Infinite Free Resolutions over Certain Families of Numerical Semigroup Algebras

Timothy Cho, Kieran Favazza, Aleister Jones, Nzingha Joseph, and Molly MacDonald

Mentor: Chris O'Neill
TA: Tara Gomes

UMN Twin Cities Algebra and Combinatorics REU Program

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What is a Numerical Semigroup?

Definition (Numerical Semigroups)

A *numerical semigroup* is a cofinite subset $S \subseteq \mathbb{Z}_{\geq 0}$ of the natural numbers closed under addition.

Notation

$$S = \langle n_1, ..., n_k \rangle = \{a_1 n_1 + ... + a_k n_k : a_i \in \mathbb{Z}_{\geq 0}\}$$

where n_i are generators.

Example

$S = \langle 13, 40, 54, 68, 82 \rangle = \{0, 13, 26, 39, 40, 53 \ldots\}$.

Embedding Dimension

$$S = \langle 13, 40, 54, 68, 82 \rangle = \{0, 13, 26, 39, 40, 53 \ldots \}.$$

Definition (Embedding dimension)

The *embedding dimension* of a numerical semigroup is the number of minimal generators.

Example

The embedding dimension of S is 5.

Multiplicity

$$S = \langle 13, 40, 54, 68, 82 \rangle = \{0, 13, 26, 39, 40, 53 \dots\}.$$

Definition (Multiplicity)

The *multiplicity* of a numerical semigroup is its smallest generator m.

Example

The multiplicity m of S is 13.

Apery Set

$$S = \langle 13, 40, 54, 68, 82 \rangle = \{0, 13, 26, 39, 40, 53 \dots\}.$$

Definition (Apéry Set)

The *Apéry set* of a semigroup S is given by
$\mathrm{Ap}(S) = \{n \in S : n - m \notin S\}$, where m is its smallest generator.

Example

$\mathrm{Ap}(S) = \{0, 40, 54, 68, 82, 122, 136, 150, 164, 204, 218, 232, 246\}$.
We order the elements by equivalence class mod m (here 13).

Definition (Apéry Poset)

The poset $(\text{Ap}(S), \preceq)$, where $a \preceq a'$ if and only if $a' - a \in S$.

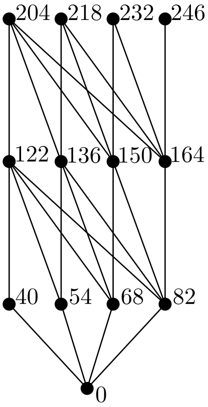


Figure: Apéry Poset of S

Factorizations

Definition (Apéry Poset)

The poset $(\mathrm{Ap}(S), \preceq)$, where $a \preceq a'$ if and only if $a' - a \in S$.

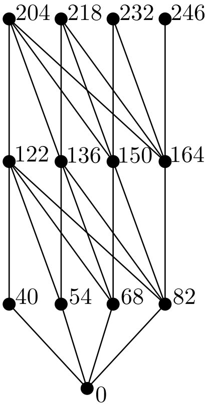


Figure: Apéry Poset of S

Kunz Poset

Definition (Kunz Poset)

The *Kunz poset* of a numerical semigroup S is the poset obtained by replacing each element of the Apéry poset with its equivalence class in $\mathbb{Z}_m$.

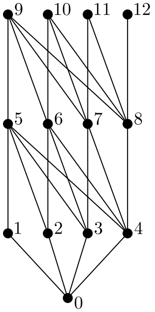


Figure: Kunz Poset of S

Additive Structure

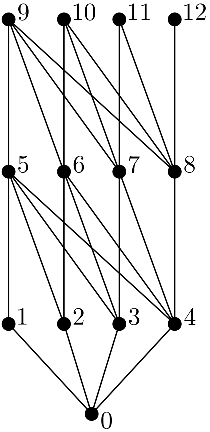
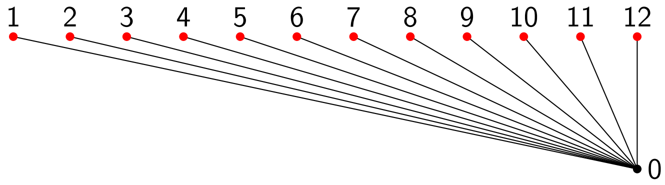


Figure: Kunz Poset of S

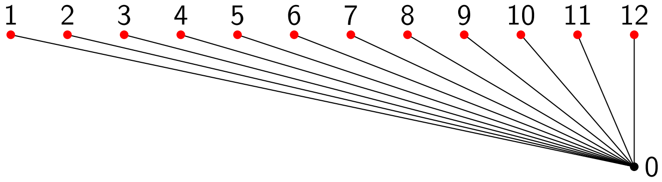
Maximal Embedding Dimension

Definition

A semigroup has *maximal embedding dimension* (MED) if it has m generators.



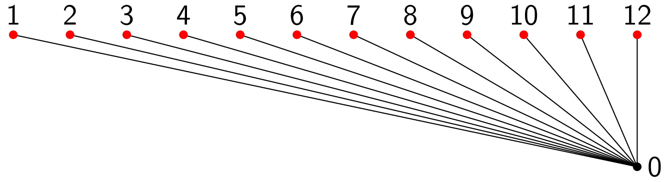
Semigroups Sharing Posets



Example

$$S = \langle 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 \rangle$$

Semigroups Sharing Posets



Example

$$S = \langle 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 \rangle$$

$$S = \langle 13, 53, 67, 55, 82, 96, 58, 72, 60, 87, 75, 50, 90 \rangle$$

Toric Ideal and Semigroup Algebra

We construct a ring that captures the structure of a semigroup S:

Example

$S = \langle 6, 9, 20 \rangle$ (minimal generating set)

$$\varphi : \mathbb{K}[y, x_1, x_2] \to \mathbb{K}[t]$$
$$y \mapsto t^6$$
$$x_1 \mapsto t^9$$
$$x_2 \mapsto t^{20}$$

- **Defining toric ideal:** $I_S = \ker(\varphi) = \langle y^3 - x_1^2, y^4 x_1^4 - x_2^3 \rangle$
- **Semigroup algebra:** $\mathbb{K}[S] = \mathrm{Im}(\varphi) \cong \mathbb{K}[y, x_1, ..., x_k]/I_S$
- $\mathbb{K}[S]$ is graded: $\deg(yx_1) = \deg(t^6 t^9) = 6 + 9 = 15$

We aim to give combinatorial descriptions for minimal infinite free resolutions of the base field $\mathbb{K}$ over the semigroup algebra $\mathbb{K}[S]$:

$$0 \longleftarrow \mathbb{K} \xleftarrow{\partial_0} F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \xleftarrow{\partial_3} \cdots$$

Theorem (Gomes, O'Neill, Sobieska, Torres Dávila '24)

Numerical semigroups with the same Kunz poset have essentially the same minimal free resolution.

Apéry Resolution

We can alternatively define $\mathbb{K}[S]$ using $\{m\} \cup \mathrm{Ap}(S)$:

- $\mathbb{K}[S] \cong \mathbb{K}[y, x_1, \ldots, x_{m-1}]/J_S$

Definition

The **Apéry resolution** is an infinite free resolution of $\mathbb{K}$ over $\mathbb{K}[S]$:

$$0 \longleftarrow \mathbb{K} \xleftarrow{\epsilon} F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \xleftarrow{\partial_3} \cdots$$

- Basis $\{e_{\mathbf{w}}\}$ of F_d corresponds to words $\mathbf{w} = (w_1, \ldots, w_d)$
- $w_i \in \{0, \ldots, m-1\}$ avoiding 0s after first position
- Grading: $\deg(e_{\mathbf{w}}) = \sum_{w_i \in W} \deg(x_i)$

Apéry Resolution

Definition

$\partial_d(e_{\mathbf{w}}) = x_{w_d} e_{\hat{\mathbf{w}}} + \sum_{i=1}^{d-1} (-1)^{d-i} y^{\bullet} e_{\tau_i \mathbf{w}}$ where

- $\hat{\mathbf{w}} = (w_1, \dots, w_{d-1})$
- $\tau_i \mathbf{w} = (w_1, \dots, w_{i-1}, w_i + w_{i+1}, w_{i+2}, \dots, w_d)$

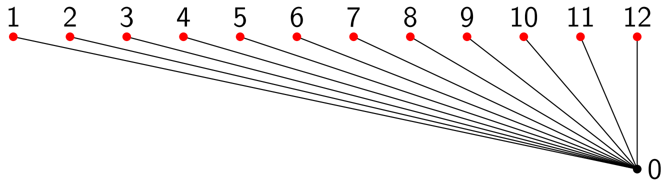
Example

$S = \langle 13, 14, 15, 16, 17 \rangle$

$$
\begin{aligned}
& e_{32121} \\
& \qquad \mapsto x_1 e_{3212} - y^{\bullet} e_{3213} + y^{\bullet} e_{3231} - y^{\bullet} e_{3321} + y^{\bullet} e_{5121} \\
& \qquad \mapsto 0
\end{aligned}
$$

Apéry Resolution

- The Apéry resolution is minimal only when S has maximal embedding dimension



- A minimal resolution in non-MED cases can be found by row-reducing the Apéry resolution

Extra-Generalized Arithmetical Numerical Semigroups

Definition

An **extra-generalized arithmetical numerical semigroup** (EGANS) is a numerical semigroup of the form

$$S = \langle m, mh + \delta, mh + 2\delta, \ldots, mh + k\delta \rangle,$$

where $m < mh + k\delta$, $h, k > 0$ and $\gcd(m, \delta) = 1$.

Example

$S = \langle 13, 47, 42, 37, 32, 27 \rangle = \langle 13, 13 \cdot 4 - 5, \ldots, 13 \cdot 4 - 5 \cdot 5 \rangle$

- $m = 13$
- $h = 4$
- $k = 5$
- $\delta = -5$

Kunz Posets for EGANS

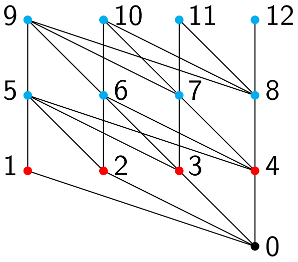
$S_1 = \langle 13, 14, 15, 16, 17 \rangle$,
$S_2 = \langle 13, 27, 28, 29, 30 \rangle$

- Same $m = 13, k = 4, \delta = 1$
- $h_1 = 1, \ h_2 = 2$

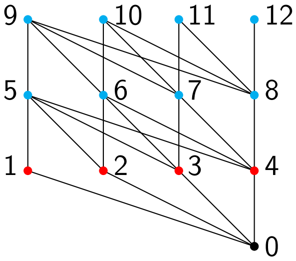
Kunz Posets for EGANS

$S_1 = \langle 13, 14, 15, 16, 17 \rangle$,
$S_2 = \langle 13, 27, 28, 29, 30 \rangle$

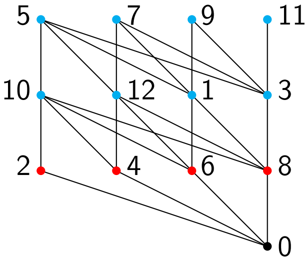
- Same $m = 13, k = 4, \delta = 1$
- $h_1 = 1,\ h_2 = 2$
- Same Kunz poset: true in general



Kunz Posets for EGANS



Poset for $S_1 = \langle 13, 14, 15, 16, 17 \rangle$
$\delta = 1$



Poset for $S_4 = \langle 13, 15, 17, 19, 21 \rangle$
$\delta = 2$

Kunz Posets for EGANS

Example

$S = \langle 15, 31, 32, 33, 34 \rangle$

- $m - 1 = q \cdot k + r$
- $15 - 1 = \mathbf{3} \cdot 4 + \mathbf{2}$
- $q = 3, \; r = 2$

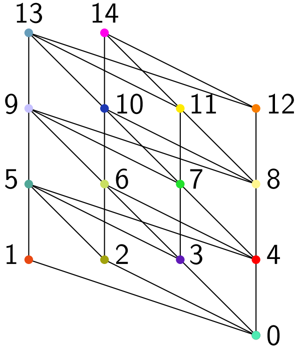
Kunz Posets for EGANS

Example

$S = \langle 15, 31, 32, 33, 34 \rangle$

- $m - 1 = q \cdot k + r$
- $15 - 1 = \mathbf{3} \cdot 4 + \mathbf{2}$
- $q = 3,\ r = 2$
- From bottom to top:

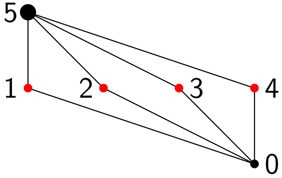
1. 0 in bottom row
2. $q = 3$ full rows with $k = 4$ elements each
3. Top row has $r = 2$ elements



- Apéry resolution: basis elements indexed by words from alphabet $\mathbb{Z}_m$, **pattern avoidance rule** $w_i \neq 0$ for all $i \geq 2$
- Apéry is minimal iff S is MED; otherwise, **too many words and letters**

Example

In $m = 13$, $k = 4$, poset relation $a_1 + a_4 = a_5 \implies$ 5 is bad



New alphabet: $\{0, 1, 2, \ldots, k, -\alpha\} \subseteq \mathbb{Z}_m$, $m \equiv \alpha \pmod{k}$

Example

$m = 15, k = 4 \implies \alpha = 3$, our new alphabet is $\{0, 1, 2, 3, 4, -3\} \subseteq \mathbb{Z}_{15}$

Kunz Posets and Pattern-Avoiding Words

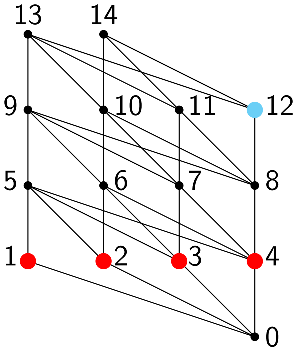
Example

$m = 15, k = 4 \implies \alpha = 3$, our new alphabet is $\{0, 1, 2, 3, 4, -3\} \subseteq \mathbb{Z}_{15}$

Pattern Avoidance Rules

Bad:

1. $w_i = 0$ for all $i \geq 2$
2. $w_1 = -3$
3. $14, 24, 34, 44$
4. $0-, 1-, 2-$



$0,\ 4-,\ 03-433,\ 3-4-1,\ 321123$

~~3210123~~, ~~−4333~~, ~~21−~~, ~~5121~~, ~~142~~

A New Minimal Resolution

A New Minimal Resolution

Theorem (CFJJM '24)

Given an EGANS $S = \langle m, mh+\delta, mh+2\delta, \ldots, mh+k\delta \rangle$ *with* $k \nmid m$*, we have a minimal free resolution*

$$\mathcal{F}'_\bullet : 0 \longleftarrow \mathbb{K} \xleftarrow{\partial'_0} F'_0 \xleftarrow{\partial'_1} F'_1 \xleftarrow{\partial'_2} F'_2 \longleftarrow \cdots,$$

wherein $F'_d = \langle e_{\mathbf{w}} : \mathbf{w} \text{ a new word of length } d \rangle$ *for all* d*, of the base field* $\mathbb{K}$ *over the semigroup algebra of* S*.*

Example

For $m = 13$ and $k = 4$: $S = \langle 13, 14, 15, 16, 17 \rangle$,
Alphabet: $\{0, 1, 2, 3, 4, -1\}$

$$
\begin{array}{cccccc}
\partial(e_{32121}) = & x_1 e_{3212} & -y^{\bullet} e_{3213} & +y^{\bullet} e_{3231} & -y^{\bullet} e_{3321} & +y^{\bullet} e_{5121} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\partial'(e_{32121}) = & x_1 e_{3212} & -y^{\bullet} e_{3213} & +y^{\bullet} e_{3231} & -y^{\bullet} e_{3321} & +x_4 e_{1121}
\end{array}
$$

Example

For $m = 13$ and $k = 4$: $S = \langle 13, 14, 15, 16, 17 \rangle$,
Alphabet: $\{0, 1, 2, 3, 4, -1\}$

$$\begin{array}{cccccc}
\partial(e_{32121}) = & x_1 e_{3212} & -y^{\bullet} e_{3213} & +y^{\bullet} e_{3231} & -y^{\bullet} e_{3321} & +y^{\bullet} e_{5121} \\
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\partial'(e_{32121}) = & x_1 e_{3212} & -y^{\bullet} e_{3213} & +y^{\bullet} e_{3231} & -y^{\bullet} e_{3321} & +x_4 e_{1121}
\end{array}$$

$$\begin{array}{cccccc}
\partial(e_{123-2}) = & x_2 e_{123-} & -y^{\bullet} e_{1231} & +y^{\bullet} e_{1222} & -y^{\bullet} e_{15-2} & +y^{\bullet} e_{33-2} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\partial'(e_{123-2}) = & x_2 e_{123-} & -y^{\bullet} e_{1231} & +y^{\bullet} e_{1222} & -y^{\bullet} e_{1132} & +y^{\bullet} e_{33-2}
\end{array}$$

From Apéry to Minimal

Theorem (CFJJM '24)

Given an EGANS $S = \langle m, mh+\delta, mh+2\delta, \ldots, mh+k\delta \rangle$ *with* $k \nmid m$, *the sequence*

$$\mathcal{F}'_\bullet : 0 \longleftarrow \mathbb{K} \xleftarrow{\partial'_0} F'_0 \xleftarrow{\partial'_1} F'_1 \xleftarrow{\partial'_2} F'_2 \longleftarrow \cdots,$$

wherein $F'_d = \langle e_{\mathbf{w}} : \mathbf{w}$ *a new word of length* $d\rangle$ *and the maps* ∂' *are given by the Apéry maps together with our translation rules, is a minimal free resolution of* $\mathbb{K}$ *over the semigroup algebra of* S.

From Apéry to Minimal

Theorem (CFJJM '24)

Given an EGANS $S = \langle m, mh+\delta, mh+2\delta, \ldots, mh+k\delta \rangle$ with $k \nmid m$, the sequence

$$\mathcal{F}'_\bullet : 0 \longleftarrow \mathbb{K} \xleftarrow{\partial'_0} F'_0 \xleftarrow{\partial'_1} F'_1 \xleftarrow{\partial'_2} F'_2 \longleftarrow \cdots,$$

wherein $F'_d = \langle e_{\mathbf{w}} : \mathbf{w}$ a new word of length $d \rangle$ and the maps ∂' are given by the Apéry maps together with our translation rules, is a minimal free resolution of $\mathbb{K}$ over the semigroup algebra of S.

Our translation rules amount to applying a chain map to the Apéry resolution:

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathbb{K} & \xleftarrow{\partial_0} & R & \xleftarrow{\partial_1} & F_1 & \xleftarrow{\partial_2} & F_2 & \longleftarrow & \cdots \\
 & & & & \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \\
0 & \longleftarrow & \mathbb{K} & \xleftarrow{\partial'_0} & R & \xleftarrow{\partial'_1} & F'_1 & \xleftarrow{\partial'_2} & F'_2 & \longleftarrow & \cdots
\end{array}$$

Counting the Betti Numbers

Corollary

Let $S = \langle m, mh+\delta, \ldots, mh+k\delta \rangle$ with $m \equiv \alpha \pmod{k}$ and $\alpha \neq 0$. The Betti numbers (ranks) of the modules resolving $\mathbb{K}$ over $\mathbb{K}[S]$ are given by $\beta_0 = 1, \beta_1 = k+1$, and, for all $n \geq 2$,

$$\beta_n = k\beta_{n-1} - (\alpha - 1)\beta_{n-2}.$$

As a Poincaré series:

$$P_{\mathbb{K}}^{\mathbb{K}[S]}(z) = \sum_{n=1}^{\infty} \beta_n z^n = \frac{1+z}{1 - kz + (\alpha - 1)z^2}$$

Counting the Betti Numbers

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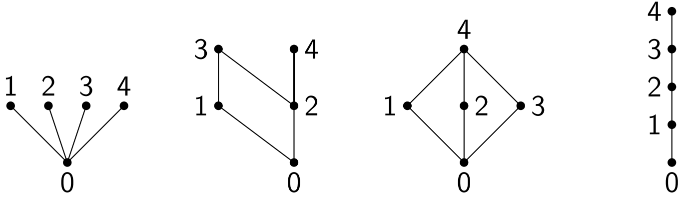
As a Poincaré series:

$$P_{\mathbb{K}}^{\mathbb{K}[S]}(z) = \sum_{n=1}^{\infty} \beta_n z^n = \frac{1+z}{1 - kz + (\alpha-1)z^2}$$

Example

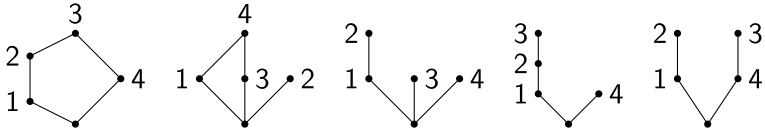
For $S = \langle 13, 14, 15, 16, 17 \rangle$,
Apéry resolution: $\beta_0 = 1, \beta_1 = 13, \beta_2 = 156, \beta_3 = 1872, \ldots$
New (minimal) resolution: $\beta_0 = 1, \beta_1 = 5, \beta_2 = 20, \beta_3 = 80, \ldots$

Enumerating the posets with m=5



Kunz posets for $m = 5$ for which infinite free resolutions are known: the leftmost one describes an MED semigroup, and the other three are posets which describe extra-generalized arithmetical semigroups.

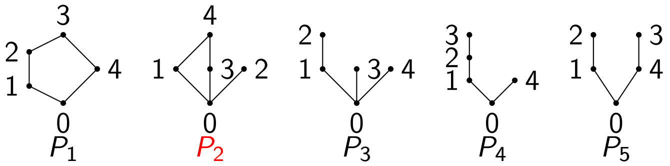
Enumerating the posets with $m = 5$



P_1 P_2 P_3 P_4 P_5

All of the posets of multiplicity 5 that do not correspond to EGANS semigroups.

Forbidden Words

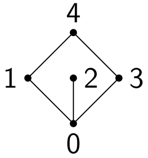


Conjecture

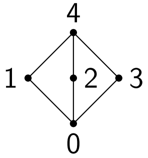
Poset	*Forbidden subwords*
P_1	$\{*0\} \cup \{2, 3, 11, 144\}$
P_2	$\{*0\} \cup \{4, 13\}$
P_3	$\{*0\} \cup \{02, 11, 22, 32, 42\}$
P_4	$\{*0\} \cup \{2, 00, 03, 11, 33, 43\}$
P_5	$\{*0\} \cup \{02, 03, 11, 13, 22, 23, 32, 33, 42, 44\}$

Interesting Observation

P_2 Similar EGANS Poset



$\langle 5, 11, 17, 8 \rangle$



$\langle 5, 6, 7, 8 \rangle$

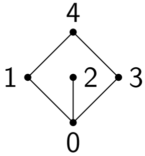
Second Matrix of Free Resolution:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 01 & 02 & 03 & 11 & 12 & 21 & 22 & 23 & 31 & 32 & 33 \end{array} \\ \begin{bmatrix} x_1 & x_2 & x_3 & & & & & -y^{\bullet} & & -y^{\bullet} & \\ -y & & & x_1 & x_2 & & & & x_3 & & -y^{\bullet} \\ & -y & & -y^{\bullet} & & x_1 & x_2 & x_3 & & & \\ & & -y & & -y^{\bullet} & -y^{\bullet} & -y^{\bullet} x_1 & & x_1 & x_2 & x_3 \end{bmatrix} \end{array}$$

$$\longleftarrow$$

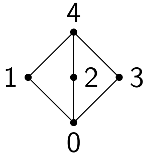
Interesting Observation (Continued)

P_2



$\langle 5, 11, 17, 8 \rangle$

Similar EGANS Poset



$\langle 5, 6, 7, 8 \rangle$

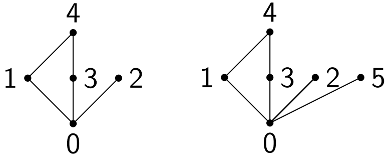
Relation in P_2 is $x_2^2 - y^3 x_1 x_3 = 0$ because:

- $2 \cdot 17 = 34, 3 \cdot 5 + 11 + 8 = 34$

Relation in Similar EGANS Posets is $x_2^2 - x_1 x_3 = 0$ because:

- $2 \cdot 7 = 14, 6 + 8 = 14$

Recursive Definition



P_2 Expanded Poset

Conjecture

Poset	*Forbidden subwords*
P_2	$\{4, 00, 10, 13, 20, 30\}$
Expanded Poset	$\{4, 00, 10, 13, 20, 30, 50\}$

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References II

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Kunz Cone

Definition (Kunz cone)

The *Kunz cone* P_m is a pointed rational cone whose integer points correspond to numerical semigroups with smallest generator m.

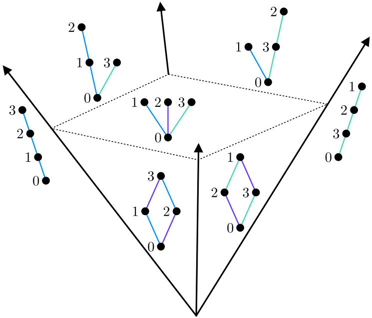

Figure: Kunz cone for semigroups with smallest element 4

Kunz Cone

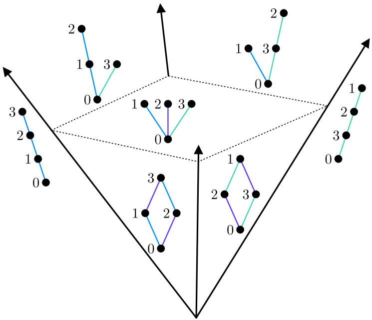

Figure: Kunz cone for semigroups with smallest element 4

Fact

Two numerical semigroups lie in the same face of P_m if and only if they have identical Kunz posets.