcal{T}_0$'s in [Che20, Section 9]. For the general case, first note that by Lemma 3.10, we have $r_{j+2} = r_j \oplus^{\pm 2} r_{j+1}$; therefore, if the numerator of r_j is even, then the numerator of r_{j+2} is also even. This implies that $\mathcal{T}_0(r_{j+2}) = \mathcal{T}(r_{j+2})/w$, so $\mathcal{T}_0(r_{j+2})(1, z) = \mathcal{T}(r_{j+2})(1, z)$. Now we have

$$\mathcal{T}(r_{j+2}) = -\mathcal{T}(r_j) + \mathcal{T}(r_{j+1})\mathcal{T}(r_j \oplus^{\pm 1} r_{j+1})$$

and since $\mathcal{T}(r_{j+1})(1, z) = 0$ by (1), we have $\mathcal{T}_0(r_{j+2})(1, z) = \mathcal{T}(r_{j+2})(1, z) = -\mathcal{T}(r_j)(1, z) = (-1)^{\frac{j+2}{2}}$ by the induction hypothesis. ☐

Lemma 3.23. *We have $P_0(1, z) = 0$, $P_1(1, z) = \pm z$, $P_2(1, z) = (-1)^{m/2}P_1(1, z)$, and for all $k \geq 3$ we have*

$$P_k(1, z) = (-1)^{\frac{m}{2}+1}P_{k-3}(1, z).$$

Proof. By Lemma 3.22, since m is even, we know that $\mathcal{T}(r_m)(1, z) = (-1)^{m/2}$, and that $P_0(1, z) = \mathcal{T}(r_{m-1})(1, z) = 0$. To compute $P_1(1, z) = P_{1,m}(1, z)$, we induct on m. First note that by Lemma 3.10, we have $r_m = r_{m-2} \oplus^{\pm 2} r_{m-1}$; therefore

$$\begin{aligned} P_{1,m} &= \mathcal{T}(r_{m-1} \oplus^{\pm 1} r_m) \\ &= \begin{cases} \mathcal{T}(r_{m-1} \oplus (r_{m-2} \oplus^{\pm 2} r_{m-1})), & \eta_{m-1}\eta_m = 1; \\ \mathcal{T}(r_{m-1} \oplus^{-1} (r_{m-2} \oplus^{\pm 2} r_{m-1})), & \eta_{m-1}\eta_m = -1, \end{cases} \\ &= \mathcal{T}(r_{m-2} \oplus^{\pm 3} r_{m-1}) \text{ or } \mathcal{T}(r_{m-2} \oplus^{\pm 1} r_{m-1}) \end{aligned}$$

and since

$$\begin{aligned} \mathcal{T}(r_{m-2} \oplus^{\pm 3} r_{m-1}) &= -\mathcal{T}(r_{m-2} \oplus^{\pm 1} r_{m-1}) + \mathcal{T}(r_{m-1})\mathcal{T}(r_{m-2} \oplus^{\pm 2} r_{m-1}) \\ &= -\mathcal{T}(r_{m-2} \oplus^{\pm 1} r_{m-1}) \end{aligned}$$

we always have $P_1 = \pm\mathcal{T}(r_{m-2} \oplus^{\pm 1} r_{m-1})$. Again, by Lemma 3.10, we have $r_{m-1} = r_{m-3} \oplus^{\pm 3} r_{m-2}$, so

$$\begin{aligned} \mathcal{T}(r_{m-2} \oplus^{\pm 1} r_{m-1}) &= \begin{cases} \mathcal{T}(r_{m-2} \oplus (r_{m-3} \oplus^{\pm 3} r_{m-2})), & \eta_{m-2}\eta_{m-1} = 1; \\ \mathcal{T}(r_{m-2} \oplus^{-1} (r_{m-3} \oplus^{\pm 3} r_{m-2})), & \eta_{m-2}\eta_{m-1} = -1, \end{cases} \\ &= \mathcal{T}(r_{m-3} \oplus^{\pm 4} r_{m-2}) \text{ or } \mathcal{T}(r_{m-3} \oplus^{\pm 2} r_{m-2}) \end{aligned}$$

and again, by using Farey recursion and the fact that $\mathcal{T}(r_{m-2}) = (-1)^{\frac{m-2}{2}}$, we have

$$\begin{aligned} \mathcal{T}(r_{m-3} \oplus^{\pm 4} r_{m-2}) &= -\mathcal{T}(r_{m-3} \oplus^{\pm 2} r_{m-2}) + (-1)^{\frac{m-2}{2}}\mathcal{T}(r_{m-3} \oplus^{\pm 3} r_{m-2}) \\ &= -\mathcal{T}(r_{m-3} \oplus^{\pm 2} r_{m-2}) + \\ &\qquad (-1)^{\frac{m-2}{2}}[-\mathcal{T}(r_{m-3} \oplus^{\pm 1} r_{m-2}) + (-1)^{\frac{m-2}{2}}\mathcal{T}(r_{m-3} \oplus^{\pm 2} r_{m-2})] \\ &= (-1)^{\frac{m}{2}}\mathcal{T}(r_{m-3} \oplus^{\pm 1} r_{m-2}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{T}(r_{m-3} \oplus^{\pm 2} r_{m-2}) &= -\mathcal{T}(r_{m-3}) + \mathcal{T}(r_{m-2})\mathcal{T}(r_{m-3} \oplus^{\pm 1} r_{m-2}) \\ &= (-1)^{\frac{m-2}{2}}\mathcal{T}(r_{m-3} \oplus^{\pm 1} r_{m-2}) \end{aligned}$$

so we conclude that $\mathcal{T}(r_{m-1} \oplus^{\pm 1} r_m) = \pm\mathcal{T}(r_{m-3} \oplus^{\pm 1} r_{m-2})$. Note that all the possible values for $r_1 \oplus^{\pm 1} r_2$ are $[\pm 3, \pm 2, \pm 1] = \{\pm 3/10, \pm 1/4, \pm 3/8, \pm 1/2\}$, and it can be verified from [Che20, Section 9] that we have $\mathcal{T}(r_1 \oplus^{\pm 1} r_2)(1, z) = \pm z$ in all these cases. It now follows from induction on m that $P_{1,m}(1, z) = \pm z$ for all values of m.

Finally, by Farey recursion, for $k \geq 2$ we have

$$P_k(1, z) = \mathcal{T}(r_m)(1, z)P_{k-1}(1, z) - P_{k-2}(1, z)$$

If $m/2$ is even, then $\mathcal{T}(r_m)(1, z) = 1$, so we get

$$P_0(1, z) = 0,\ P_1(1, z) = \pm z,\ P_2(1, z) = P_1(1, z),$$

and $P_k(1, z) = -P_{k-3}(1, z)$ for all $k \geq 3$. If $m/2$ is odd, then $\mathcal{T}(r_m)(1, z) = -1$, so we get $P_k(1, z) = P_{k-3}(1, z)$. This proves the lemma.

□

Corollary 3.24. *In the case that* $r = [3, 2, 3, 2, ..., 3, 2]$ *with length* m*, we have* $P_{1,m}(1, z) = (-1)^{\lfloor m/4 \rfloor} z$.

Proof. In this case, it follows from the calculation in the previous lemma that

$$\begin{aligned} P_{1,m} &= \mathcal{T}(r_{m-1} \oplus r_m) = -\mathcal{T}(r_{m-2} \oplus r_{m-1}) \\ &= (-1)(-1)^{\frac{m}{2}}\mathcal{T}(r_{m-3} \oplus r_{m-2}) = (-1)^{\frac{m+2}{2}} P_{1,m-2} \end{aligned}$$

and when $m = 2$, we have $P_{1,2}(1, z) = \mathcal{T}([3, 2, 1])(1, z) = \mathcal{T}(3/10)(1, z) = z$. The general formula then follows by induction on m. □

Lemma 3.25. *In the case that $r = [3, 2, 3, 2, ..., 3, 2]$ with length $m + 2$, we have*

$$F_{0,m+2} = (-1)^{m/2+1} F_{0,m} + (-1)^{\lfloor m/4 \rfloor} \frac{\mathcal{T}(r_m)^2 - 1}{w^2 - 1}(1, z),$$

$$\frac{\mathcal{T}(r_{m+2})^2 - 1}{w^2 - 1}(1, z) = \frac{\mathcal{T}(r_m)^2 - 1}{w^2 - 1}(1, z) + (-1)^{\lfloor m/4 \rfloor} 2z^2 F_{0,m+2}.$$

Proof. Since $r_{m+1} = r_{m-1} \oplus^3 r_m$, we have

$$\begin{aligned} F_{0,m+2} &= \frac{\mathcal{T}(r_{m+1})}{z(w^2 - 1)}(1, z) \\ &= \frac{\mathcal{T}(r_{m-1} \oplus^3 r_m)}{z(w^2 - 1)}(1, z) \qquad (3.3) \\ &= \frac{P_{3,m}}{z(w^2 - 1)}(1, z) = F_{1,m}. \end{aligned}$$

By Lemma 3.22, Lemma 3.21, and Corollary 3.24, we know

$$F_{1,m} = (-1)^{m/2+1} F_{0,m} + (-1)^{\lfloor m/4 \rfloor} \frac{\mathcal{T}(r_m)^2 - 1}{w^2 - 1}(1, z)$$

which proves the first equation. To prove the second equation, first note that since $r_{m+2} = r_m \oplus^2 r_{m+1}$, we have

$$\mathcal{T}(r_{m+2}) = \mathcal{T}(r_{m+1})\mathcal{T}(r_m \oplus r_{m+1}) - \mathcal{T}(r_m).$$

Since $r_{m+1} = r_{m-1} \oplus^3 r_m$ and $r_m \oplus r_{m+1} = r_{m-1} \oplus^4 r_m$, we have

$$\mathcal{T}(r_{m+2}) = \mathcal{T}(r_{m-1} \oplus^3 r_m)\mathcal{T}(r_{m-1} \oplus^4 r_m) - \mathcal{T}(r_m) = P_3 P_4 - \mathcal{T}(r_m).$$

Therefore

$$\begin{aligned} \frac{\mathcal{T}(r_{m+2})^2 - 1}{w^2 - 1}(1, z) &= \frac{(P_3 P_4 - \mathcal{T}(r_m))^2 - 1}{w^2 - 1}(1, z) \\ &= \frac{P_3^2 P_4^2}{w^2 - 1}(1, z) - \frac{2\mathcal{T}(r_m) P_3 P_4}{w^2 - 1}(1, z) + \frac{\mathcal{T}(r_m)^2 - 1}{w^2 - 1}(1, z) \end{aligned}$$

We can rewrite $\frac{P_3^2 P_4^2}{w^2-1}(1, z) = \frac{P_3}{w^2-1}(1, z) \cdot P_3(1, z) \cdot P_4^2(1, z)$. By Lemma 3.23 $P_3(1, z) = 0$, so

$$\frac{P_3^2 P_4^2}{w^2 - 1}(1, z) - \frac{2\mathcal{T}(r_m) P_3 P_4}{w^2 - 1}(1, z) + \frac{\mathcal{T}(r_m)^2 - 1}{w^2 - 1}(1, z)$$

is equal to

$$\frac{\mathcal{T}(r_m)^2 - 1}{w^2 - 1}(1, z) - 2\frac{P_3}{w^2 - 1}(1, z)\mathcal{T}(r_m)(1, z) P_4(1, z).$$

Now by Lemma 3.22 and Lemma 3.23, we have $\mathcal{T}(r_m)(1,z) = (-1)^{m/2}$ and $P_4(1,z) = (-1)^{(m+2)/2}P_1(1,z)$. Substituting these two and Equation (3.3) into the above, we get

$$\begin{aligned}\frac{\mathcal{T}(r_{m+2})^2-1}{w^2-1}(1,z) &= \frac{\mathcal{T}(r_m)^2-1}{w^2-1}(1,z) - 2zF_{0,m+2}(-1)^{m/2}(-1)^{m/2+1}P_1(1,z)\\ &= \frac{\mathcal{T}(r_m)^2-1}{w^2-1}(1,z) + (-1)^{\lfloor m/4\rfloor}2z^2F_{0,m+2}.\end{aligned}$$

□

Corollary 3.26. *Let $r = [3,2,3,2,...,3,2]$ with length m. If we define*

$$a_n = \frac{\mathcal{T}(r_{2n})^2-1}{w^2-1}(1,z)$$

$$b_n = (-1)^{\lceil \frac{n}{2}\rceil+1}F_{0,2n}$$

then for all $1 \leq n \leq m/2$ we have the following formulae for a_n and b_n:

$$\begin{aligned}a_n &= c_1(A_1-1)A_1^{n-1} + c_2(A_2-1)A_2^{n-1}\\ b_n &= c_1A_1^{n-1} + c_2A_2^{n-1}\end{aligned}$$

where

$$c_1 = \frac{\sqrt{z^4+2z^2}+z^2+1}{2\sqrt{z^4+2z^2}}, \qquad c_2 = \frac{\sqrt{z^4+2z^2}-z^2-1}{2\sqrt{z^4+2z^2}},$$

$$A_1 = \sqrt{z^4+2z^2}+z^2+1, \qquad A_2 = z^2+1-\sqrt{z^4+2z^2}.$$

Alternatively, a_n and b_n can also be expressed as:

$$a_n = \sum_{i=0}^{n-1}(2z^2)^{n-i-1}\left(\binom{2n-i-2}{i}(2z^2+1)+\binom{2n-i-3}{i}\right)$$

$$b_n = \sum_{i=1}^{n-1}(2z^2)^{n-i-1}\left(\binom{2n-i-2}{i-1}(2z^2+1)+\binom{2n-i-3}{i-1}\right)$$

Proof. First of all, it follows from Lemma 3.25 that

$$a_{n+1} = a_n + (-1)^{\lfloor \frac{n}{2}\rfloor}2z^2\cdot(-1)^{\lceil\frac{n+1}{2}\rceil+1}b_{n+1}$$

$$(-1)^{\lceil\frac{n+1}{2}\rceil+1}b_{n+1} = (-1)^{n+1}\cdot(-1)^{\lceil\frac{n}{2}\rceil+1}b_n + (-1)^{\lfloor\frac{n}{2}\rfloor}a_n$$

Since $\lceil\frac{n+1}{2}\rceil+1 \equiv \lceil\frac{n}{2}\rceil + n \pmod 2$ and $\lfloor\frac{n}{2}\rfloor + \lceil\frac{n}{2}\rceil \equiv n \pmod 2$, the above two equations simplify to

$$\begin{aligned}a_{n+1} &= a_n + 2z^2b_{n+1}\\ b_{n+1} &= b_n + a_n\end{aligned}$$

and substituting the first equation into the second yields $b_{n+2} = (2z^2+2)b_{n+1} - b_n$. The solution to this linear recurrence relation is then $b_n = c_1A_1^{n-1} + c_2A_2^{n-1}$, where A_1, A_2 are the roots of the quadratic equation $x^2 - (2z^2+2)x + 1 = 0$, and c_1, c_2 are constants such that $c_1 + c_2 = b_1$, $c_1A_1 + c_2A_2 = b_2$. By explicit calculation

$$b_1 = F_{0,2} = \frac{\mathcal{T}(1/3)}{z(w^2-1)}(1,z) = 1$$

$$b_2 = F_{0,4} = \frac{\mathcal{T}(7/24)}{z(w^2-1)}(1,z) = 2z^2+2.$$

Then solving for c_1, c_2 gives the first formula for b_n. Substituting into $a_n = b_{n+1} - b_n$ gives us the first formula for a_n.

The second version of the formulae follows from induction. Since $a_1 = \frac{\mathcal{T}(2/7)^2-1}{w^2-1}(1,z) = 2z^2+1$ and $b_1 = 1$, the base case holds. Suppose that the second formulae holds for a_n and b_n; then we have

$$\begin{aligned} b_{n+1} &= a_n + b_n \\ &= \sum_{i=0}^{n-1}(2z^2)^{n-i-1}\left(\binom{2n-i-1}{i}(2z^2+1) + \binom{2n-i-2}{i}\right) \\ &= \sum_{i=1}^{n}(2z^2)^{n-i}\left(\binom{2n-i}{i-1}(2z^2+1) + \binom{2n-i-1}{i-1}\right) \end{aligned}$$

$$\begin{aligned} a_{n+1} &= a_n + 2z^2b_{n+1} \\ &= \sum_{i=1}^{n}(2z^2)^{n-i}\left(\binom{2n-i-1}{i-1}(2z^2+1) + \binom{2n-i-2}{i-1}\right) \\ &\qquad + \sum_{i=0}^{n-1}(2z^2)^{n-i}\left(\binom{2n-i-1}{i}(2z^2+1) + \binom{2n-i-2}{i}\right) \\ &= \sum_{i=0}^{n}(2z^2)^{n-i}\left(\binom{2n-i}{i}(2z^2+1) + \binom{2n-i-1}{i}\right) \end{aligned}$$

which shows that the formulae also hold for a_{n+1} and b_{n+1}.

□

Having calculated all of the terms in Equation (3.1) explicitly, we can now give a general formula for F_k.

Theorem 3.27. *Fix $r = [3,2,3,2,\cdots,3,2]$ with length m, and let $n = m/2$. Then for any $k \geq 0$, we have*

$$F_k = (-1)^{(k-1)(n+1)}(kC + (-1)^{n+1}F_0)$$

where C, $F_0 \in \mathbb{Z}[z]$ are given by

$$C = (-1)^{\lfloor \frac{n}{2} \rfloor}\left(c_1(A_1-1)A_1^{n-1} + c_2(A_2-1)A_2^{n-1}\right)$$

$$F_0 = (-1)^{\lceil \frac{n}{2} \rceil + 1}\left(c_1A_1^{n-1} + c_2A_2^{n-1}\right)$$

with c_1, c_2, A_1, A_2 defined as in Corollary 3.26.

Proof. By Lemma 3.21, we have

$$F_k = \frac{\mathcal{T}(r_{2n})^2 - 1}{w^2 - 1}(1, z) \cdot \frac{P_{3k-2}(1, z)}{z} - \mathcal{T}(r_{2n})(1, z)F_{k-1}.$$

By Lemma 3.22 we know that $\mathcal{T}(r_{2n})(1, z) = (-1)^n$. Additionally, by Lemma 3.23 we know that $P_{3k-2}(1, z)/z = (-1)^{\lfloor n/2 \rfloor}(-1)^{(k-1)(n+1)}$. So, by substituting $a_n = \frac{\mathcal{T}(r_{2n})^2-1}{w^2-1}(1, z)$ as in Corollary 3.26, we have:

$$F_k = (-1)^{(k-1)(n+1)}(-1)^{\lfloor n/2 \rfloor}a_n + (-1)^{n+1}F_{k-1}.$$

By Corollary 3.26 setting $C = (-1)^{\lfloor n/2 \rfloor}a_n$ is equivalent to the definition of C in the theorem statement. Then $F_k = (-1)^{(k-1)(n+1)}C + (-1)^{n+1}F_{k-1}$, which implies that $F_k = (-1)^{(k-1)(n+1)}(kC + (-1)^{n+1}F_0)$. The formulae for C and F_0 come from Corollary 3.26. □

Remark 3.28. Fix $q/p = [3, 2, 3, 2, \cdots, 3, 2, 3k]$ with n-many 2's and $k \geq 0$. For p to be odd (and yield a two-bridge knot), then $n + k$ must be odd. So by Theorem 3.27 $F_k = kC + (-1)^{n+1}F_0$.

Example 3.29. In the following table we compute C, F_0 and F_k for $[3, 2]$, $[3, 2, 3, 2]$, and $[3, 2, 3, 2, 3, 2]$. The calculations have been computed using SageMath.

r_m	C	F_0
$[3, 2]$	$2z^2 + 1$	1
$[3, 2, 3, 2]$	$-4z^4 - 6z^2 - 1$	$2z^2 + 2$
$[3, 2, 3, 2, 3, 2]$	$-8z^6 - 20z^4 - 12z^2 - 1$	$-4z^4 - 8z^2 - 3$

TABLE 3.1. Data

It then follows from Theorem 3.27 that

$$\begin{aligned}
\tilde{p}(\mathcal{K}(1, k)) &= 2kz^2 + k + 1 \\
\tilde{p}(\mathcal{K}(2, k)) &= -4kz^4 - 2z^2(3k + 1) - (k + 2) \\
\tilde{p}(\mathcal{K}(3, k)) &= -8kz^6 - 4z^4(5k + 1) - 4z^2(3k + 2) - (k + 3)
\end{aligned}$$

◇

We now prove a final lemma about discrete valuations before proving Theorem 1.1.

Lemma 3.30. *Let α be an algebraic number but not an algebraic integer, and let $f(x) \in \mathbb{Z}[x]$ be its minimal polynomial over $\mathbb{Q}$. If F is a number field containing α, then there exists a discrete valuation v on F such that $v(\alpha) < 0$.*

Proof. We write $f(x) = \sum_{i=0}^{n} a_i x^i$ where $a_n \in \mathbb{Z}\backslash\{0, 1\}$, and let $\beta = a_n\alpha$. We first show that β is an algebraic integer. Compute

$$0 = f(\alpha) = \sum_{i=0}^{n} a_i\alpha^i = \sum_{i=0}^{n} a_i \left(\frac{\beta}{a_n}\right)^i = \sum_{i=0}^{n} \frac{a_i}{a_n^i}\beta^i,$$

and multiplying both sides of the above equation by a_n^{n-1} we get

$$0 = a_n^{n-1} \sum_{i=0}^{n} \frac{a_i}{a_n^i} \beta^i = \sum_{i=0}^{n} a_i a_n^{n-i-1} \beta^i,$$

where $a_i a_n^{n-i-1} = 1$ when $i = n$, and $a_n^{n-i-1} \in \mathbb{Z}$ when $0 \leq i < n$. Therefore $g(x) = \sum_{i=0}^{n} a_i a_n^{n-i-1} x^i \in \mathbb{Z}[x]$ is a monic polynomial with root β, so β is an algebraic integer.

For the field $F \supset \mathbb{Q}[\alpha]$, because $\mathcal{O}_F$ is a Dedekind domain, we can factor $\beta\mathcal{O}_F$ and $a_n\mathcal{O}_F$ into unique products of prime ideals. We write $\beta\mathcal{O}_F = \prod_{i=1}^{m} P_i^{e_i}$ and $a_n\mathcal{O}_F = \prod_{i=1}^{m'} P_i'^{e_i'}$, where the P_i's are distinct prime ideals, as are the P_i''s. Note that we also assume $e_i, e_i' > 0$. If there exists $1 \leq j \leq m'$ such that either $\mathcal{P} := P_j' \neq P_i$ for all $1 \leq i \leq m$, or $\mathcal{P} = P_h$ and $e_j' > e_h$, then as defined in Definition 2.33, we have either $v_{\mathcal{P}}(a_n) = e_j' > 0 = v_{\mathcal{P}}(\beta)$, or $v_{\mathcal{P}}(a_n) = e_j' > e_h = v_{\mathcal{P}}(\beta)$. In both cases we have $v_{\mathcal{P}}(\alpha) = v_{\mathcal{P}}(\beta) - v_{\mathcal{P}}(a_n) < 0$.

Suppose otherwise, there exists no such j, which means for all $1 \leq i \leq m'$, there exists $1 \leq j \leq m$ such that $P_j = P_i'$ and $e_j \geq e_i'$. In other words, we can write $\beta\mathcal{O}_F$ as a product of $a_n\mathcal{O}_F$ and another ideal I, which means $\beta\mathcal{O}_F \subset a_n\mathcal{O}_F$, $\beta \in a_n\mathcal{O}_F$. This implies $\alpha = \beta/a_n \in \mathcal{O}_F$, contradicting the fact that $\alpha \notin \mathcal{O}_F$. □

Theorem 1.1. *For every two-bridge knot $K_r = \mathcal{K}(n,k)$, there exists an epimorphism $\Gamma_r \to \Gamma_{1/3}$. Moreover, for every $(x_0, y_0) \in \mathbb{C}^2$ that is an intersection point of $X_0(\Gamma_r)$ and the irreducible component $x^2 - y - 1$ of $X(\Gamma_r)$, and for any $\mathrm{SL}_2(\mathbb{C})$-representation ρ of Γ_r corresponding to (x_0, y_0),*
(1) There exists a number field F such that the image of ρ is in $\mathrm{SL}_2(F)$;
(2) There exists a prime ideal $\mathcal{P}$ of $\mathcal{O}_F$ such that ρ is an ANI-representation of Γ_r with respect to the discrete valuation $v_{\mathcal{P}}$.

Proof. Recall from Remark 3.20 that we can obtain $\tilde{p}(\mathcal{K}(n,k))$ from $F_{k,2n}$ by substituting $z^2 = 3 - x^2$. We first claim that all the coefficients of $\tilde{p}(\mathcal{K}(n,k)) \in \mathbb{Z}[x]$ but the constant term are even, and the constant term is odd. By Theorem 3.27, we have

$$F_{k,2n} = \pm kC \pm F_0$$

Using the formula for C and F_0 in Corollary 3.26 and substituting $z^2 = 3 - x^2$, we have

$$\begin{aligned} C &\equiv \sum_{i=0}^{n-1} (2z^2)^{n-i-1} \left(\binom{2n-i-2}{i}(2z^2+1) + \binom{2n-i-3}{i} \right) \\ &\equiv \sum_{i=0}^{n-1} (6-2x^2)^{n-i-1} \left(\binom{2n-i-2}{i}(7-2x^2) + \binom{2n-i-3}{i} \right) \pmod{2\mathbb{Z}[x]}. \end{aligned}$$

If $i \neq n-1$ then $(6-2x^2)^{n-i-1} \in 2\mathbb{Z}[x]$. Therefore, it suffices to consider only $i = n-1$. Therefore,

$$C \equiv \left(\binom{2n-i-2}{i}(7-2x^2) + \binom{2n-i-3}{i} \right) \equiv 7 \equiv 1 \pmod{2\mathbb{Z}[x]}.$$

Similarly, we can calculate (again substituting $z^2 = 3 - x^2$ in the second step):

$$\begin{aligned} F_{0,2n} &\equiv \sum_{i=1}^{n-1} (2z^2)^{n-i-1} \left(\binom{2n-i-2}{i-1}(2z^2+1) + \binom{2n-i-3}{i-1} \right) \\ &\equiv \sum_{i=1}^{n-1} (6-2x^2)^{n-i-1} \left(\binom{2n-i-2}{i-1}(7-2x^2) + \binom{2n-i-3}{i-1} \right) \\ &\equiv 7(n-1)+1 \equiv n \pmod{2\mathbb{Z}[x]}, \end{aligned}$$

Therefore we have $\tilde{p}(\mathcal{K}(n,k)) \equiv F_{0,2n} + kC \equiv n + k \pmod{2\mathbb{Z}[x]}$. Since the knot $\mathcal{K}(n,k)$ is only defined when $n+k$ is odd, we conclude that $\tilde{p}(\mathcal{K}(n,k)) \equiv 1 \pmod{2\mathbb{Z}[x]}$, which proves the first claim.

Given a polynomial F such that all its coefficients but the constant term are even, we claim that F cannot be written as $F = fg$ where either f or g has degree ≥ 1 and has odd leading term coefficient. Suppose by contradiction that this is the case; then we can write $F = \sum_{i=0}^{l} c_i x^i$, $f = \sum_{i=0}^{m} a_i x^i$ where a_m is odd, and $g = \sum_{i=0}^{n} b_i x^i$. Let $j = \max\{0 \leq i \leq n : b_i \text{ is odd}\}$ (note that there must exist some b_i that is odd, otherwise the constant term of F would be even). Consider the coefficient of $F = fg$ in degree $j + m \geq 1$:

$$c_{j+m} = \sum_{(i,k):\ i+k=j+m} a_i b_k = \sum_{k \geq j} a_{j+m-k} b_k$$

Since c_{j+m} is even, and b_k is even for $k > j$, we know that b_j is also even, contradicting the definition of b_j.

For any intersection point (x_0, y_0) between $X_0(\Gamma_{\mathcal{K}(n,k)})$ and $x^2 - y - 1 = 0$, we always have $\tilde{p}(\mathcal{K}(n,k))(x_0) = 0$. Thus $\tilde{p}(\mathcal{K}(n,k)) = q \cdot q'$ for some $q' \in \mathbb{Q}[x]$ where $q(x)$ is the minimal polynomial of x_0 over $\mathbb{Q}$. If x_0 is an algebraic integer, then $q(x)$ is a monic polynomial of $\mathbb{Z}[x]$, and because $\tilde{p}(\mathcal{K}(n,k)) \in \mathbb{Z}[x]$, this means $q' \in \mathbb{Z}[x]$ (see [Fra67, Theorem 23.11]). But then q has odd leading term coefficient, contradicting the last claim we proved. Hence x_0 is an algebraic number non-integer.

It then follows from the remark after Lemma 2.14 that for any number field F containing all roots α of the equation $\alpha + 1/\alpha = x_0$ (where x_0 runs through all intersection points between $X_0(\Gamma_{\mathcal{K}(n,k)})$ and $x^2 - y - 1 = 0$), $\mathrm{SL}_2(F)$ contains the image of any representation $\rho : \Gamma_{\mathcal{K}(n,k)} \to \mathrm{SL}_2(\mathbb{C})$ corresponding to (x_0, y_0). Finally, it follows from Lemma 3.30 that there exists a prime ideal $\mathcal{P}$ of $\mathcal{O}_F$ such that $v_{\mathcal{P}}(x_0) < 0$. □

4. Boundary slopes of $\mathcal{K}(n,k)$

This section addresses a method for computing all of the boundary slopes corresponding to a continued fraction $r = \mathcal{K}(n,k)$ and the corresponding boundary slopes.

Definition 4.1. An *edge path* from $1/0$ to p/q is a sequence of rightward moves across vertices of triangles in a Farey graph (see Figure 4.2.5) given by a unique tuple $(b_1, \ldots, b_k)$ for $b_i \in \mathbb{Z}$. Each $b_i < 0$ corresponds to a move across i triangles on the top edge of the diagram and $b_i > 0$ corresponds to a move across i triangles on the bottom edge of the diagram.

As in [HT85] pg. 229 a path is *minimal* if no edge is immediately retraced and no two edges of a triangle are traversed in succession. This requires each $|b_i| \geq 2$.

Lemma 4.2 (Pg. 229 in [HT85]). *Every fraction $q/p \in \mathbb{Q}$ has a unique continued fraction decomposition*

$$q/p = [a_1, a_2, a_3, a_4, \ldots, a_k] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}},$$

where $a_i > 0$, $a_k > 1$. These numbers a_i determine the number of smaller triangles in each larger triangle in Figure 4.2.5. All minimal edge paths for q/p are contained in the bolded lines of the finite subcomplex of the Farey graph as in Figure 4.2.5.

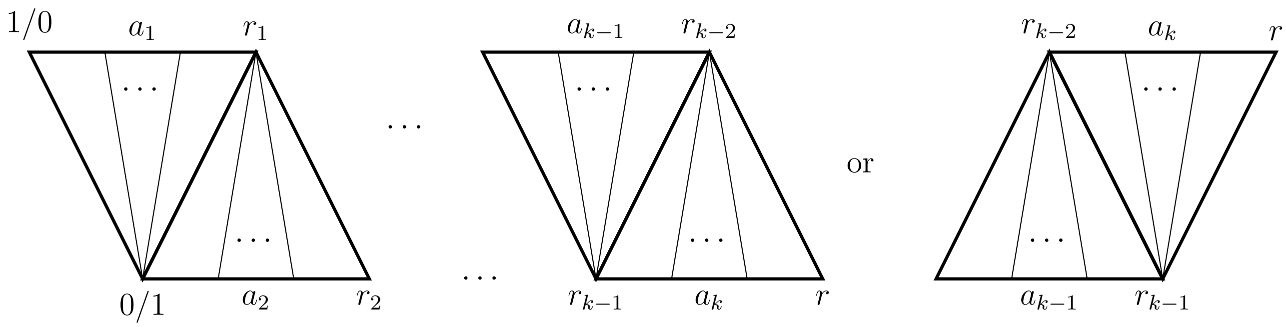

FIGURE 4.2.5. Subcomplexes of the Farey graph for $r = [a_1, a_2, \ldots, a_k]$.

Remark 4.3. Intuitively, the minimal edge paths for q/p correspond to moving to the right, along horizontal and diagonal bolded edges in the subcomplex of the Farey graph in Figure 3.3.4.

Theorem 4.4 (Restatement of Proposition 2 in [HT85]). *Each boundary slope of an essential surface corresponds to a minimal edge path $(b_1, \ldots, b_k)$, which wraps around $K_{q/p}$ once longitudinally and $m(b_1, \ldots, b_k)$ times meridionally, with m being the function*

$$m(b_1, \ldots, b_k) = 2[(n^+ - n^-) - (n_0^+ - n_0^-)],$$

where n^+ and n^- are the number of positive and negative b_i's and n_0^+ and n_0^- are the corresponding numbers for the unique edge path $(b'_1, \ldots, b'_n)$ with each b'_i even. In other words, the boundary slope of this surface corresponds to $\mu^{m(b_1,\ldots,b_k)}\lambda \in \pi_1(\partial M(K))$.

Remark 4.5. In particular, 0, which corresponds to the longitude, is always a boundary slope.

Remark 4.6. See [HT85] for the topological properties of $S_n(n_1, \ldots, n_{k-1})$.

Notation 4.7. For a continued fraction $r = [a_1, \ldots, a_k]$, let M_r denote the set of minimal edge paths from 1/0 to r. Furthermore, let $r_j = [a_1, \ldots, a_j]$. Let $N(j, r)$ be the set of all values $n^+ - n^-$ for the minimal edge paths in M_{r_j}. Let $T(j,r) = n_0^- - n_0^+$ for the partial sum, with n^+, n^-, n_0^+, n_0^- as in Theorem 4.4. Let $B(j,r) = \{2(n + T(j,r)) \mid n \in N(j,r)\}$ be the set of all boundary slopes $m(b_1, \dots, b_\ell)$ as in Theorem 4.4.

Lemma 4.8 (Pg. 230 in [HT85])**.** *For a fraction $q/p \in \mathbb{Q}$, with continued fraction decomposition $q/p = [r_1, \dots, r_k]$, the number of minimal edge paths from $1/0$ to partial sum p_i/q_i can be counted recursively as:*

$$|M_{r_i}| = \begin{cases} |M_{r_{i-1}}| + |M_{r_{i-2}}|, & r_i > 1 \\ |M_{r_{i-3}}| + |M_{r_{i-2}}|, & r_i = 1 \end{cases}$$

with $|M_{r_0}| = |M_{r_{-1}}| = |M_{r_{-2}}| = 0$.

Lemma 4.9. *For a continued fraction $r = [a_1, \dots, a_k]$, such that k is even, we have $p \in M_r$ if and only if one of the following holds:*

- *$p = (q, -a_k)$ for some $q \in M_{r_{k-1}}$ and the last entry of q is positive,*
- *$p = (q, -a_k - 1)$ for some $q \in M_{r_{k-1}}$ and the last entry of q is negative,*
- *$p = (q, 2 + a_{k-1}, \underbrace{2, 2, \dots, 2}_{a_k - 1})$ for some $q \in M_{r_{k-2}}$ and the last entry of q is positive,*
- *$p = (q, 1 + a_{k-1}, \underbrace{2, 2, \dots, 2}_{a_k - 1})$ for some $q \in M_{r_{k-2}}$ and the last entry of q is negative.*

Proof. We begin by showing that the given edge paths are contained in M_r. Let $q \in M_{r_{k-1}}$. If the final entry of q is positive, by inspection of the finite subcomplex of the Farey graph corresponding to r, we see that $(q, -a_k) \in M_r$. Furthermore, if the final entry of q is negative, then $(q, -a_k - 1) \in M_r$. This corresponds to the first two bullet points.

Suppose $q \in M_{r_{k-2}}$. By a similar argument to above, if the last entry of q is positive, then

$$(q, 2 + a_{k-1}, \underbrace{2, 2, \dots, 2}_{a_k - 1}) \in M_r.$$

Additionally, if the last entry of q is negative, then

$$(q, 1 + a_{k-1}, \underbrace{2, 2, \dots, 2}_{a_k - 1}) \in M_r.$$

Therefore we have shown containment of $|M_{r_{k-1}}| + |M_{r_{k-2}}|$ elements in M_r. Therefore, by Lemma 4.8 the result follows. □

Lemma 4.10. *For a continued fraction $r = [a_1, \dots, a_k]$, such that k is odd, we have $p \in M_r$ if and only if:*

- *$p = (q, a_k + 1)$ for some $q \in M_{r_{k-1}}$ and the last entry of q is positive,*
- *$p = (q, a_k)$ for some $q \in M_{r_{k-1}}$ and the last entry of q is negative,*
- *$p = (q, -1 - a_{k-1}, \underbrace{-2, -2, \dots, -2}_{a_k - 1})$ for some $q \in M_{r_{k-2}}$ and the last entry of q is positive,*
- *$p = (q, -2 - a_{k-1}, \underbrace{-2, -2, \dots, -2}_{a_k - 1})$ for some $q \in M_{r_{k-2}}$ and the last entry of q is negative.*

Proof. Follows similarly to Lemma 4.9 by inspection of the subcomplex of the Farey graph corresponding to r. ☐

Corollary 4.11. *For a continued fraction $r = [a_1, \dots, a_j]$:*

- *if j is odd:*
$$N(j, r) = \{n + 1 \mid n \in N(j-1, r)\} \cup \{n - a_j \mid n \in N(j-2, r)\},$$
- *if j is even:*
$$N(j, r) = \{n - 1 \mid n \in N(j-1, r)\} \cup \{n + a_j \mid n \in N(j-2, r)\}.$$

Proof. Follows from Lemma 4.9 when k is even and from Lemma 4.10 when k is odd. ☐

Corollary 4.12. *For a knot $q/p = \mathcal{K}(n, k)$, the unique minimal edge path with only even entries in the continued fraction decomposition will be of the form:*

$$(\underbrace{-2, -2, -4, -2, -2, -4, \dots, -2, -2, -4}_{m-\text{many } 4\text{'s}}, \underbrace{-2, \dots, -2}_{3k-1})$$

This gives $T(2n+1, q/p) = 3(n+k) - 1$.

Proof. Existence follows from Lemma 4.10 and uniqueness follows from [HT85]. Note that every entry in the edge path is negative so $n_0^- - n_0^+ = 3(n+k) - 1$, the length of the edge path. ☐

Proposition 4.13. *For $q/p = \mathcal{K}(n, k)$, then*

$$B(2n+1, q/p) = \{6k + 6a + 10b \mid a + b \le n\} \cup \{6a + 10b \mid a + b \le n, 0 < a\} \cup \{0\}.$$

Proof. We prove this result by induction on n, with the additional assumption that

$$N(2n, q/p) = \bigcup_{j=0}^{n} \{m + 2j - 1 \mid m \in N(2(n-j-1)+1, q/p)\}$$

where we take $N(-1, q/p)$ to be $\{1\}$.

For the base case, consider $n = 0$. That is, $q/p = [3k]$. Note that the minimal edge paths for $[3k]$ are $(3k)$ and $(-2, -2, \dots, -2)$ with $3k - 1$ repetitions of -2. So $N(1, [3k]) = \{1, -3k+1\}$, and $T(1, [3k]) = 3k - 1$. Therefore $B(1, [3k]) = \{0, 6k\}$. Note

$$\{6k + 6a + 10b \mid a + b \le 0\} \cup \{6a + 10b \mid a + b \le 0, 0 < a\} \cup \{0\} = \{0, 6k\} = B(1, [3k]).$$

Additionally,

$$N(0, [3k]) = \{0\} = \{m - 1 \mid m \in N(-1, [3k])\}.$$

Suppose that the theorem holds for all $n' < n$. Fix a knot $q/p = [3, 2, \dots, 3, 2, 3k]$ with n-many 2's and $k \in \mathbb{Z}_{>0}$. Since $2n$ is even, by Corollary 4.11,

$$N(2n, q/p) = \{m - 1 \mid m \in N(2n-1, q/p)\} \cup \{m + 2 \mid m \in N(2(n-1), q/p)\}.$$

By the inductive hypothesis,

$$N(2(n-1), q/p) = \bigcup_{j=0}^{n-1} \{m + 2j - 1 \mid m \in N(2(n-j-1)+1, q/p)\}.$$

Therefore, we have

$$\{m+2 \mid m \in N(2(n-1))\} = \left\{ m'+2 \;\middle|\; m' \in \bigcup_{j=0}^{n-1} \{m+2j-1 \mid m \in N(2(n-j-1)+1, q/p)\} \right\}$$
$$= \bigcup_{j=1}^{n} \{m+2j-1 \mid m \in N(2(n-j-1)+1, q/p)\}.$$

Additionally,

$$\{m-1 \mid m \in N(2n-1, q/p)\} = \{m-1 \mid m \in N(2(n-1)+1, q/p)\}.$$

So taken together,

$$N(2n, q/p) = \bigcup_{j=0}^{n} \{m+2j-1 \mid m \in N(2(n-j-1)+1, q/p)\}.$$

Next we show that all $c \in B(2n+1, q/p)$ have the desired for.

We know $T(2n+1, q/p) = 3(n+k)-1$ by Corollary 4.12. Since $2n+1$ is odd, by Corollary 4.11,

$$N(2n+1, q/p) = \{m+1 \mid m \in N(2n, q/p)\} \cup \{m-3k \mid m \in N(2(n-1)+1, q/p)\}.$$

Since $B(2n+1, q/p) = \{2(c+T(2n+1, q/p)) \mid c \in N(2n+1, q/p)\}$, it suffices to show for each $c \in N(2n+1, q/p)$ that $2(c+T(2n+1, q/p))$ has the desired form. We will consider each set in the union separately.

Case 1. Suppose

$$c \in \{m-3k \mid m \in N(2(n-1)+1, q/p)\}.$$

So $2(c+T(2n+1, q/p)) \in B(2n+1, q/p)$. Furthermore, since for $2(n-1)+1$ we know that the corresponding partial sum is $[3, 2, 3, 2, \ldots, 3, 2, 3]$, so

$$T(2(n-1)+1, q/p) = 3 + 3(n-1) - 1 = 3n-1,$$

so $T(2(n-1)+1, q/p)+3k = T(2n+1, q/p)$. Additionally, since $c+3k \in N(2(n-1)+1, q/p)$ we have

$$2(c+3k+T(2(n-1)+1, q/p)) \in B(2(n-1)+1, q/p).$$

This then simplifies to

$$2(c+T(2n+1, q/p)) \in B(2(n-1)+1, q/p).$$

By the inductive hypothesis

$$B(2(n-1)+1, q/p) = \{6+6a+10b \mid a+b \leq n-1\} \cup \{6a+10b \mid a+b \leq n-1, 0 < a\} \cup \{0\}.$$

Since $n-1 < n$ it follows that

$$2(c+T(2n+1, q/p)) \in \{6+6a+10b \mid a+b \leq n\} \cup \{6a+10b \mid a+b \leq n, 0 < a\} \cup \{0\}.$$

Case 2. Suppose

$$c \in \{m+1 \mid m \in N(2n, q/p)\}.$$

In particular, by the inductive hypothesis

$$c-1\in\bigcup_{j=0}^{n}\{m+2j-1\mid m\in N(2(n-j-1)+1,q/p)\}.$$

Choose $0\le c_1\le n$, and $c_2\in N(2(n-c_1-1)+1,q/p)$ such that $c=c_2+2c_1$. By Corollary 4.12 it follows that $T(2(n-c_1-1)+1,q/p)=3(n-c_1)-1$. So

$$T(2n+1,q/p)=3c_1+3k+T(2(n-c_1-1)+1,q/p).$$

Thus

$$\begin{aligned}2(c+T(2n+1,q/p))&=2(c_2+2c_1+3c_1+3k+T(2(m-c_1-1)+1,q/p))\\&=2(c_2+5c_1+3k+T(2(n-c_1-1)+1,q/p))\\&=10c_1+6k+2(c_2+T(2(n-c_1-1)+1,q/p)).\end{aligned}$$

Since $c_2\in N(2(n-c_1-1)+1,q/p)$, we have

$$2(c_2+T(2(n-c_1-1)+1,q/p))\in B(2(n-c_1-1)+1,q/p).$$

Then by the inductive hypothesis

$$B(2(n-c_1-1)+1,q/p)=\{6+6a+10b\mid a+b\le n-c_1-1\}\cup\{6a+10b\mid a+b\le n-c_1-1,0<a\}\cup\{0\}.$$

So we can choose $a',b'\in\mathbb{Z}_{\ge0}$, such that $2(c_2+T(2(n-c_1-1)+1,q/p))=6a'+10b'$ with $a'+b'=n-c_1-1$ and $0<a'$, or $6+6a'+10b'$ with $a'+b'\le n-c_1-1$ (or of course 0), according to the inductive hypothesis. Therefore

$$2(c+T(2n+1,q/p))=10c_1+6k+6a'+10b'=6a'+10(b'+c_1)+6k.$$

Thus

$$2(c+T(2n+1,q/p))\in\{6k+6a+10b\mid a+b\le n\}.$$

Therefore

$$B(2n+1,q/p)\subseteq\{6k+6a+10b\mid a+b\le n\}\cup\{6a+10b\mid a+b\le n,0<a\}\cup\{0\}.$$

To show the other containment, we have three cases:

Case 1. Suppose $c=0$. Since $T(2n+1,q/p)\in N(2n+1,q/p)$, it follows that $0\in B(2n+1,q/p)$.

Case 2. Suppose $2c\in\{6k+6a+10b\mid a+b\le n\}$. Choose $a,b\in\mathbb{Z}_{\ge0}$ with $a+b\le n$ such that $c=3a+5b+3k$. Since $T(2n+1,q/p)=3(n+k)-1$, it suffices to show that $3a+5b+3k-3(n+k)+1\in N(2n+1,q/p)$. Thus, by Corollary 4.11 it suffices to show that $3(a-n)+5b\in N(2n,q/p)$. Note that $3(a-n)+5b=3a+5(b-n)+2b$. Since

$$N(2n,q/p)=\bigcup_{j=0}^{n}\{m+2j-1\mid m\in N(2(n-j-1)+1,q/p)\},$$

it suffices to show that $3a+3(b-n)+1=m$ for some m in $N(2(n-b-1)+1,q/p)$. Since $T(2(n-b-1)+1,q/p)=3(n-b)-1$, by the inductive hypothesis

$$\begin{aligned}N(2(n-b-1)+1,q/p)=&\{3a'+5b'+3(b-n)+4\mid a'+b'\le n-b-1\}\\&\cup\{3a'+5b'+3(b-n)+1\mid a'+b'\le n-b-1,0<a'\}\cup\{3(b-n)+1\},\end{aligned}$$

so $3a + 3(b - n) + 1 \in N(2(n - b - 1) + 1, q/p)$.

Case 3. Suppose $2c \in \{6a + 10b \mid a + b \leq n, 0 < a\}$. Choose $a, b \in \mathbb{Z}_{\geq 0}$ with $a + b \leq n$ and $a > 0$ such that $c = 3a + 5b$. Since $T(2n + 1, q/p) = 3(n + k) - 1$, it suffices to show that $3a + 5b - 3(n + k) + 1 \in N(2n + 1, q/p)$. So by Corollary 4.11 it suffices to show that $3(a - n) + 5b + 1 \in N(2(n - 1) + 1, q/p)$. By the inductive hypothesis

$$N(2(n-1)+1, q/p) = \{3(a'-n) + 5b' + 4 \mid a' + b' \leq n - 1\} \\ \cup \{3(a'-n) + 5b' + 1 \mid a' + b' \leq n - 1, 0 < a'\} \cup \{-3n + 1\},$$

so $3(a - n) + 5b + 1 \in N(2(n - b - 1) + 1, q/p)$, completing the second containment. □

Example 4.14. The knot $q/p = [3, 2, 3k] = \frac{6k+1}{21k+3}$ (where k is even) has exactly 5 boundary slopes:

Boundary slope	Minimal edge-path
0	$(-2, -2, -4, -2, ..., -2)$ (with $3k - 1$ copies of -2 at the end)
6	$(3, -3, -2, ..., -2)$ (with $3k - 1$ copies of -2 at the end)
$6k$	$(-2, -2, -3, 3k)$
$6k + 6$	$(3, -2, 3k)$
$6k + 10$	$(4, 2, 3k + 1)$

◇

5. Determining the detected boundary slope

The following lemma determines an explicit description of the presentation for the family of two-bridge knots $[3, 2, 3k']$.

Lemma 5.1. *For the knot $q/p = [3, 2, 3k'] = \frac{6k'+1}{21k'+3}$, where $k' = 2k \in 2\mathbb{Z}_{\geq 0}$, with knot group $\Gamma_{q/p} = \langle a, b \mid wa = bw \rangle$ as in Theorem 2.8, define*

$$S_1 = babab^{-1}a^{-1}b^{-1}ababa^{-1}b^{-1}a^{-1}, \quad S_2 = bab.$$

Then $w = b^{-1}S_1^{3k}S_2$.

Proof. In order for $q/p = [3, 2, 3k'] = [3, 2, 3, 0, \cdots, 3, 0, 3]$ with $k' - 1$ zeros to be a knot p must be odd. By Lemma 2.24, this means $k' - 1$ must be even. Therefore k' is in $2\mathbb{Z}$.

We first show that for a fixed $0 \leq c < 3k'$, we have

$$\begin{cases} (-1)^{\lfloor nq/p \rfloor} = 1, & 7c \leq n \leq 7c + 3; \\ (-1)^{\lfloor nq/p \rfloor} = -1, & 7c + 4 \leq n \leq 7c + 6, \end{cases} \tag{5.1}$$

Recall that $q = 6k' + 1$ and $p = 21k' + 3$. For $7c \le n \le 7c + 3$ the following calculation yields:

$$\begin{aligned} 2cp &= 2c(21k' + 3) \le 2c(21k' + 3) + c = 7c(6k' + 1) \\ &\le n(6k' + 1) = nq \\ &\le (7c + 3)(6k' + 1) = 2c(21k' + 3) + (18k' + c + 3) \\ &< 2c(21k' + 3) + (21k' + 3) = (2c + 1)p, \end{aligned}$$

i.e. $2cp \le nq < (2c + 1)p$, which means

$$2c = \left\lfloor \frac{2cp}{p} \right\rfloor \le \left\lfloor \frac{nq}{p} \right\rfloor < \left\lfloor \frac{(2c+1)p}{p} \right\rfloor = 2c + 1,$$

so $\lfloor nq/p \rfloor = 2c$.

When $7c + 4 \le n \le 7c + 6$ we compute

$$\begin{aligned} (2c + 1)p &= (2c + 1)(21k' + 3) \le (2c + 1)(21k' + 3) + (3k' + c + 1) = (7c + 4)(6k' + 1) \\ &\le n(6k' + 1) = nq \\ &\le (7c + 6)(6k' + 1) = (2c + 1)(21k' + 3) + (15k' + c + 3) \\ &< (2c + 1)(21k' + 3) + (21k' + 3) = (2c + 2)p, \end{aligned}$$

so

$$2c + 1 = \left\lfloor \frac{(2c+1)p}{p} \right\rfloor \le \left\lfloor \frac{nq}{p} \right\rfloor < \left\lfloor \frac{(2c+2)p}{p} \right\rfloor = 2c + 2.$$

For the convenience of this proof, we define

$$w' := bw = b^{\epsilon_0} a^{\epsilon_1} b^{\epsilon_2} \cdots b^{\epsilon_{p-1}}$$

where $\epsilon_0 = 1 = (-1)^{\lfloor 0/p \rfloor}$. By Equation (5.1), for $0 \le i < 7$, we have $b^{\epsilon_0} a^{\epsilon_1} b^{\epsilon_2} \cdots b^{\epsilon_6} = babab^{-1}a^{-1}b^{-1}$, and for $7 \le i < 14$, we have $b^{\epsilon_7} a^{\epsilon_8} b^{\epsilon_9} \cdots b^{\epsilon_{13}} = ababa^{-1}b^{-1}a^{-1}$. Because a and b alternate in w', thus for $i \in \mathbb{Z}_{\ge 0}$ such that $14i + 13 < p$, we have

$$b^{\epsilon_{14i}} a^{\epsilon_{14i+1}} \cdots a^{\epsilon_{14i+13}} = b^{\epsilon_0} a^{\epsilon_1} \cdots a^{\epsilon_{13}} = babab^{-1}a^{-1}b^{-1}ababa^{-1}b^{-1}a^{-1} = S_1.$$

Moreover, the length of w' is the length of w plus one, that is w' has length p, and the length of S_1 is 14. Therefore, there are $p = 21k' + 3 = 42k + 3 \equiv 3 \pmod{14}$ terms at the end of w' that are not included in $b^{\epsilon_{14i}} a^{\epsilon_{14i+1}} \cdots a^{\epsilon_{14i+13}}$ for any i with $14i + 13 < p$. Again by Equation (5.1) and the fact that a, b alternate in w', they are

$$b^{\epsilon_{42k}} a^{\epsilon_{42k+1}} b^{\epsilon_{42k+2}} = b^{\epsilon_0} a^{\epsilon_1} b^{\epsilon_2} = bab = S_2.$$

We also know the first $42k$ terms are powers of S_1, and by the length of S_1 they must be S_1^{3k}. Therefore, $w = b^{-1}w' = b^{-1}S_1^{3k}S_2$. □

Lemma 5.2. *For matrices $A, B, C \in SL_2(\mathbb{C})$ and $k \in \mathbb{Z}$, we have*

$$\operatorname{tr}(AB^kC) = \operatorname{tr}(B)\operatorname{tr}(AB^{k-1}C) - \operatorname{tr}(AB^{k-2}C).$$

Proof. This follows from the following two equations:

$$\operatorname{tr}(AB) = \operatorname{tr}(A)\operatorname{tr}(B) - \operatorname{tr}(AB^{-1}),$$

$$\operatorname{tr}(ABC) = \operatorname{tr}(CAB).$$

Then we calculate

$$\begin{aligned}\operatorname{tr}(AB^kC) &= \operatorname{tr}(CAB^k) = \operatorname{tr}((CAB^{k-1})B) \\ &= \operatorname{tr}(CAB^{k-1})\operatorname{tr}(B) - \operatorname{tr}(CAB^{k-1}B^{-1}) \\ &= \operatorname{tr}(AB^{k-1}C)\operatorname{tr}(B) - \operatorname{tr}(CAB^{k-2}) \\ &= \operatorname{tr}(B)\operatorname{tr}(AB^{k-1}C) - \operatorname{tr}(AB^{k-2}C).\end{aligned}$$

□

Theorem 5.3. *Under the setting of Theorem 1.1, if r is of the form $[3, 2, 3k']$, then the boundary slope of K_r detected by (x_0, y_0) is $6k' + 6$.*

Proof. First note that for q/p to be a knot, by the same argument as in Lemma 5.1, we must have $k' = 2k \in 2\mathbb{Z}$.

We show that $\operatorname{tr}(\rho(\mu^{6k'+6}\lambda)) = \operatorname{tr}(\rho(\mu^{12k+6}\lambda)) = -2 \in \mathcal{O}_{\mathfrak{K}}$. Then by Theorem 2.35, this implies $\mu^{6k'+6}\lambda$ is the boundary slope of the essential surface detected by this action.

According to Theorem 2.8, the meridian is $\mu = a$, and the longitude is $\lambda = w^*wa^{-2e(w)}$, where $e(w)$ is the sum of exponents in w.

We denote ρ_k and w_k for $q/p = [3, 2, 6k]$. From Lemma 5.1 we have $w_k = b^{-1}S_1^{3k}S_2$, and the sum of exponents in $S_1 = babab^{-1}a^{-1}b^{-1}ababa^1b^{-1}a^{-1}$ is 2, and in $S_2 = bab$ is 3. This means the sum of exponents in $w_k = b^{-1}S_1^{3k}S_2$ is $e(w_k) = -1 + 2\cdot 3k + 3 = 6k + 2$. Therefore, we have

$$\mu^{12k+6}\lambda = a^{12k+6}w_k^*w_ka^{-2e(w_k)} = a^{12k+6}[S_2^*(S_1^*)^{3k}b^{-1}][b^{-1}(S_1)^{3k}S_2]a^{-12k-4}.$$

Hence

$$\begin{aligned}\operatorname{tr}(\rho_k(\mu^{12k+6}\lambda)) &= \operatorname{tr}(\rho_k(a^{12k+6}S_2^*(S_1^*)^{3k}b^{-2}(S_1)^{3k}S_2a^{-12k-4})) \\ &= \operatorname{tr}(\rho_k(a^{12k+6})\rho_k(S_2^*(S_1^*)^{3k}b^{-2}(S_1)^{3k}S_2a^{-12k-4})) \\ &= \operatorname{tr}(\rho_k(S_2^*(S_1^*)^{3k}b^{-2}(S_1)^{3k}S_2a^{-12k-4})\rho_k(a^{12k+6})) \\ &= \operatorname{tr}(\rho_k(S_2^*(S_1^*)^{3k}b^{-2}(S_1)^{3k}S_2a^{-12k-4}a^{12k+6})) \\ &= \operatorname{tr}(\rho_k(S_2^*(S_1^*)^{3k}b^{-2}(S_1)^{3k}S_2a^2)).\end{aligned}$$

Let (x_k, y_k) be an intersection point of the irreducible component of the trefoil knot and the canonical component of the knot $[3, 2, 6k]$. Since the irreducible component of the trefoil knot is $(y - x^2 + 1)$, we must have $y_k - x_k^2 + 1 = 0$, $y_k = x_k^2 - 1$. Because the corresponding representation is given by

$$\rho_k(a) = \begin{bmatrix}\alpha_k & 1 \\ 0 & \frac{1}{\alpha_k}\end{bmatrix}, \quad \rho_k(b) = \begin{bmatrix}\alpha_k & 0 \\ t_k & \frac{1}{\alpha_k}\end{bmatrix}$$

where $\alpha_k + 1/\alpha_k = x_k$ and $y_k = 2 - t_k$ by Lemma 2.11 and Lemma 2.14, α_k, t_k satisfies the relation

$$t_k = 2 - y_k = 2 - (x_k^2 - 1) = 2 - ((\alpha_k + 1/\alpha_k)^2 - 1) = 1 - \alpha_k^2 - 1/\alpha_k^2,$$

so we may write

$$\rho_k(b) = \begin{bmatrix} \alpha_k & 0 \\ 1 - \alpha_k^2 - 1/\alpha_k^2 & \frac{1}{\alpha_k} \end{bmatrix}.$$

With this substitution, we can write ρ for ρ_k and α, t for α_k, t_k instead of assigning them specific values depending on k. Denote $M(k_1, k_2) = \mathrm{tr}(\rho(S_2^*(S_1^*)^{3k_1} b^{-2} (S_1)^{3k_2} S_2 a^2))$, then by Lemma 5.2, we have the following relations:

$$\begin{aligned} M(k,k) &= \mathrm{tr}(\rho(S_2^*(S_1^*)^{3k} b^{-2} (S_1)^{3k} S_2 a^2)) \\ &= \mathrm{tr}(\rho(S_2^*(S_1^*)^{3(k-1)} b^{-2} (S_1)^{3k} S_2 a^2))\,\mathrm{tr}(\rho(S_1^*)^3) - \mathrm{tr}(\rho(S_2^*(S_1^*)^{3(k-2)} b^{-2} (S_1)^{3k} S_2 a^2)) \\ &= \mathrm{tr}(\rho(S_1^*)^3) M(k-1,k) - M(k-2,k), \end{aligned}$$

and similarly:

$$\begin{aligned} M(k-1,k) &= \mathrm{tr}(\rho(S_1)^3) M(k-1,k-1) - M(k-1,k-2); \\ M(k-2,k) &= \mathrm{tr}(\rho(S_1)^3) M(k-2,k-1) - M(k-2,k-2); \\ M(k,k-1) &= \mathrm{tr}(\rho(S_1^*)^3) M(k-1,k-1) - M(k-2,k-1). \end{aligned}$$

In SageMath, we can compute that $\mathrm{tr}(\rho(S_1)^3) = \mathrm{tr}(\rho(S_1^*)^3) = -2$. Therefore we claim that $M(k,k) = -2$ and $M(k-1,k) = M(k,k-1) = 2$.

We prove $\mathrm{tr}(\rho(\mu^{12k+6}\lambda)) = -2$ inductively. For the base case, we compute in SageMath:

$$M(0,0) = M(1,1) = \mathrm{tr}(-I) = -2;$$

$$M(0,1) = M(1,0) = \mathrm{tr}(I) = 2.$$

Suppose our claim holds for values less than k. Then we have

$$\begin{aligned} M(k-1,k) &= -2M(k-1,k-1) - M(k-1,k-2) = -2 \cdot (-2) - 2 = 2; \\ M(k-2,k) &= -2M(k-2,k-1) - M(k-2,k-2) = -2 \cdot 2 - (-2) = -2; \\ M(k,k-1) &= -2M(k-1,k-1) - M(k-2,k-1) = -2 \cdot (-2) - 2 = 2; \\ M(k,k) &= -2M(k-1,k) - M(k-2,k) = -2 \cdot 2 - (-2) = -2. \end{aligned}$$

Therefore, we concluded that

$$\mathrm{tr}(\rho(\mu^{6k'+6}\lambda)) = \mathrm{tr}(\rho(\mu^{12k+6}\lambda)) = \mathrm{tr}(\rho(S_2^*(S_1^*)^{3k} b^{-2} (S_1)^{3k} S_2 a^2)) = M(k,k) = -2,$$

and by Theorem 2.35 this means $\mu^{6k'+6}\lambda$ is the boundary slope of the essential surface detected by this action. □

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Appendix A. Data for minimal polynomials

The following table records information about the character varieties of some two bridge knots whose knot groups surject onto the trefoil knot. The table records intersection points between 1) the irreducible component that is shared with the trefoil and 2) each other irreducible component in the character variety. The polynomial in "product of min. poly." is the product of minimal polynomials of each intersection point.

knot	# irr. comp.	intersection points	product of min. poly.
$(27, 5)$	1	$\pm 3/2$	$4x^2 - 9$
$(33, 5)$	1	$\pm\sqrt{11}/2$	$4x^2 - 11$
$(39, 7)$	1	$\pm\sqrt{13}/2$	$4x^2 - 13$
$(45, 7)$	1	$\pm\sqrt{15}/2$	$4x^2 - 15$
$(45, 19)$	2	$\pm\sqrt{6}/2, \pm\sqrt{10}/2$	$4x^4 - 16x^2 + 15$
$(69, 19)$	1	$\pm\sqrt{\pm\sqrt{2}/2 + 5/2}$	$4x^4 - 20x^2 + 23$
$(75, 29)$	2	$\pm\sqrt{10}/2, \pm\sqrt{10}/2$	$4x^2 - 20x^2 + 25$
$(99, 29)$	1	$\pm\sqrt{\pm\sqrt{3}/2 + 3}$	$4x^4 - 24x^2 + 33$
$(105, 29)$	2	$\pm\sqrt{10}/2, \pm\sqrt{14}/2$	$4x^4 - 24x^2 + 35$
$(105, 41)$	2	$\pm\sqrt{14}/2, \pm\sqrt{10}/2$	$4x^4 - 24x^2 + 35$
$(111, 31)$	1	$\pm\sqrt{\pm\sqrt{-1}/2 + 3}$	$4x^4 - 24x^2 + 37$
$(141, 41)$	1	$\pm\sqrt{\pm\sqrt{2}/2 + 7/2}$	$4x^4 - 28x^2 + 47$
$(147, 41)$	2	$\pm\sqrt{14}/2, \pm\sqrt{14}/2$	$4x^4 - 28x^2 + 49$

Table A.1. Epimorphisms onto $(3, 1)$

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