On triangulations of order polytopes for snake posets

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UMN Twin Cities REU Final Presentation

July 31, 2024
Table of contents

1  Background

2  A partial order on triangulations

3  Quadrilaterals, pentagons, and hexagons

4  Valence-regularity of the flip graph

5  The twist action

6  Twist eigenbasis
Table of contents

1 Background

2 A partial order on triangulations

3 Quadrilaterals, pentagons, and hexagons

4 Valence-regularity of the flip graph

5 The twist action

6 Twist eigenbasis
Definition

Given a point configuration \( \mathcal{A} \subseteq \mathbb{R}^d \) with convex hull \( \text{conv}(\mathcal{A}) \), a **triangulation** of \( \mathcal{A} \) is a collection \( \mathcal{T} \) of \( d \)-simplices such that

1. The union of all simplices in \( \mathcal{T} \) is \( \text{conv}(\mathcal{A}) \).
2. The collection \( \mathcal{T} \) forms a geometric simplicial complex.
3. Every simplex in \( \mathcal{T} \) has vertices only from \( \mathcal{A} \).

We say that \( \mathcal{T} \) is **regular** if there are heights \( h_1, \ldots, h_d \in \mathbb{R} \) such that the projection of the upper convex hull of \( \hat{\mathcal{A}} = \left\{ \begin{bmatrix} a_1 \\ h_1 \end{bmatrix}, \ldots, \begin{bmatrix} a_d \\ h_d \end{bmatrix} \right\} \subseteq \mathbb{R}^{d+1} \) to \( \mathbb{R}^d \) is \( \mathcal{T} \).
The flip property

Figure: These two triangulations differ by a flip.

Example

We say the set of vertices \{A, C, E, F\} is a circuit: in particular, it is minimally dependent and has precisely two triangulations.
Generalized snake posets

- Generalized snake posets take the form $P(w)$ where $w$ is a string that starts with $\varepsilon$ and consists of $R$’s and $L$’s thereafter.
- We let $Q_w = \text{Irr} \wedge P(w)$, the poset of meet-irreducibles.

![Diagram of posets](image)

**Figure:** The posets $P(\varepsilon RLLR)$ and $Q_{\varepsilon RLLR} = S_2$

- If $w$ avoids $RLR$ and $LRL$, we say $w \in \mathcal{V}$. From this point onwards, we assume all $w \in \mathcal{V}$. 
The poset of order filters

An order filter of $Q_w$ is a subset of elements closed under going upwards. We denote by $J(Q_w) \cong \hat{P}(w)$ the poset of order filters ordered by inclusion.

\[ Q_w = S_1 \]

\[ J(S_1) \]
Definition

The order polytope $\mathcal{O}(Q_w)$ is $\text{conv}\{v_A : A \in J(Q_w)\}$.
**Definition**

The order polytope $\mathcal{O}(Q_w)$ is $\text{conv}(\{v_A : A \in J(Q_w)\})$.

**Example**

The five maximal simplices of the canonical triangulation of the order polytope of $\mathcal{O}(S_1)$:
Theorem ([Bel+22, Theorem 4.5])

There exists a bijection between a certain class of connected induced subgraphs and circuits of $O(Q_w)$.

We follow the sign convention of [Bel+22] where the top vertex is +.
The secondary polytope

**Theorem ([GKZ08, §7, Theorem 1.7])**

Let \( \mathcal{A} \) be a point configuration. Then, there exists a polytope called the *secondary polytope*, \( \Sigma_\mathcal{A} \), such that

- Vertices of \( \Sigma_\mathcal{A} \) ↔ regular triangulations of \( \mathcal{A} \)
- Edges ↔ flips between them

**Figure:** The secondary polytope of \( \mathcal{O}(S_1) \).
Conjecture ([Bel+22], Conjecture 6.1)

The flip graph of regular triangulations for $O(Q_w)$ is $k$-regular, where $k$ is the dimension of the secondary polytope of $O(Q_w)$. 
Conjectures

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All triangulations of $O(Q_w)$ are regular.
Conjectures

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**Conjecture ([Bel+22], Conjecture 6.5)**

The number of regular triangulations of $O(S_k)$ is $2^{k+1} \cdot \text{Cat}(2k + 1)$. 
Table of contents

1 Background

2 A partial order on triangulations

3 Quadrilaterals, pentagons, and hexagons

4 Valence-regularity of the flip graph

5 The twist action

6 Twist eigenbasis
Recall that a circuit $Z = (Z_+, Z_-)$ is a minimal dependent set.
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**Lemma**

There are two triangulations of a circuit $Z$:

$$T^+_Z = \{Z \setminus \{z\} : z \in Z_+\}, \quad T^-_Z = \{Z \setminus \{z\} : z \in Z_-\}$$
Triangulating a square

Example

Consider the following square circuit

\[ Z = \begin{array}{c}
+ \\
- \\
- \\
+ 
\end{array} \]

Then:

\[ \mathcal{T}_Z^+ = \left\{ \begin{array}{c}
\begin{array}{c}

\end{array} \\
\begin{array}{c}

\end{array}
\end{array} \right\}, \quad \mathcal{T}_Z^- = \left\{ \begin{array}{c}
\begin{array}{c}

\end{array} \\
\begin{array}{c}

\end{array}
\end{array} \right\} \]
Partial ordering of triangulations

Definition

Given two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of $\mathcal{O}(Q_w)$, we say $\mathcal{T} \preceq \mathcal{T}'$ if there exists a flip between $\mathcal{T}$ and $\mathcal{T}'$ at some circuit $Z$ such that $\mathcal{T}$ contains the triangulation $\mathcal{T}_Z^-$ and $\mathcal{T}'$ contains $\mathcal{T}_Z^+$. We say $\mathcal{T} \preceq T \preceq \mathcal{T}'$ if there is a sequence of negative to positive flips from $\mathcal{T}$ to $\mathcal{T}'$. Covering relations correspond to edges of the flip graph.

Theorem [BCJTT24+] This relation is a partial ordering of the triangulations of $\mathcal{O}(Q_w)$. 

UMN Twin Cities REU
Snake Posets and Order Polytopes
July 31, 2024
15 / 39
Partial ordering of triangulations

Definition

Given two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of $\mathcal{O}(Q_w)$, we say $\mathcal{T} \prec \mathcal{T}'$ if there exists a flip between $\mathcal{T}$ and $\mathcal{T}'$ at some circuit $Z$ such that $\mathcal{T}$ contains the triangulation $\mathcal{T}^-_Z$ and $\mathcal{T}'$ contains $\mathcal{T}^+_Z$.

- We say $\mathcal{T} \preceq \mathcal{T}'$ if there is a sequence of negative to positive flips from $\mathcal{T}$ to $\mathcal{T}'$.
- Covering relations correspond to edges of the flip graph.
Definition

Given two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of $\mathcal{O}(Q_w)$, we say $\mathcal{T} \prec \mathcal{T}'$ if there exists a flip between $\mathcal{T}$ and $\mathcal{T}'$ at some circuit $Z$ such that $\mathcal{T}$ contains the triangulation $\mathcal{T}_Z^-$ and $\mathcal{T}'$ contains $\mathcal{T}_Z^+$.

- We say $\mathcal{T} \preceq \mathcal{T}'$ if there is a sequence of negative to positive flips from $\mathcal{T}$ to $\mathcal{T}'$.
- Covering relations correspond to edges of the flip graph.

Theorem [BCJTT24+]

This relation is a partial ordering of the triangulations of $\mathcal{O}(Q_w)$. 
Partial ordering of triangulations

Figure: Partial order on triangulations of $\mathcal{O}(S_1)$. 
Partial ordering of triangulations

Pay attention to the highest vertex in a circuit $Z$: it always appears in every simplex of $\mathcal{T}_Z^-$, but is missing from one simplex of $\mathcal{T}_Z^+$.

\[
\mathcal{T}_Z^+ = \left\{ \begin{array}{c}
\end{array} \right\}
\]

\[
\mathcal{T}_Z^- = \left\{ \begin{array}{c}
\end{array} \right\}
\]
Partial ordering of triangulations

Pay attention to the highest vertex in a circuit $Z$: it always appears in every simplex of $\mathcal{T}_Z^-$, but is missing from one simplex of $\mathcal{T}_Z^+$.

\[ \mathcal{T}_Z^+ = \begin{cases} & \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \end{cases} \]

\[ \mathcal{T}_Z^- = \begin{cases} & \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \end{cases} \]

- Implies that if $\mathcal{T} \leq \mathcal{T}'$ for $\mathcal{T}' \neq \mathcal{T}$, we cannot also have $\mathcal{T}' \leq \mathcal{T}$
A partial ordering is called a *lattice* if every pair of elements has a unique least upper bound and a unique greatest lower bound.
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**Conjecture**

The partial ordering we have defined is a lattice.
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An acyclic orientation of the 1-skeleton of a polytope is said to have good orientation if the graph of every nonempty face of the polytope has exactly one sink (vertex with all edges oriented inwards).
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**Conjecture**

The partial ordering we have defined is a lattice.

An acyclic orientation of the 1-skeleton of a polytope is said to have *good orientation* if the graph of every nonempty face of the polytope has exactly one sink (vertex with all edges oriented inwards).

**Conjecture**

The partial ordering we have defined is a good orientation of the 1-skeleton of the secondary polytope.
Quadrilaterals, pentagons, and hexagons

**Conjecture**

The 2-dimensional faces of $\Sigma_{\mathcal{O}(S_k)}$ are quadrilaterals, pentagons and hexagons.

*Figure: $\Sigma_{\mathcal{O}(S_1)}$.***
Commuting circuits

Theorem [BCJTT24+]

Let $Z_1$ and $Z_2$ be circuits of $\mathcal{O}(Q_w)$. Then $Z_1$ and $Z_2$ commute at $\mathcal{T}$ if and only if $\mathcal{T}$ can be flipped at $Z_1$ and $Z_2$ and at least one of the following hold:

(i) $Z_1$ and $Z_2$ appear on different maximal simplices in $\mathcal{T}$, or
(ii) $Z_1$ and $Z_2$ share no vertex
Pentagons and hexagons

\[ T_1 \rightarrow Z_1 \rightarrow T_3 \rightarrow Z_2 \rightarrow T_1 \]
\[ T_5 \rightarrow Z_3 \rightarrow T_5 \]
\[ T_2 \rightarrow Z_2 \rightarrow T_4 \rightarrow Z_3 \rightarrow T_4 \]
\[ T_6 \rightarrow Z_1 \rightarrow T_6 \]

\[ T_2 \rightarrow Z_3 \rightarrow T_2 \]

\[ T_1 \rightarrow Z_1 \rightarrow T_1 \]
# Table of contents

1. Background

2. A partial order on triangulations

3. Quadrilaterals, pentagons, and hexagons

4. Valence-regularity of the flip graph

5. The twist action

6. Twist eigenbasis
Valence $k$-regularity of the flip graph

**Conjecture ([Bel+22], Conjecture 6.1)**

The flip graph of regular triangulations for $O(Q_w)$ is $k$-regular, where $k$ is the dimension of the secondary polytope of $O(Q_w)$.

**Figure:** Flip graph for $O(S_1)$, where each vertex has 3 edges
Valence-regularity for subset of triangulations

**Theorem [BCJTT24+]**

Let $\mathcal{T}$ be a triangulation of $O(Q_w)$ obtained by applying one flip to the canonical triangulation $\mathcal{T}_w$, where $w = w_0w_1\cdots w_n \in \mathcal{V}$. Then, $\mathcal{T}$ admits $n+1$ flips.
Proof strategy

Figure: Circuits flippable at the canonical triangulation $\mathcal{T}_w$ and $f_{Sq(w_2)}(\mathcal{T}_w)$
# Table of contents

1. Background
2. A partial order on triangulations
3. Quadrilaterals, pentagons, and hexagons
4. Valence-regularity of the flip graph
5. The twist action
6. Twist eigenbasis
Ladders are the diagonal lines of squares in $\hat{P}(w)$. 

$L^1$:

$L^2$:

$L^3$:
Consider $Q_w = S_1$ and $J(S_1)$ below and the ladder $L^1$. 

\begin{center}
\begin{tikzpicture}
  \node (0) at (0,0) {0};
  \node (1) at (1,-1) {1};
  \node (2) at (2,-1) {2};
  \node (3) at (1,-2) {3};
  \node (4) at (2,-2) {4};
  \node (5) at (1,-3) {5};
  \draw (0) -- (1);
  \draw (0) -- (2);
  \draw (1) -- (3);
  \draw (1) -- (4);
  \draw (2) -- (4);
  \draw (2) -- (5);
  \draw (3) -- (5);
  \node (11) at (3,1) {1};
  \node (12) at (4,1) {111110};
  \node (13) at (5,1) {111100};
  \node (14) at (6,1) {111010};
  \node (15) at (7,1) {110000};
  \node (16) at (8,1) {100000};
  \node (17) at (9,1) {100000};
  \node (18) at (10,1) {101000};
  \node (19) at (11,1) {101010};
  \node (20) at (12,1) {110000};
  \node (21) at (13,1) {111010};
  \node (22) at (14,1) {111100};
  \node (23) at (15,1) {111110};
  \node (24) at (16,1) {1};
  \draw (11) -- (12);
  \draw (12) -- (13);
  \draw (13) -- (14);
  \draw (14) -- (15);
  \draw (15) -- (16);
  \draw (16) -- (17);
  \draw (17) -- (18);
  \draw (18) -- (19);
  \draw (19) -- (20);
  \draw (20) -- (21);
  \draw (21) -- (22);
  \draw (22) -- (23);
  \draw (23) -- (24);
  \end{tikzpicture}
\end{center}
The twist group

Generally, elementary twists $\tau_i$ act on $\hat{P}(w)$ by swapping along ladder $L^i$.

**Definition**

For $w \in \mathcal{V}$, the twist group $\mathcal{Z}(w) \leq \mathfrak{S}_{|P(w)|}$ is defined by:

$$\mathcal{Z}(w) = \langle \tau_i \mid \tau_i^2 = 1, \tau_i \tau_j = \tau_j \tau_i \rangle \cong (\mathbb{Z}/2\mathbb{Z})^d,$$

where $d$ is the number of ladders.
The twist group

Generally, elementary twists $\tau_i$ act on $\hat{P}(w)$ by swapping along ladder $L^i$.

**Definition**

For $w \in \mathcal{V}$, the twist group $\mathcal{I}(w) \leq \mathfrak{S}_{|P(w)|}$ is defined by:

$$\mathcal{I}(w) = \langle \tau_i \mid \tau_i^2 = 1, \tau_i \tau_j = \tau_j \tau_i \rangle \cong (\mathbb{Z}/2\mathbb{Z})^d,$$

where $d$ is the number of ladders.

Observe that $\mathcal{I}(w)$ acts on the vertices of $\mathcal{O}(Q_w)$ by:

$$\tau \cdot \mathbf{v}_A = \mathbf{v}_{\tau \cdot A}$$
The twist group

Generally, elementary twists $\tau_i$ act on $\hat{P}(w)$ by swapping along ladder $L^i$.

**Definition**

For $w \in \mathcal{V}$, the twist group $\mathfrak{T}(w) \leq \mathfrak{S}_{|P(w)|}$ is defined by:

$$\mathfrak{T}(w) = \langle \tau_i \mid \tau_i^2 = 1, \tau_i \tau_j = \tau_j \tau_i \rangle \cong (\mathbb{Z}/2\mathbb{Z})^d,$$

where $d$ is the number of ladders.

Observe that $\mathfrak{T}(w)$ acts on the vertices of $\mathcal{O}(Q_w)$ by:

$$\tau \cdot \mathbf{v}_A = \mathbf{v}_{\tau \cdot A}$$

**Theorem (von Bell et al.)**

$\mathfrak{T}(w)$ acts on regular triangulations of $\mathcal{O}(Q_w)$. 
We recall Conjecture 6.5 by von Bell et al.

**Conjecture 6.5**

The number of regular triangulations of $\mathcal{O}(S_k)$ is $2^{k+1} \cdot \text{Cat}(2k + 1)$. 
We recall Conjecture 6.5 by von Bell et al.

Conjecture 6.5

The number of regular triangulations of $\mathcal{O}(S_k)$ is $2^{k+1} \cdot \text{Cat}(2k + 1)$.

- The twist group for $S_k$ has size $2^{k+1}$
- Conjecturally, $2^{k+1}$ divides the number of triangulations
Proposition Theorem [BCJTT24+]

Let $Q_w$ be a generalized snake poset and let $\mathcal{T}_w$ be its canonical triangulation. For any twist $\tau \in \mathcal{T}(w)$, we have $\tau(\mathcal{T}_w) \neq \mathcal{T}_w$
Freeness of the twist action

Proposition Theorem [BCJTT24+]

Let $Q_w$ be a generalized snake poset and let $\mathcal{T}_w$ be its canonical triangulation. For any twist $\tau \in \mathcal{S}(w)$, we have $\tau(\mathcal{T}_w) \neq \mathcal{T}_w$

With the proposition above, we were able to show the following theorem.

Theorem [BCJTT24+]
The twist group acts freely on regular triangulations of $\mathcal{O}(Q_w)$
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Background</td>
</tr>
<tr>
<td>2</td>
<td>A partial order on triangulations</td>
</tr>
<tr>
<td>3</td>
<td>Quadrilaterals, pentagons, and hexagons</td>
</tr>
<tr>
<td>4</td>
<td>Valence-regularity of the flip graph</td>
</tr>
<tr>
<td>5</td>
<td>The twist action</td>
</tr>
<tr>
<td>6</td>
<td>Twist eigenbasis</td>
</tr>
</tbody>
</table>
Twist Eigenbasis

Remember: the twist group acts on the set of regular triangulations

On $\Sigma_{O(S_1)}$ the elementary twists are $180^\circ$ rotations
We can define vectors in the space containing $\Sigma O(Q_w)$ by giving a coefficient to each vertex of $\hat{P}(w)$

We want a subspace $V \cong \mathbb{R}^{n+1}$ in which the twists are linear

We define one basis element $v_i$ for each square $w_i$ in $\hat{P}(w)$:

- corner or not next to any corners
- next to 1 corner
- between 2 corners
**Theorem [BCJTT24+]**

Let \( \mathbf{w} = \varepsilon w_1 \cdots w_n \), and let \( V \cong \mathbb{R}^{n+1} \) be the linear subspace of \( \mathbb{R}^{2n+6} \) parallel to the subspace containing \( \Sigma_{O(Q_w)} \cdot v_1, v_2, \ldots, v_{n+1} \) is an eigenbasis of \( V \) and each elementary twist \( \tau_i \) negates exactly the basis elements that correspond to the \( w_i \) in the ladder \( \tau_i \) reflects.

Since \( \tau_i \tau_i = 1 \), each twist \( \tau_i \) has only 1 and \(-1\) for eigenvalues.
Next steps

1. Finish proving quadrilaterals, pentagons, and hexagons conjecture.
2. Generalize valence-regularity proof.
3. Catalan numbers: bijection or recurrence relation?
4. Boundary of dual polytope: flag simplicial complex?
Acknowledgements

This project was supported in large part by a grant from the D.E. Shaw group, and also by NSF grant DMS-2053288. It was supervised as part of the University of Minnesota School of Mathematics Summer 2024 REU program. The authors would like to thank Vic Reiner for mentoring the project and for his valuable ideas and suggestions. The authors would also like to thank Kaelyn Willingham for his helpful feedback and advice, and Ayah Almousa for co-coordinating the REU.