Free Resolutions and Hilbert Series for Skew Specht Ideals

2024 Twin Cities REU in Combinatorics & Algebra

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Diagrams and Tableaux

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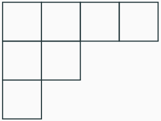
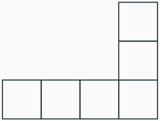
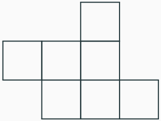
 ,  , and 

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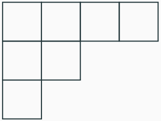
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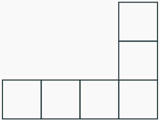
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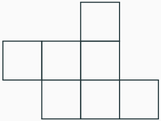
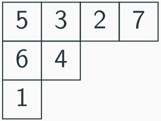


A tableau of a diagram D with n boxes is a filling of the boxes of D with numbers $1, \ldots, n$, such that each number appears only once:



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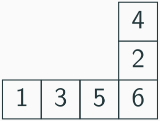

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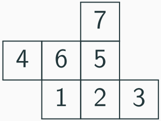
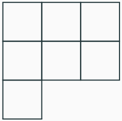


are tableaux of the above diagrams.

Diagrams and Tableaux

For $n \in \mathbb{N}$, a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a sequence of integers such that $\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0$ and $\lambda_1 + \cdots + \lambda_\ell = n$. A partition corresponds to a Young diagram in which the i^{th} row has λ_i boxes, aligned to the left.



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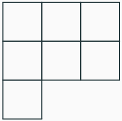
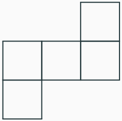


corresponds to $\lambda = (3, 3, 1)$.

A skew partition λ/μ is the diagram obtained by removing boxes of μ from the diagram of λ.



corresponds to $s = (3, 3, 1)/(2)$.

Specht Polynomials

For a polynomial ring $R = \mathbf{k}[x_1, \ldots, x_n]$ and a tableau T of size n, we define the Specht polynomial $f_T \in R$:

$$T = \begin{array}{ccc} & & \boxed{3} \\ \boxed{1} & \boxed{4} & \boxed{5} \\ \boxed{2} & & \end{array} \quad \Longrightarrow \quad f_T =$$

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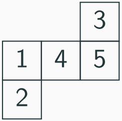
$T =$ 

$$\implies \quad f_T = (x_3 - x_5)(x_1 - x_2)x_1x_2x_4$$

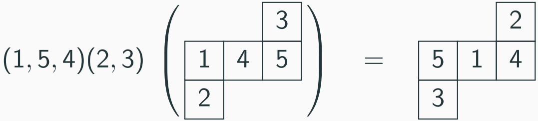
In general, the Specht polynomial of T is

$$f_T = \prod_{\substack{1 \leq i,j \leq n \\ i \text{ above } j \text{ in } T}} (x_i - x_j) \cdot \prod_{1 \leq i \leq n} x_i^{p_i},$$

where p_i is the number of empty spaces above i in T.

Specht Polynomials

For a diagram D with n boxes, the symmetric group $\mathfrak{S}_n$ acts on the tableau of D by permuting the labels $1, \ldots, n$:



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$$(1,5,4)(2,3)\left(\begin{array}{ccc} & & \boxed{3}\\ \boxed{1} & \boxed{4} & \boxed{5}\\ \boxed{2} & & \end{array}\right) \quad = \quad \begin{array}{ccc} & & \boxed{2}\\ \boxed{5} & \boxed{1} & \boxed{4}\\ \boxed{3} & & \end{array}$$

This extends to an action on Specht polynomials.

The Specht module of D is the $\mathbf{k}$-linear span of the Specht polynomials of D:

$$\mathcal{S}_D = \operatorname{span}_{\mathbf{k}}\{f_T \mid T \text{ is a tableau of shape } D\}$$

The Specht modules of partitions λ of n are precisely the *irreducible representations* of $\mathfrak{S}_n$ over $\mathbb{C}$.

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Specht ideals have appeared in work related to subspace arrangements [ZGS14; Bro+16; BPS05], graph theory [LL81; Lov94; Loe95], combinatorial Hilbert schemes [Woo05; DK24], and symmetric systems of equations [MRV21].

Our research: What can we say about the homological structure of a Specht ideal I_D in terms of the combinatorics of the diagram D?

Free Resolutions

Let M be a finitely generated graded module over a graded $\mathbf{k}$-algebra R, and suppose that we have a finite group G acting on R by graded k-algebra automorphisms and on M so that $g(rm) = g(r)g(m)$.

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1. Goal: Understand the structure of M.

Generators and Relations

We can't find a basis for M, but we can do the next best thing: find a (finite) collection $g_1, \ldots, g_{n_0} \in M$ of elements of M that generate M as an R-module.

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If we can find suitable $g_1, \ldots, g_{n_0}$, then we will have reduced the problem of understanding M to understanding the relations between the $g_1, \ldots, g_{n_0}$.

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Just Like M

1. So we can repeat this strategy recursively!
2. Under a few technical conditions, we are guaranteed to end up with the 0 module after finitely many steps.

Free Resolutions

If we do this, we end up with a sequence of maps,

$$\mathcal{F}_\bullet : 0 \longrightarrow F_d \xhookrightarrow{\partial_d} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \longrightarrow 0$$

Such that

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- If our generators are all Homogenous, then we can introduce degree shifts d_i for each F_i so that the ∂_i preserve the grading.
- If our choices of generators each have k-spans that are Representations of G, then we can give each F_i a G-module structure so that the ∂_i are G-equivariant.
- If we have all of the above, then we've got a G-**equivariant (graded) minimal free resolution** on our hands!

Example: The Koszul Complex

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- Maps:

$$\partial_2 \left(\ytableaushort{\none 3, 1, 2} \right) = \ytableaushort{\none 31, 2} \otimes x_1 - \ytableaushort{\none 32, 1} \otimes x_2$$

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Definition
Let M be a graded R-module. The *Hilbert series* of M over R is the formal power series

$$\mathrm{HS}_R(M, t) = \sum_{j \in \mathbb{N}} \dim_{\mathbf{k}} M_j \, t^j.$$

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Let M be a graded R-module with a grading preserving action of G. The *G-equivariant Hilbert series* of M over R is the formal power series

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Example

$$\mathrm{HS}_{eq,\mathbf{k}[x_1,x_2,x_3]}(I_{\ydiagram{2,1}},t) = \frac{\chi_{\ydiagram{2,1}}t - \chi_{\ydiagram{1,1,1}}t^2}{(1-t)^3}$$

Two-row ribbons

Two row shapes

- In [SY23b], Shibata and Yanagawa found the minimal free resolution for $I_{(n-d,d)}$.

Two row shapes

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- Can we generalize the resolution for two-row skew shapes?

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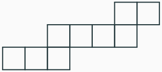
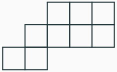
The diagram  is a ribbon, but  is not

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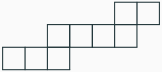
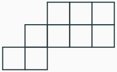
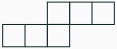
The diagram  is a ribbon, but  is not

Notation

Given a composition $(\alpha_1, \ldots, \alpha_k)$, let $\mathrm{Ribb}(\alpha_1, \ldots, \alpha_k)$ denote the unique ribbon having α_i boxes in row i.

The Free Modules, an example

Example

Consider the ribbon $\mathrm{Ribb}(3,3) =$  and let $R = \mathbf{k}[x_1, \ldots, x_6]$.
The free R-modules in the resolution are:

Example

Consider the ribbon $\mathrm{Ribb}(3,3) = \ydiagram{2+3,3}$ and let $R = \mathbf{k}[x_1, \ldots, x_6]$. The free R-modules in the resolution are:

$$\begin{array}{ccccc}
\mathcal{S}_{\ydiagram{1,1,1,1,1,1}} \otimes R(-8) \longrightarrow \mathcal{S}_{\ydiagram{2,1,1,1,1}} \otimes R(-7) & & & & \\
\downarrow & & & & \\
\mathcal{S}_{\ydiagram{2+1,3,1,1}} \otimes R(-5) & \longrightarrow & \mathcal{S}_{\ydiagram{2+2,3,1}} \otimes R(-4) & \longrightarrow & \mathcal{S}_{\ydiagram{2+3,3}} \otimes R(-3)
\end{array}$$

Definition

More generally, for $\mathrm{Ribb}(k, \ell)$, we have

$$\mathcal{F}_\bullet^{\mathrm{Ribb}(k,\ell)} : 0 \longrightarrow F_{k+\ell-2} \xrightarrow{\partial_{k+\ell-2}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0,$$

Definition

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where if $0 \leq i \leq k-1$

$$F_i = \mathcal{S}_{\mathrm{Ribb}(k-i,\ell,1^i)} \otimes R(-\ell-i),$$

The Free Modules

Definition

More generally, for $\mathrm{Ribb}(k,\ell)$, we have

$$\mathcal{F}_\bullet^{\mathrm{Ribb}(k,\ell)} : 0 \longrightarrow F_{k+\ell-2} \xrightarrow{\partial_{k+\ell-2}} \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \longrightarrow 0,$$

where if $0 \leq i \leq k-1$

$$F_i = \mathcal{S}_{\mathrm{Ribb}(k-i,\ell,1^i)} \otimes R(-\ell - i),$$

and if $k \leq i \leq k+\ell-2$

$$F_i = \mathcal{S}_{\mathrm{Ribb}(k+\ell-i-1,1^{i+1})} \otimes R(-\ell - i - 1).$$

Boundary Maps

Example

For $\mathrm{Ribb}(2,2)$, we have

$$0 \longrightarrow \mathcal{S}_{\ydiagram{1,1,1,1}} \otimes R(-5) \xrightarrow{\partial_2} \mathcal{S}_{\ydiagram{1+1,2,1}} \otimes R(-3) \xrightarrow{\partial_1} \mathcal{S}_{\ydiagram{1+2,2}} \otimes R(-2) \longrightarrow I_{\ydiagram{1+2,2}} \longrightarrow 0.$$

Example

For $\mathrm{Ribb}(2,2)$, we have

$$0 \longrightarrow \mathcal{S}_{\begin{ytableau} \\ \\ \\ \\ \end{ytableau}} \otimes R(-5) \xrightarrow{\partial_2} \mathcal{S}_{\begin{ytableau} \none & \\ & \\ \\ \end{ytableau}} \otimes R(-3) \xrightarrow{\partial_1} \mathcal{S}_{\begin{ytableau} \none & & \\ & \\ \end{ytableau}} \otimes R(-2) \longrightarrow I_{\begin{ytableau} \none & & \\ & \\ \end{ytableau}} \longrightarrow 0.$$

Then

$$\partial_2 \left(\begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau} \otimes 1 \right) = \begin{ytableau} \none & 1 \\ 3 & 2 \\ 4 \end{ytableau} \otimes x_1 x_2 - \begin{ytableau} \none & 1 \\ 2 & 3 \\ 4 \end{ytableau} \otimes x_1 x_3 + \begin{ytableau} \none & 1 \\ 2 & 4 \\ 3 \end{ytableau} \otimes x_1 x_4$$
$$+ \begin{ytableau} \none & 2 \\ 1 & 3 \\ 4 \end{ytableau} \otimes x_2 x_3 - \begin{ytableau} \none & 2 \\ 1 & 4 \\ 3 \end{ytableau} \otimes x_2 x_4 + \begin{ytableau} \none & 3 \\ 1 & 4 \\ 2 \end{ytableau} \otimes x_3 x_4,$$

Example

For $\mathrm{Ribb}(2,2)$, we have

$$0 \longrightarrow \mathcal{S}_{\ydiagram{1,1,1,1}} \otimes R(-5) \xrightarrow{\partial_2} \mathcal{S}_{\ydiagram{1+1,2,1}} \otimes R(-3) \xrightarrow{\partial_1} \mathcal{S}_{\ydiagram{1+2,2}} \otimes R(-2) \longrightarrow I_{\ydiagram{1+2,2}} \longrightarrow 0.$$

Then

$$\partial_2\left(\begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau} \otimes 1\right) = \begin{ytableau} \none & 1 \\ 3 & 2 \\ 4 \end{ytableau} \otimes x_1x_2 - \begin{ytableau} \none & 1 \\ 2 & 3 \\ 4 \end{ytableau} \otimes x_1x_3 + \begin{ytableau} \none & 1 \\ 2 & 4 \\ 3 \end{ytableau} \otimes x_1x_4$$

$$+ \begin{ytableau} \none & 2 \\ 1 & 3 \\ 4 \end{ytableau} \otimes x_2x_3 - \begin{ytableau} \none & 2 \\ 1 & 4 \\ 3 \end{ytableau} \otimes x_2x_4 + \begin{ytableau} \none & 3 \\ 1 & 4 \\ 2 \end{ytableau} \otimes x_3x_4,$$

and

$$\partial_1\left(\begin{ytableau} \none & 1 \\ 2 & 3 \\ 4 \end{ytableau} \otimes 1\right) = \begin{ytableau} \none & 1 & 2 \\ 4 & 3 \end{ytableau} \otimes x_2 - \begin{ytableau} \none & 1 & 4 \\ 2 & 3 \end{ytableau} \otimes x_4.$$

Conjecture

Conjecture

$\mathcal{F}_\bullet^{\mathrm{Ribb}(k,\ell)}$ *is a minimal free resolution for* $I_{\mathrm{Ribb}(k,\ell)}$

Progress

We make partial progress towards proving the conjecture above.

Theorem

The maps ∂_i *are well-defined.*

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The maps ∂_i are well-defined.

Theorem

$\mathcal{F}_\bullet^{\mathrm{Ribb}(k,\ell)}$ *is a chain complex, i.e.* $\partial_{i-1}\partial_i = 0$.

Towards exactness

In the direction of proving exactness, we proved the prime decomposition of two-row ribbon Specht ideals.

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In the direction of proving exactness, we proved the prime decomposition of two-row ribbon Specht ideals.

Theorem

A two-row ribbon Specht ideal has the following prime decomposition:

$$I_{\mathrm{Ribb}(k,\ell)} = \langle x_i - x_j \mid 1 \le i < j \le k+\ell \rangle \cap \left(\bigcap_{\substack{\#F = d-1 \\ F \subseteq [k+\ell]}} P_F \right),$$

where $P_F = \langle x_i \mid i \notin F \rangle$ *for* $F \subseteq \{1, \dots, k+\ell\}$.

$\mathfrak{S}_n$-equivariant Hilbert series

	$(3,3)/(2)$	$(4,3)/(2)$	$(5,3)/(2)$
0	(4)	(5)	(6)
1	$(3,1) + (4)$	$(4,1) + (5)$	$(5,1) + (6)$
2	$(2,2) + 2\,(3,1) + 2\,(4)$	$(3,2) + 2\,(4,1) + 2\,(5)$	$(4,2) + 2\,(5,1) + 2\,(6)$
3	$(2,1,1) + 2\,(2,2) + 4\,(3,1) + 4\,(4)$	$(3,1,1) + 2\,(3,2) + 4\,(4,1) + 4\,(4)$	$(4,1,1) + 2\,(4,2) + 4\,(5,1) + 4\,(6)$

Table 1: Stabilization of $R/I_{\mathrm{Ribb}(n,3)}$

$\mathfrak{S}_n$-equivariant Hilbert series

	$\ydiagram{2+1,3}$	$\ydiagram{2+2,3}$	$\ydiagram{2+3,3}$
0	$\ydiagram{4}$	$\ydiagram{5}$	$\ydiagram{6}$
1	$\ydiagram{3,1} + \ydiagram{4}$	$\ydiagram{4,1} + \ydiagram{5}$	$\ydiagram{5,1} + \ydiagram{6}$
2	$\ydiagram{2,2} + 2\,\ydiagram{3,1} + 2\,\ydiagram{4}$	$\ydiagram{3,2} + 2\,\ydiagram{4,1} + 2\,\ydiagram{5}$	$\ydiagram{4,2} + 2\,\ydiagram{5,1} + 2\,\ydiagram{6}$
3	$\ydiagram{2,1,1} + 2\,\ydiagram{2,2} + 4\,\ydiagram{3,1} + 4\,\ydiagram{4}$	$\ydiagram{3,1,1} + 2\,\ydiagram{3,2} + 4\,\ydiagram{4,1} + 4\,\ydiagram{4}$	$\ydiagram{4,1,1} + 2\,\ydiagram{4,2} + 4\,\ydiagram{5,1} + 4\,\ydiagram{6}$

Table 1: Stabilization of $R/I_{\mathrm{Ribb}(n,3)}$

	$\ydiagram{3+1,4}$	$\ydiagram{3+2,4}$	$\ydiagram{3+3,4}$
0	$\ydiagram{5}$	$\ydiagram{6}$	$\ydiagram{7}$
1	$\ydiagram{4,1} + \ydiagram{5}$	$\ydiagram{5,1} + \ydiagram{6}$	$\ydiagram{6,1} + \ydiagram{7}$
2	$\ydiagram{3,2} + 2\,\ydiagram{4,1} + 2\,\ydiagram{5}$	$\ydiagram{4,2} + 2\,\ydiagram{5,1} + 2\,\ydiagram{6}$	$\ydiagram{5,2} + 2\,\ydiagram{6,1} + 2\,\ydiagram{7}$
3	$\ydiagram{3,1,1} + 2\,\ydiagram{3,2} + 4\,\ydiagram{4,1} + 3\,\ydiagram{5}$	${\color{blue}\ydiagram{3,3}} + \ydiagram{4,1,1} + 2\,\ydiagram{4,2} + 4\,\ydiagram{5,1} + 2\,\ydiagram{6}$	${\color{blue}\ydiagram{4,3}} + \ydiagram{5,1,1} + 2\,\ydiagram{5,2} + 4\,\ydiagram{6,1} + 2\,\ydiagram{7}$

Table 2: Stabilization of $R/I_{\mathrm{Ribb}(n,4)}$

Representation Stability

Definition

The phenomenon described above is known as *representation stability*

Conjecture

Let k be a fixed positive integer. For each $n \geq 1$, let

$$V_n^i = (R/I_{\mathrm{Ribb}(n-k,k)})_i,$$

considered as a $\mathfrak{S}_n$-representation. Then, $\{V_n^i\}$ is representation stable for all $i \geq 0$. Moreover, it stabilizes at $n \geq 2(k-1)$.

Generalized Hooks

The Eagon-Northcott complex can be used to construct the minimal free $\mathfrak{S}_n$ equivariant resolutions of hooks.

Example

$$0 \longrightarrow \mathcal{S}_{(1,1,1,1,1)} \otimes (R(-5) \oplus R(-6) \oplus R(-7))$$

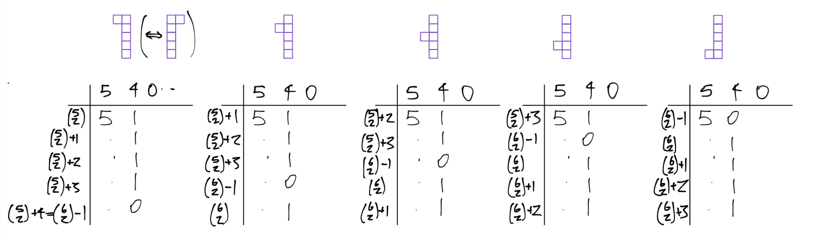
$$\downarrow \partial_3$$

$$\mathcal{S}_{(2,1,1,1)} \otimes (R(-4) \oplus R(-5))$$

$$\downarrow \partial_2$$

$$\mathcal{S}_{(3,1,1)} \otimes R(-3) \xrightarrow{\quad \partial_1 \quad} I_{(3,1,1)}$$

We also investigated generalizations of hooks, and spotted clear patterns in their Betti tables.



In the 2 column case, we were able to find explicit conjectural minimal free resolutions which explain the pattern in their Betti Tables.

Vandermonde Determinants and the Jacobi Bialternant Formula

Given variables $z_1, \ldots, z_n$, we let

$$\mathrm{VD}(z_1, \ldots, z_n) := \begin{vmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_i & \cdots & z_n \\ z_1^2 & z_2^2 & \cdots & z_i^2 & \cdots & z_n^2 \\ \vdots & \vdots & & \vdots & & \vdots \\ z_1^{n-1} & z_2^{n-1} & \ldots & z_i^{n-1} & \ldots & z_n^{n-1} \end{vmatrix}$$

Theorem (Jacobi Bialternant formula)
Let $\lambda = (\lambda_1, \ldots, \lambda_n)$, with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Then we have

$$\begin{vmatrix} z_1^{\lambda_n} & z_2^{\lambda_n} & \ldots & z_i^{\lambda_n} & \ldots & z_n^{\lambda_n} \\ z_1^{\lambda_{n-1}+1} & z_2^{\lambda_{n-1}+1} & \ldots & z_i^{\lambda_{n-1}+1} & \ldots & z_n^{\lambda_{n-1}+1} \\ z_1^{\lambda_{n-2}+2} & z_2^{\lambda_{n-2}+2} & \ldots & z_i^{\lambda_{n-}+2} & \ldots & z_n^{\lambda_{n-2}+2} \\ \vdots & \vdots & & \vdots & & \vdots \\ z_1^{\lambda_1+n-1} & z_2^{\lambda_1+n-1} & \ldots & z_i^{\lambda_1+n-1} & \ldots & z_n^{\lambda_1+n-1} \end{vmatrix} = \mathrm{VD}(z_1, \ldots, z_n) s_\lambda(z_1, \ldots, z_n)$$

2-Column Hooks: Jacobi Bialternant Relations

The g_i's are equal to the Specht polynomials of 2-column generalized hooks with row $d+1$ of the diagram being the 2-box row

$$g_i = x_i^d \cdot \begin{vmatrix} 1 & 1 & \cdots & \widehat{1} & \cdots & 1 \\ x_1 & x_2 & \cdots & x_i & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_i^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_i^{n-2} & \cdots & x_n^{n-2} \end{vmatrix}$$

$$\sum_{i=1}^{n} (-1)^{i-1} g_i \cdot x_i^j = \begin{vmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_i & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_i^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_i^{n-2} & \cdots & x_n^{n-2} \\ x_1^{d+j} & x_2^{d+j} & \cdots & x_i^{d+j} & \cdots & x_n^{d+j} \end{vmatrix}$$

2-Column Hooks: Jacobi Bialternant Relations

The g_i's are equal to the Specht polynomials of 2-column generalized hooks with row $d+1$ of the diagram being the 2-box row

$$g_i = x_i^d \cdot \begin{vmatrix} 1 & 1 & \cdots & \widehat{1} & \cdots & 1 \\ x_1 & x_2 & \cdots & x_i & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_i^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_i^{n-2} & \cdots & x_n^{n-2} \end{vmatrix}$$

$$\sum_{i=1}^{n} (-1)^{i-1} g_i \cdot x_i^j = \begin{vmatrix} 1 & 1 & \cdots & 1 & \cdots \\ x_1 & x_2 & \cdots & x_i & \cdots \\ x_1^2 & x_2^2 & \cdots & x_i^2 & \cdots \\ \vdots & \vdots & & \vdots & \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_i^{n-2} & \cdots \\ x_1^{n-1+(d+j-n+1)} & x_2^{n-1+(d+j-n+1)} & \cdots & x_i^{n-1+(d+j-n+1)} & \cdots \end{vmatrix}$$

2-Column Hooks: Jacobi Bialternant Relations

The g_i's are equal to the Specht polynomials of 2-column generalized hooks with row $d+1$ of the diagram being the 2-box row

$$g_i = x_i^d \cdot \begin{vmatrix} 1 & 1 & \cdots & \widehat{1} & \cdots & 1 \\ x_1 & x_2 & \cdots & x_i & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_i^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_i^{n-2} & \cdots & x_n^{n-2} \end{vmatrix}$$

$$\sum_{i=1}^{n} (-1)^{i-1} g_i \cdot x_i^j = \mathrm{VD}(x_1, \ldots, x_n) s_{(d+j-n+1)}(x_1, \ldots, x_n)$$

$$= \mathrm{VD}(x_1, \ldots, x_n) h_{d+j-n+1}(x_1, \ldots, x_n)$$

J.B. relations example

- $D =$ 
- $f_1 := x_1 \begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ x_2^2 & x_3^2 & x_4^2 \end{vmatrix} = x_1 \,\mathrm{VD}(x_2, x_3, x_4)$
- $f_2 := x_2 \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_3 & x_4 \\ x_1^2 & x_3^2 & x_4^2 \end{vmatrix} = x_2 \,\mathrm{VD}(x_1, x_3, x_4)$
- $\dots$

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ x_1 & -x_2 & x_3 & -x_4 \\ x_1^2 & -x_2^2 & x_3^2 & -x_4^2 \\ x_1^3 & -x_2^3 & x_3^3 & -x_4^3 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \mathrm{VD}(x_1, x_2, x_3, x_4) \begin{bmatrix} 0 \\ 0 \\ 1 \\ x_1 + x_2 + x_3 + x_4 \end{bmatrix}$$

Further directions

Generalizations and Explorations

1. Proving the Betti numbers for MFR of $\mathrm{Ribb}(k, \ell)$.
2. Approach representation stability conjecture.
3. Generalizations of $(d, d, 1)$
   - Skew shape $(d, d, 1)/(a)$
   - $(d_1, d_2, 1)$
4. MFR for two-column generalized hooks.

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References

[LL81] Shuo-Yen Robert Li and Wen Ch'ing Winnie Li. **"Independence numbers of graphs and generators of ideals".** In: *Combinatorica* 1.1 (1981), pp. 55–61. ISSN: 0209-9683. DOI: 10.1007/BF02579177. URL: https://doi.org/10.1007/BF02579177.

[Lov94] L. Lovász. **"Stable sets and polynomials".** In: vol. 124. 1-3. Graphs and combinatorics (Qawra, 1990). 1994, pp. 137–153. DOI: 10.1016/0012-365X(92)00057-X. URL: https://doi.org/10.1016/0012-365X(92)00057-X.

[Loe95] Jesus Antonio de Loera. **Triangulations of polytopes and computational algebra.** Thesis (Ph.D.)–Cornell University. ProQuest LLC, Ann Arbor, MI, 1995, p. 188. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:9527412.

[BPS05] Anders Björner, Irena Peeva, and Jessica Sidman. **"Subspace arrangements defined by products of linear forms".** In: *J. London Math. Soc. (2)* 71.2 (2005), pp. 273–288. ISSN: 0024-6107,1469-7750. DOI: 10.1112/S0024610705006356. URL: https://doi.org/10.1112/S0024610705006356.

[Woo05] Alexander Kar-Man Woo. **Ideals of the polynomial ring generated by irreducible symmetric group representations and Ellingsrud-Stromme cells on the Hilbert scheme.** Thesis (Ph.D.)–University of California, Berkeley. ProQuest LLC, Ann Arbor, MI, 2005, p. 66. ISBN: 978-0542-62127-7. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:3211578.

[ZGS14] Christine Berkesch Zamaere, Stephen Griffeth, and Steven V Sam. **"Jack Polynomials as Fractional Quantum Hall States and the Betti Numbers of the (k + 1)-Equals Ideal".** eng. In: *Communications in mathematical physics* 330.1 (2014), pp. 415–434. ISSN: 0010-3616.

[Bro+16] Aaron Brookner et al. **"On Cohen-Macaulayness of S_n-invariant subspace arrangements".** In: *Int. Math. Res. Not. IMRN* 7 (2016), pp. 2104–2126. ISSN: 1073-7928,1687-0247. DOI: 10.1093/imrn/rnv200. URL: https://doi.org/10.1093/imrn/rnv200.

[MRV21] Philippe Moustrou, Cordian Riener, and Hugues Verdure. **"Symmetric ideals, Specht polynomials and solutions to symmetric systems of equations".** In: *J. Symbolic Comput.* 107 (2021), pp. 106–121. ISSN: 0747-7171,1095-855X. DOI: 10.1016/j.jsc.2021.02.002. URL: https://doi.org/10.1016/j.jsc.2021.02.002.

[SY23a] Kosuke Shibata and Kohji Yanagawa. **"Elementary construction of minimal free resolutions of the Specht ideals of shapes $(n-2,2)$ and $(d,d,1)$".** In: *J. Algebra Appl.* 22.9 (2023), Paper No. 2350199, 26. ISSN: 0219-4988,1793-6829. DOI: 10.1142/S0219498823501992. URL: https://doi.org/10.1142/S0219498823501992.

[SY23b] Kosuke Shibata and Kohji Yanagawa. **“Elementary construction of the minimal free resolution of the Specht ideal of shape $(n-d, d)$”.** In: *J. Algebra* 634 (2023), pp. 563–584. ISSN: 0021-8693,1090-266X. DOI: 10.1016/j.jalgebra.2023.07.028. URL: https://doi.org/10.1016/j.jalgebra.2023.07.028.

[DK24] Sebastian Debus and Andreas Kretschmer. **“Symmetric Ideals and Invariant Hilbert Schemes”.** In: *arXiv preprint arXiv:2404.15240* (2024).

$(d, d, 1)/(a)$

Example

Consider $(3,3,1)/(2) = \ydiagram{2+1,3,1}$. The MFR for R/I is

$$0 \longrightarrow \mathcal{S}_{\ydiagram{1+1,1,1,1,1}} \otimes R(-7) \xrightarrow{\partial_3} \mathcal{S}_{\ydiagram{2+1,2,1,1}} \otimes R(-6) \xrightarrow{\partial_2} \mathcal{S}_{\ydiagram{2+1,3,1}} \otimes R(-5) \xrightarrow{\partial_1} R \longrightarrow 0.$$

$(d, d, 1)/(a)$

Example

Consider $(3,3,1)/(2) = \ydiagram{2+1,3,1}$. The MFR for R/I is

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Conjecture

The minimal free resolution of $R/I_{(d,d,1)/(a)}$ for $a < d$ has free modules $F_0 = R$ and

$$F_i = \mathcal{S}_{(d,d-i+1,1^i)/(a)} \otimes R(-d-i-1)$$

for $1 \leq i \leq d$. Furthermore, the maps ∂_i are defined the same way as in [SY23a].

$(d_1, d_2, 1)$

Here is the MFR for the partition $(3,2,1)$:

$$0 \longrightarrow \mathcal{S}_{(1,1,1,1,1,1)} \otimes R(-9)$$

$$\downarrow$$

$$\mathcal{S}_{(2,1,1,1,1)} \otimes (R(-6) \oplus R(-7)) \longrightarrow (\mathcal{S}_{(3,1,1,1)} \oplus \mathcal{S}_{(2,2,1,1)} \oplus \mathcal{S}_{(2,2,2)}) \otimes R(-5)$$

$$\downarrow$$

$$\mathcal{S}_{(3,2,1)} \otimes R(-4) \longrightarrow I_{(3,2,1)}$$

$$\downarrow$$

$$0.$$

There is *almost* a pattern in these partitions in terms of moving boxes, but it fails in an interesting way.