# (K)not detecting boundary slopes via intersections in the character variety arising from epimorphisms

Isidora Bailly-Hall, Karina Dovgodko, Akash Ganguly, Jiachen Kang, and Jishi Sun

Twin Cities REU in Algebra and Combinatorics

August 4, 2023

## Definition

A *knot* $K$ is an embedding of $S^1$ into $S^3$.

The *knot complement* of $K$ is the 3-manifold $M(K) = S^3 \setminus N(K)$.

The *knot group* of $K$ is $\Gamma_K = \pi_1(M(K))$.

## Goal

To find essential surfaces in the complement of two-bridge knots.

# A Bird's Eye View

Two-bridge knots

↓

Character varieties

↓ epimorphisms

Intersection points

↓ Farey recursion

Algebraic non-integral representations

↓ $SL_2$-trees

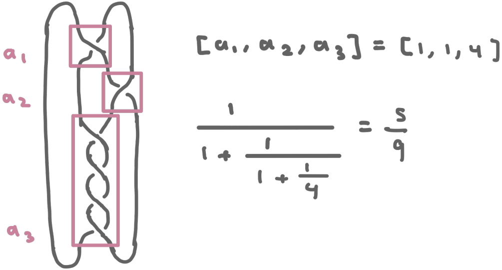
Detected essential surfaces

# Two-Bridge Knots

## Definition

A *two-bridge knot* is a knot with diagram having two local maxima.

- Every two-bridge knot can be associated to a reduced fraction $q/p \in (0,1)$ with $p$, $q$ both odd, called its *two-bridge normal form*.
- $q/p$ is given by the continued fraction expansion $[a_1, \dots, a_k] = q/p$



# Presentation of Two-Bridge Knot Groups

## Theorem (Maylands, 1974)

*Given a two-bridge knot $K = (p, q)$, $\Gamma_{q/p}$ has the following canonical presentation:*

$$\Gamma_{q/p} = \langle a, b \mid wa = bw \rangle$$

*where $w$ is determined by $p$ and $q$, and $a$ and $b$ are conjugate.*

## Example

For $q/p = [1, 1, 4] = 5/9$ we have

$$w = ab^{-1}a^{-1}bab^{-1}a^{-1}b$$

with

$$\Gamma_{q/p} = \langle a, b \mid ab^{-1}a^{-1}bab^{-1}a^{-1}ba = bab^{-1}a^{-1}bab^{-1}a^{-1}b \rangle$$

# Representations of Two-Bridge Knot Groups

## Corollary

*Every irreducible representation* $\ : \Gamma_{q=p}$ *!* $\mathrm{SL}_2(\mathbf{C})$ *is determined by* $\ (a)$ *and* $\ (b)$*, which (up to conjugation) has the form*

$$ (a) = \begin{bmatrix} & 1 \\ 0 & 1= \end{bmatrix} \quad \text{and} \quad (b) = \begin{bmatrix} & 0 \\ t & 1= \end{bmatrix}$$

*Therefore every representation* *of* $\Gamma_{q=p}$ *corresponds to a point* $( \ ; t)$ *2* $\mathbf{C}^2$ *that satis es* $\ (wa) = \ (bw)$*.*

We can rewrite the polynomial relation $(wa) = \ (bw)$ in terms of the ***traces*** of $(a)$ and $(ab^{\ 1})$: we define

$$x := \mathrm{tr}(\ (a)) = \quad + 1=$$

$$y := \mathrm{tr}(\ (ab^{\ 1})) = 2 \quad t$$

### De nition

The algebraic set $X(\Gamma_{q=p})$ in $\mathbb{C}^2$ defined by this polynomial in $x$ and $y$ is called the ***character variety*** of $\Gamma_{q=p}$.

### Example

The defining polynomial of $X(\Gamma_{1=3})$ is $x^2 \quad y \quad 1 = 0$.

# Epimorphisms onto the trefoil knot

## Definition

The rational number

$$q/p = [\underbrace{3,2,\ldots,3,2}_{n \text{ many } 2\text{'s}},3k]$$

is the two-bridge normal form of a knot whenever $n+k$ is odd. We denote this knot by $K(n,k)$.

## Theorem (Ohtsuki-Riley-Sakuma, 2008)

*For all $n,k > 0$ there exists an epimorphism*

$$\Gamma_{K(n,k)} \to \Gamma_{1/3}$$

*where $\Gamma_{1/3}$ is the knot group of the trefoil knot.*

# Intersection Points

Given an epimorphism $\Gamma_{K(n;k)} \quad \Gamma_{1=3}$, every representation $\Gamma_{1=3}$ ! $SL_2(\mathsf{C})$ will induce a representation $\Gamma_{K(n;k)}$ ! $SL_2(\mathsf{C})$. This implies the following:

## Corollary

$X(K(n;k))$ always contain an irreducible component $x^2 \quad y \quad 1 = 0$, which corresponds to $X(\Gamma_{1=3})$.

## Goal

To describe the intersection points between $x^2 \quad y \quad 1 = 0$ and other components of $X(K(n;k))$.

However this is **HARD!**

# Horri c Example

Character variety of $K(1; 2) = [3; 2; 6]$

# Horri c Example

Character variety of $K(1; 2) = [3; 2; 6]$

### Moral of the story

This sucks. New approach needed.

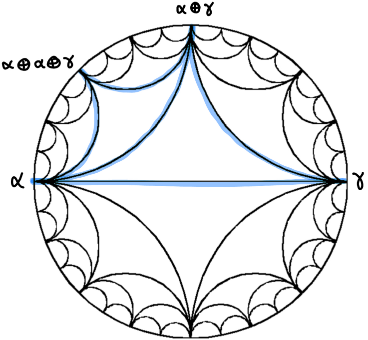
# Farey Recursion

## Definition

For any $p/q,\ r/s \in \hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$, we call them a ***Farey pair*** if $ps - qr = \pm 1$;

For any Farey pair $(p/q, r/s)$, we define their ***Farey sum*** to be $\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}$.

This operation has a geometric explanation on the ***Farey graph***:



# Farey Recursion

## De nition

Let $R$ be any commutative ring. A function $F : \hat{\mathbb{Q}} \to R$ is called a ***Farey recursive function*** if for every Farey pair $( \ ; \ )$ we have

$$F(\qquad\qquad) = \quad F(\ ) + F(\ )F(\qquad)$$

## Cool stu ! (Chesebro 2019)

The defining polynomial of $X(\Gamma_{K(n;k)})$ can be generated recursively using Farey recursion.

# Farey Recursion

## Example

If we substitute $y = x^2 \quad 1$ into the rest of the defining polynomial of $K(n; k)$, we get a polynomial $\tilde{p}(x)$ that describes the intersection points:

| Knot | $\tilde{p}(x)$ |
|---|---|
| $K(1; 2)$ | $4x^2 \quad 15$ |
| $K(1; 4)$ | $8x^2 \quad 29$ |
| $K(1; 6)$ | $12x^2 \quad 43$ |
| $K(2; 1)$ | $4x^4 \quad 32x^2 + 63$ |
| $K(2; 3)$ | $12x^4 \quad 92x^2 + 173$ |
| $K(2; 5)$ | $20x^4 \quad 152x^2 + 283$ |

## Upshot

Using Farey recursion, we found a general formula for $\tilde{p}(x)$; it follows that for all $K(n; k)$, all coefficients of $\tilde{p}(x)$ but the constant term are even.

## $P$-adic valuation

### Definition

Let $F$ be a number field, and let $O_F$ denote the ring of integers of $F$. Let $P$ be a prime ideal of $O_F$.
A discrete valuation $v_P$ on $F$ as follows:

- For any $x \in O_F$, let $v_P(x) = \max\{n \in \mathbb{Z}_{\geq 0} : x \in P^n\}$;
- For $x \in F \setminus O_F$, write $x = a/b$ where $a, b \in O_F$, and define $v_P(x) = v_P(a) - v_P(b)$.

The discrete valuation $v_P$ is called the $P$-adic valuation on $F$.

### Example

For $F = \mathbb{Q}$ we have $O_F = \mathbb{Z}$ consider $P = 2\mathbb{Z}$, then

$$v_2(2) = 1; \; v_2\left(\frac{4}{5}\right) = 2; \; v_2(5) = 0; \; v_2\left(\frac{1}{2}\right) = -1$$

# Algebraic non-integral representations

## Definition

Let $\rho : \Gamma_K \to \mathrm{SL}_2(F)$ be a representation of $\Gamma_K$ where $F$ is a number field. We call $\rho$ an ***algebraic non-integral (ANI)*** representation if there exists some $\gamma \in \Gamma_K$ such that $\mathrm{tr}(\rho(\gamma))$ is not an algebraic integer. That is, there is a $P$-adic valuation $v_P$ such that $v_P(\mathrm{tr}(\rho(\gamma))) < 0$.

## Fact (Culler-Shalen, 1983)

Every ANI-representation of $\Gamma_K$ can detect essential surfaces in the knot complement of $K$ (via $\mathrm{SL}_2$-tree actions from Bass-Serre theory).

# Algebraic non-integral representations

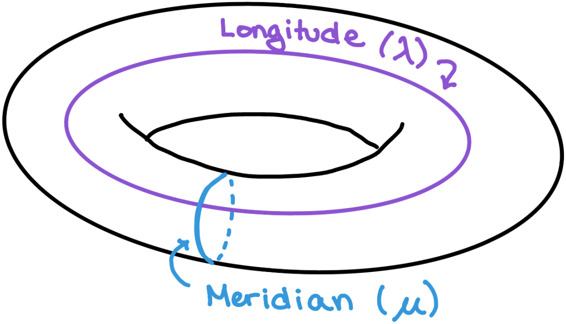
This leads to our first main theorem:

**Theorem (B-D-G-K-S, 2023+)**

*For every two-bridge knot $K(n; k)$, and every $(x_0; y_0)$ 2 $\mathbb{C}^2$ that is an intersection point between $x^2 \quad y \quad 1 = 0$ and another component of $X(\Gamma_{K(n;k)})$, every $\mathrm{SL}_2(\mathbb{C})$-representation of $\Gamma_{K(n;k)}$ corresponding to $(x_0; y_0)$ is an ANI-representation.*

In other words, every intersection point will detect essential surfaces for $K(n; k)$.

# Boundary slope




**Definition**

A *slope* of $K$ is an element $a/b \in \mathbb{Q} \cup \{\infty\}$, which corresponds to the element $\mu^a \lambda^b \in \pi_1(\partial M(K))$.

A *boundary slope* of $K$ is a slope that appears in $\partial S$ for an essential surface $S$ in $M(K)$.

# Detecting boundary slopes

Although the detected essential surfaces may not be unique, their boundary slope is unique:

**Theorem (Schanuel-Zhang, 2001)**

*Let* $: \Gamma_K$ *!* $\mathrm{SL}_2(F)$ *be an ANI-representation of* $\Gamma_K$ *with respect to a* $P$*-adic valuation* $v_P$*. Then there exists a unique boundary slope of* $K$ *such that* $v_P(\mathrm{tr}(\ (\ )))\ \ 0$*, and is the detected boundary slope.*

For a fixed two-bridge knot $K$, (Hatcher-Thurston, 1985) gives an explicit description of all the boundary slopes of $K$, so we can calculate their traces and find the unique one with integral trace.

# Detecting boundary slopes

## Proposition (B-D-G-K-S, 2023+)

The set of all boundary slopes of $K(n,k)$ is

$$\{6k+6a+10b \mid a+b \leq n\} \cup \{6a+10b \mid a+b \leq n, 0<a\} \cup \{0\}.$$

## Example

The knot $K(1,k)$ (where $k$ is even) has exactly 5 boundary slopes:

$$0, 6, 6k, 6k+6, 6k+10$$

# Detecting boundary slopes

Our second main theorem:

**Theorem (B-D-G-K-S, 2023+)**

*For $K(n;k)$, the detected boundary slope is $6n + 6k$.*
*That is, $\phantom{x}^{6(n+k)}\phantom{x}$ is a loop in the boundary corresponding to a detected essential surface.*

# Thank you!

We would like to thank our project mentor Michelle Chu and TAs Tori Braun and Lilly Webster. Also, we thank our parents for giving birth to us.

This project was partially supported by RTG grant NSF/DMS-1745638.