On triangulations of order polytopes for snake posets

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UMN Twin Cities REU Final Presentation

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Triangulations

Definition

Given a point configuration $\mathcal{A} \subseteq \mathbb{R}^d$ with convex hull $\mathrm{conv}(\mathcal{A})$, a *triangulation* of $\mathcal{A}$ is a collection $\mathcal{T}$ of d-simplices such that

1. The union of all simplices in $\mathcal{T}$ is $\mathrm{conv}(\mathcal{A})$.
2. The collection $\mathcal{T}$ forms a geometric simplicial complex.
3. Every simplex in $\mathcal{T}$ has vertices only from $\mathcal{A}$.

We say that $\mathcal{T}$ is *regular* if there are heights $h_1, \ldots, h_d \in \mathbb{R}$ such that the projection of the upper convex hull of $\hat{A} = \left\{ \begin{bmatrix} a_1 \\ h_1 \end{bmatrix}, \ldots, \begin{bmatrix} a_d \\ h_d \end{bmatrix} \right\} \subseteq \mathbb{R}^{d+1}$ to $\mathbb{R}^d$ is $\mathcal{T}$.

The flip property

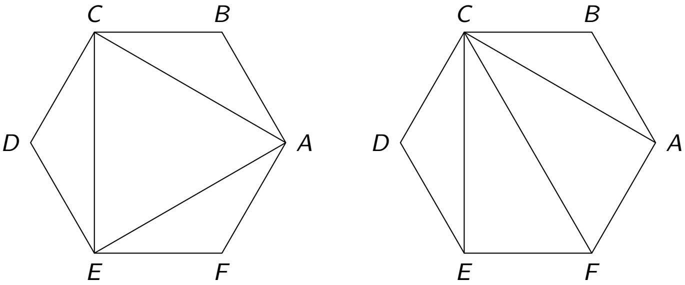


Figure: These two triangulations differ by a flip.

Example

We say the set of vertices $\{A, C, E, F\}$ is a *circuit*: in particular, it is minimally dependent and has precisely two triangulations.

Generalized snake posets

- Generalized snake posets take the form $P(\mathbf{w})$ where $\mathbf{w}$ is a string that starts with ε and consists of R's and L's thereafter.
- We let $Q_{\mathbf{w}} = \mathrm{Irr}_{\wedge} P(\mathbf{w})$, the poset of meet-irreducibles.

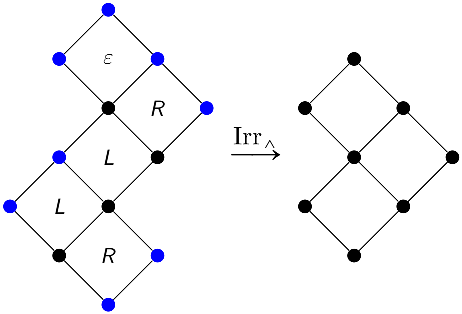
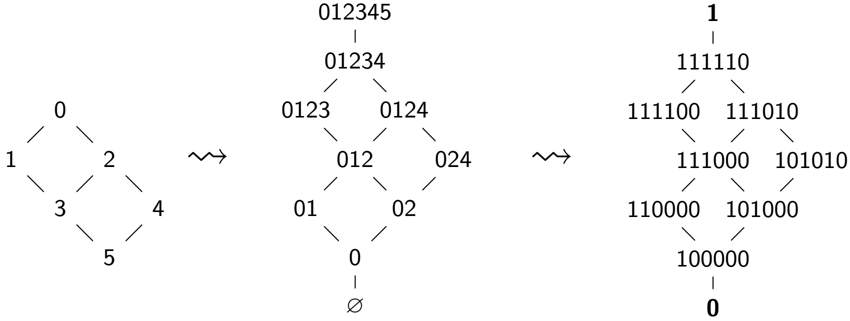


Figure: The posets $P(\varepsilon RLLR)$ and $Q_{\varepsilon RLLR} = S_2$

- If $\mathbf{w}$ avoids RLR and LRL, we say $\mathbf{w} \in \mathcal{V}$. From this point onwards, we assume all $\mathbf{w} \in \mathcal{V}$.

The poset of order filters

An order filter of $Q_{\mathbf{w}}$ is a subset of elements closed under going upwards. We denote by $J(Q_{\mathbf{w}}) \cong \hat{P}(\mathbf{w})$ the poset of order filters ordered by inclusion.



$Q_{\mathbf{w}} = S_1$ $J(S_1)$

Triangulations of the order polytope

Definition

The order polytope $\mathcal{O}(Q_{\mathbf{w}})$ is $\mathrm{conv}(\{\mathbf{v}_A : A \in J(Q_{\mathbf{w}})\})$.

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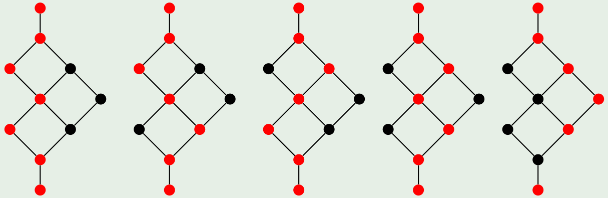
Example

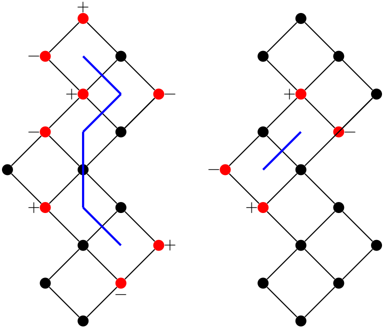
The five maximal simplices of the canonical triangulation of the order polytope of $\mathcal{O}(S_1)$:



Circuits

Theorem ([Bel+22, Theorem 4.5])

There exists a bijection between a certain class of connected induced subgraphs and circuits of $\mathcal{O}(Q_{\mathbf{w}})$.



We follow the sign convention of [Bel+22] where the top vertex is $+$.

The secondary polytope

Theorem ([GKZ08, §7, Theorem 1.7])

Let $\mathcal{A}$ be a point configuration. Then, there exists a polytope called the *secondary polytope*, $\Sigma_{\mathcal{A}}$, such that

- Vertices of $\Sigma_{\mathcal{A}} \longleftrightarrow$ regular triangulations of $\mathcal{A}$
- Edges $\longleftrightarrow$ flips between them

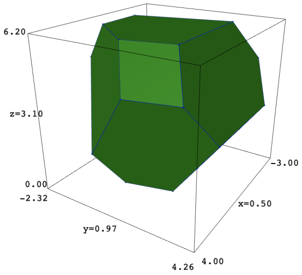


Figure: The secondary polytope of $\mathcal{O}(S_1)$.

Conjectures

Conjecture ([Bel+22], Conjecture 6.1)

The flip graph of regular triangulations for $\mathcal{O}(Q_{\mathbf{w}})$ is k-regular, where k is the dimension of the secondary polytope of $\mathcal{O}(Q_{\mathbf{w}})$.

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All triangulations of $\mathcal{O}(Q_{\mathbf{w}})$ are regular.

Conjecture ([Bel+22], Conjecture 6.5)

The number of regular triangulations of $\mathcal{O}(S_k)$ is $2^{k+1} \cdot \mathrm{Cat}(2k+1)$.

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Triangulating circuits

Recall that a circuit $Z = (Z_+, Z_-)$ is a minimal dependent set.

Triangulating circuits

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Lemma

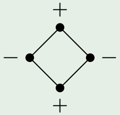
There are two triangulations of a circuit Z:

$$\mathcal{T}_Z^+ = \{Z \backslash \{z\} : z \in Z_+\}, \quad \mathcal{T}_Z^- = \{Z \backslash \{z\} : z \in Z_-\}$$

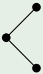
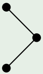
Triangulating a square

Example

Consider the following square circuit

$Z =$  .

Then:

$\mathcal{T}_Z^+ = \{$  ,  $\}$, $\mathcal{T}_Z^- = \{$  ,  $\}$

Partial ordering of triangulations

Definition

Given two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of $\mathcal{O}(Q_{\mathbf{w}})$, we say $\mathcal{T} \lessdot_{\mathcal{O}} \mathcal{T}'$ if there exists a flip between $\mathcal{T}$ and $\mathcal{T}'$ at some circuit Z such that $\mathcal{T}$ contains the triangulation $\mathcal{T}_Z^-$ and $\mathcal{T}'$ contains $\mathcal{T}_Z^+$.

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- We say $\mathcal{T} \leqslant_{\mathcal{O}} \mathcal{T}'$ if there is a sequence of negative to positive flips from $\mathcal{T}$ to $\mathcal{T}'$.
- Covering relations correspond to edges of the flip graph.

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Theorem [BCJTT24+]

This relation is a partial ordering of the triangulations of $\mathcal{O}(Q_{\mathbf{w}})$.

Partial ordering of triangulations

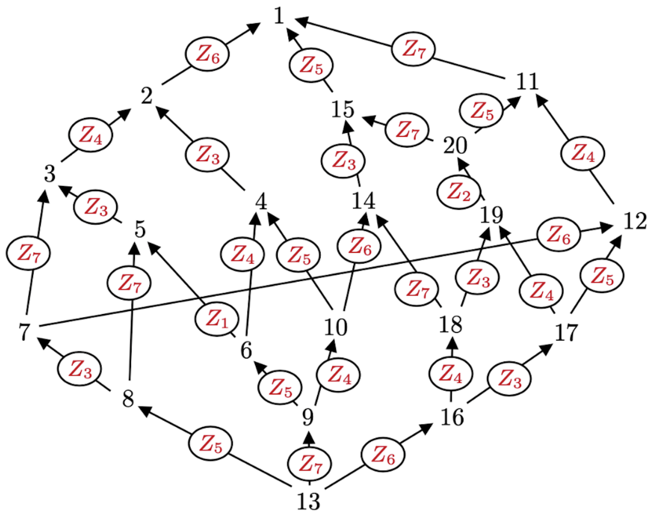
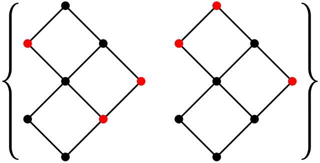
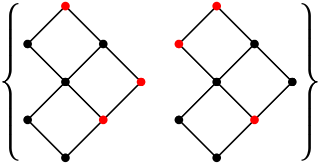


Figure: Partial order on triangulations of $\mathcal{O}(S_1)$.

Partial ordering of triangulations

Pay attention to the highest vertex in a circuit Z: it always appears in every simplex of $\mathcal{T}_Z^-$, but is missing from one simplex of $\mathcal{T}_Z^+$

$\mathcal{T}_{Z_3}^+ =$ 

$\mathcal{T}_{Z_3}^- =$ 

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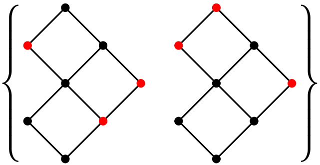

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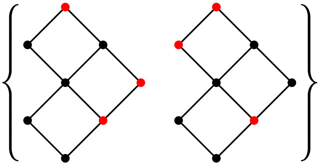

- Implies that if $\mathcal{T} \leqslant_{\mathcal{O}} \mathcal{T}'$ for $\mathcal{T}' \neq \mathcal{T}$, we cannot also have $\mathcal{T}' \leqslant_{\mathcal{O}} \mathcal{T}$

Partial ordering of triangulations

- A partial ordering is called a *lattice* if every pair of elements has a unique least upper bound and a unique greatest lower bound

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Conjecture

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- An acyclic orientation of the 1-skeleton of a polytope is said to have *good orientation* if the graph of every nonempty face of the polytope has exactly one sink (vertex with all edges oriented inwards)

Partial ordering of triangulations

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- An acyclic orientation of the 1-skeleton of a polytope is said to have *good orientation* if the graph of every nonempty face of the polytope has exactly one sink (vertex with all edges oriented inwards)

Conjecture

The partial ordering we have defined is a good orientation of the 1-skeleton of the secondary polytope.

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Quadrilaterals, pentagons, and hexagons

Conjecture

The 2-dimensional faces of $\Sigma_{\mathcal{O}(S_k)}$ are quadrilaterals, pentagons and hexagons.

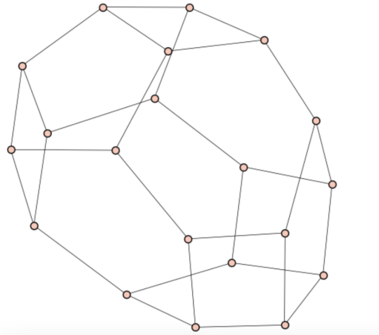


Figure: $\Sigma_{\mathcal{O}(S_1)}$.

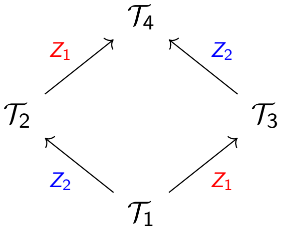
Commuting circuits

Theorem [BCJTT24+]

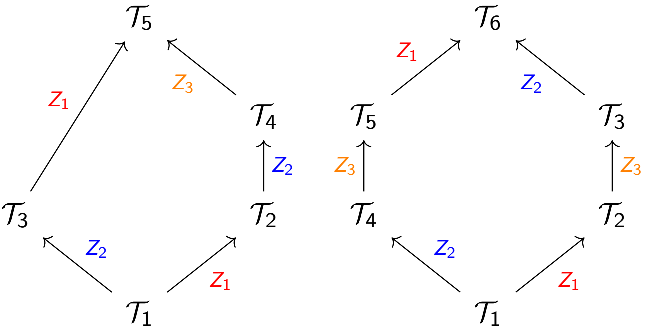
Let Z_1 and Z_2 be circuits of $\mathcal{O}(Q_{\mathbf{w}})$.Then Z_1 and Z_2 commute at $\mathcal{T}$ if and only if $\mathcal{T}$ can be flipped at Z_1 and Z_2 and at least one of the following hold:

(i) Z_1 and Z_2 appear on different maximal simplices in $\mathcal{T}$, or

(ii) Z_1 and Z_2 share no vertex



Pentagons and hexagons



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Valence k-regularity of the flip graph

Conjecture ([Bel+22], Conjecture 6.1)

The flip graph of regular triangulations for $\mathcal{O}(Q_{\mathbf{w}})$ is k-regular, where k is the dimension of the secondary polytope of $\mathcal{O}(Q_{\mathbf{w}})$.

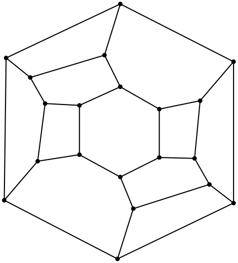

Figure: Flip graph for $\mathcal{O}(S_1)$, where each vertex has 3 edges

Valence-regularity for subset of triangulations

Theorem [BCJTT24+]

Let $\mathcal{T}$ be a triangulation of $\mathcal{O}(Q_{\mathbf{w}})$ obtained by applying one flip to the canonical triangulation $\mathcal{T}_{\mathbf{w}}$, where $\mathbf{w} = w_0 w_1 \cdots w_n \in \mathcal{V}$. Then, $\mathcal{T}$ admits $n+1$ flips.

Proof strategy

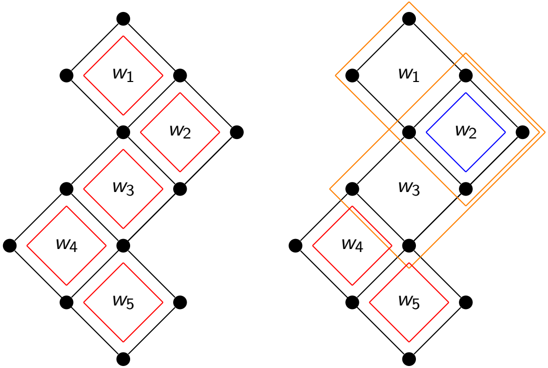

Figure: Circuits flippable at the canonical triangulation $\mathcal{T}_{\mathbf{w}}$ and $f_{\mathrm{Sq}(w_2)}(\mathcal{T}_{\mathbf{w}})$

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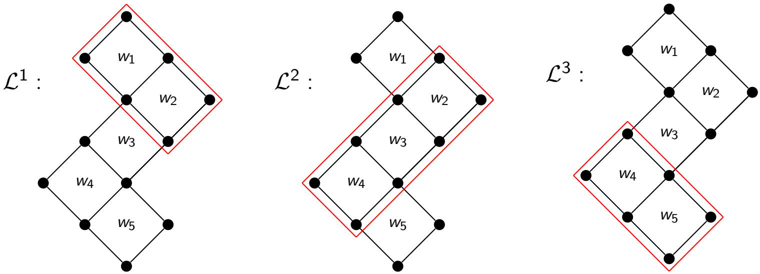
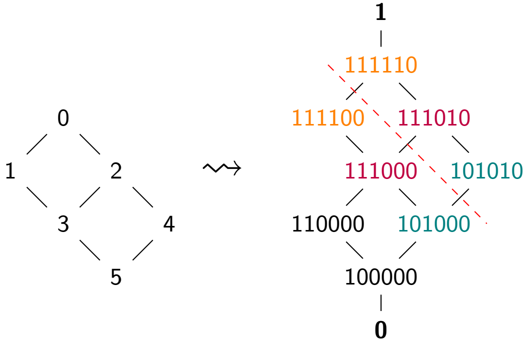
Ladders on $\hat{P}(w)$

Ladders are the diagonal lines of squares in $\hat{P}(\mathbf{w})$.



Elementary twists

Consider $Q_w = S_1$ and $J(S_1)$ below and the ladder $\mathcal{L}^1$.



The twist group

Generally, elementary twists τ_i act on $\hat{P}(\mathbf{w})$ by swapping along ladder $\mathcal{L}^i$.

Definition

For $w \in \mathcal{V}$, the twist group $\mathfrak{T}(w) \leqslant \mathfrak{S}_{|P(w)|}$ is defined by:

$$\mathfrak{T}(w) = \langle \tau_i \mid \tau_i^2 = 1, \tau_i\tau_j = \tau_j\tau_i \rangle \cong (\mathbb{Z}/2\mathbb{Z})^d,$$

where d is the number of ladders.

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Observe that $\mathfrak{T}(w)$ acts on the vertices of $\mathcal{O}(Q_w)$ by:

$$\tau \cdot \mathbf{v}_A = \mathbf{v}_{\tau \cdot A}$$

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Observe that $\mathfrak{T}(w)$ acts on the vertices of $\mathcal{O}(Q_w)$ by:

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Theorem (von Bell et al.)

$\mathfrak{T}(\mathbf{w})$ acts on regular triangulations of $\mathcal{O}(Q_{\mathbf{w}})$.

Freeness: motivation

We recall Conjecture 6.5 by von Bell et al.

Conjecture 6.5

The number of regular triangulations of $\mathcal{O}(S_k)$ is $2^{k+1} \cdot \mathrm{Cat}(2k+1)$.

Freeness: motivation

We recall Conjecture 6.5 by von Bell et al.

Conjecture 6.5

The number of regular triangulations of $\mathcal{O}(S_k)$ is $2^{k+1} \cdot \mathrm{Cat}(2k+1)$.

- The twist group for S_k has size 2^{k+1}
- Conjecturally, 2^{k+1} divides the number of triangulations

Freeness of the twist action

Proposition Theorem [BCJTT24+]

Let $Q_{\mathbf{w}}$ be a generalized snake poset and let $\mathcal{T}_{\mathbf{w}}$ be its canonical triangulation. For any twist $\tau \in \mathfrak{T}(\mathbf{w})$, we have $\tau(\mathcal{T}_{\mathbf{w}}) \neq \mathcal{T}_{\mathbf{w}}$

Freeness of the twist action

Proposition Theorem [BCJTT24+]

Let $Q_{\mathbf{w}}$ be a generalized snake poset and let $\mathcal{T}_{\mathbf{w}}$ be its canonical triangulation. For any twist $\tau \in \mathfrak{T}(\mathbf{w})$, we have $\tau(\mathcal{T}_{\mathbf{w}}) \neq \mathcal{T}_{\mathbf{w}}$

With the proposition above, we were able to show the following theorem.

Theorem [BCJTT24+]

The twist group acts freely on regular triangulations of $\mathcal{O}(Q_{\mathbf{w}})$

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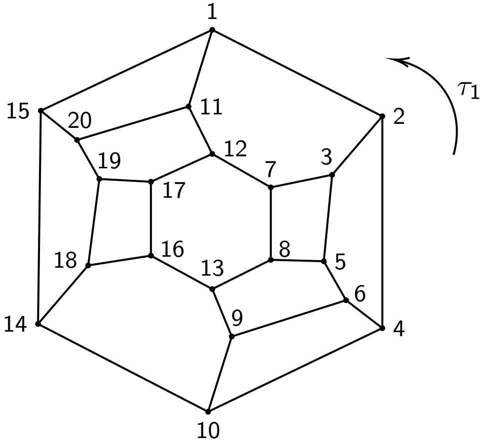
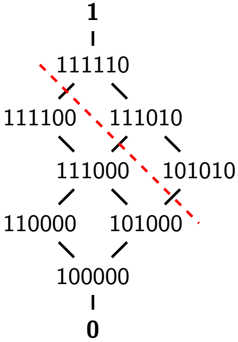
Twist Eigenbasis

Remember: the twist group acts on the set of regular triangulations

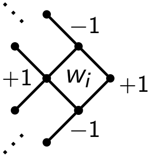




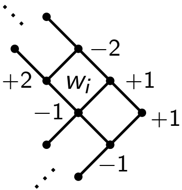
On $\Sigma_{\mathcal{O}(S_1)}$ the elementary twists are 180° rotations

Twist eigenbasis

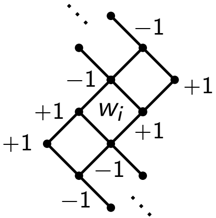
- We can define vectors in the space containing $\Sigma_{\mathcal{O}(Q_{\mathbf{w}})}$ by giving a coefficient to each vertex of $\hat{P}(\mathbf{w})$
- We want a subspace $V \cong \mathbb{R}^{n+1}$ in which the twists are linear
- We define one basis element v_i for each square w_i in $\hat{P}(\mathbf{w})$:



corner or not next to any corners



next to 1 corner

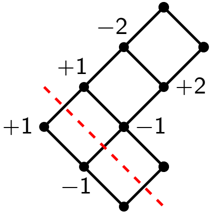


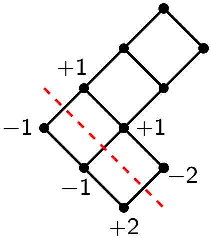
between 2 corners

Twist eigenbasis

Theorem [BCJTT24+]

Let $\mathbf{w} = \varepsilon w_1 \cdots w_n$, and let $V \cong \mathbb{R}^{n+1}$ be the linear subspace of $\mathbb{R}^{2n+6}$ parallel to the subspace containing $\Sigma_{\mathcal{O}(Q_\mathbf{w})}$. $v_1, v_2, \ldots, v_{n+1}$ is an eigenbasis of V and each elementary twist τ_i negates exactly the basis elements that correspond to the w_i in the ladder τ_i reflects.





Since $\tau_i \tau_i = 1$, each twist τ_i has only 1 and -1 for eigenvalues

Next steps

1. Finish proving quadrilaterals, pentagons, and hexagons conjecture.
2. Generalize valence-regularity proof.
3. Catalan numbers: bijection or recurrence relation?
4. Boundary of dual polytope: flag simplicial complex?

Acknowledgements

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Bibliography

[Bel+22] Matias von Bell et al. "Triangulations, order polytopes, and generalized snake posets". In: *Comb. Theory* 2.3 (2022), Paper No. 10, 34. ISSN: 2766-1334. DOI: 10.5070/c62359166.

[GKZ08] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Modern Birkhäuser Classics. Reprint of the 1994 edition. Birkhäuser Boston, Inc., Boston, MA, 2008, pp. x+523. ISBN: 978-0-8176-4770-4.