Mixed Dimers and Sub-Dimension Vectors for Type E Quiver Representations

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Outline

1. Motivation
2. Quiver representations
3. Correspondence between dimers and subrepresentations
4. Type E results

Motivation

Our project focuses on creating *dimer models* for cluster algebras from type E quivers.

Main idea: we study polynomials associated to type E quiver representations using *dimer combinatorics*.

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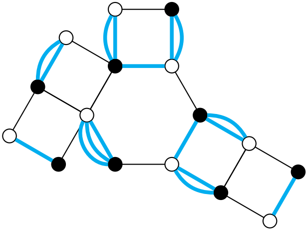
41 (1, 2, 3, 2, 2, 1) u0*u1^2*u2^3*u3^2*u4^2*u5 +
u0*u1^2*u2^3*u3^2*u4^2 + u0*u1^2*u2^3*u3^2*u4*u5 +
u0*u1*u2^3*u3^2*u4^2*u5 + u1^2*u2^3*u3^2*u4^2*u5 +
2*u0*u1^2*u2^3*u3^2*u4 + u0*u1*u2^3*u3^2*u4^2 +
u1^2*u2^3*u3^2*u4^2 + u0*u1*u2^3*u3^2*u4*u5 + u1^2*u2^3*u3^2*u4*u5
+ 2*u1*u2^3*u3^2*u4^2*u5 + u0*u1^2*u2^3*u3^2 +
u0*u1^2*u2^2*u3^2*u4 + 2*u0*u1*u2^3*u3^2*u4 + 2*u1^2*u2^3*u3^2*u4
+ 2*u1*u2^3*u3^2*u4^2 + u0*u1*u2^2*u3^2*u4*u5 +
2*u1*u2^3*u3^2*u4*u5 + u1*u2^2*u3^2*u4^2*u5 + u2^3*u3^2*u4^2*u5 +
u0*u1^2*u2^2*u3^2 + u0*u1*u2^3*u3^2 + u1^2*u2^3*u3^2 +
2*u0*u1*u2^2*u3^2*u4 + u1^2*u2^2*u3^2*u4 + 4*u1*u2^3*u3^2*u4 +
u1*u2^2*u3^2*u4^2 + u2^3*u3^2*u4^2 + 2*u1*u2^2*u3^2*u4*u5 +
u2^3*u3^2*u4*u5 + u2^2*u3^2*u4^2*u5 + 2*u0*u1*u2^2*u3^2 +
u1^2*u2^2*u3^2 + 2*u1*u2^3*u3^2 + u0*u1*u2^2*u3*u4 +
5*u1*u2^2*u3^2*u4 + 2*u2^3*u3^2*u4 + u2^2*u3^2*u4^2 +
u1*u2^2*u3*u4*u5 + 2*u2^2*u3^2*u4*u5 + u0*u1*u2^2*u3 +
u0*u1*u2*u3^2 + 4*u1*u2^2*u3^2 + u2^3*u3^2 + 2*u1*u2^2*u3*u4 +
u1*u2*u3^2*u4 + 4*u2^2*u3^2*u4 + u2^2*u3*u4*u5 + u2*u3^2*u4*u5 +
u0*u1*u2*u3 + 2*u1*u2^2*u3 + 2*u1*u2*u3^2 + 3*u2^2*u3^2 +
u1*u2*u3*u4 + 2*u2^2*u3*u4 + 2*u2*u3^2*u4 + u2*u3*u4*u5 +
2*u1*u2*u3 + 2*u2^2*u3 + 3*u2*u3^2 + 2*u2*u3*u4 + 4*u2*u3 + u3^2 +
u2 + 2*u3 + 1 (1, 1, -1, -2, 1, 1)
```



Quiver Representations

Definition

A *quiver* is a directed graph. A *quiver representation* associates a vector space to each vertex and a linear transformation to each arrow.

Example

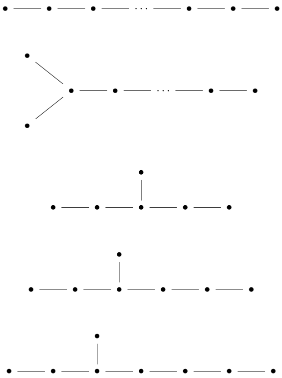
$$\begin{array}{ccccccccc} & & & & \mathbb{K} & & & & \\ & & & & \uparrow {\scriptstyle [1\ \ 1]} & & & & \\ \mathbb{K} & \xleftarrow[{[1]}]{} & \mathbb{K} & \xrightarrow[{[1\ \ 0]^T}]{} & \mathbb{K}^2 & \xleftarrow[{[0\ \ 1]^T}]{} & \mathbb{K} & \xleftarrow[{[0]}]{} & 0 \end{array}$$

Definition

A *subrepresentation* M' of a quiver representation M consists of the same linear transformations assigned to each arrow as in M, and subspaces of the originally assigned vector spaces for each vertex.

Dynkin Diagrams

The *Dynkin diagrams* are A_m ($m \geq 1$), D_n ($n \geq 4$), E_6, E_7, and E_8.



Theorem (Gabriel's Theorem)

A quiver has finitely many isomorphism classes of indecomposable representations if and only if its underlying graph is a Dynkin diagram.

The F-polynomial

F-polynomials are special polynomials associated to indecomposable representations $M(\underline{d})$ for a dimension vector $\underline{d}$:

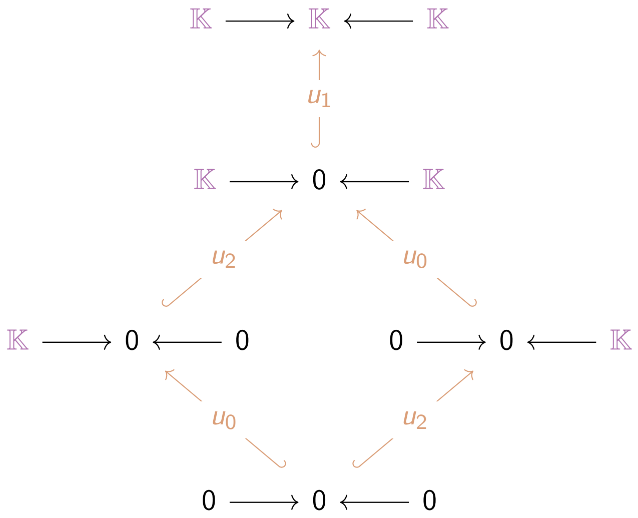
$$F_{\underline{d}} = \sum_{\underline{e}} \chi(Gr_{\underline{e}}(M(\underline{d})))\underline{u}^{\underline{e}}$$

where $\underline{e}$ is the dimension vector of a subrepresentation of M.

A_3 F-polynomial Example

$Q = \ 0 \longrightarrow 1 \longleftarrow 2$

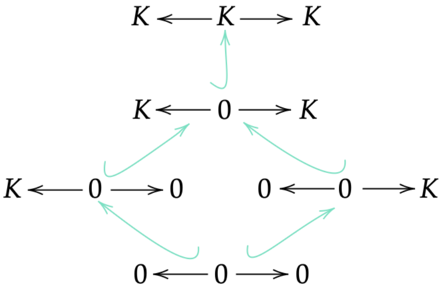
$F_{(1,1,1)}(u_0, u_1, u_2) = 1 + u_0 + u_2 + u_0u_2 + u_0u_1u_2$

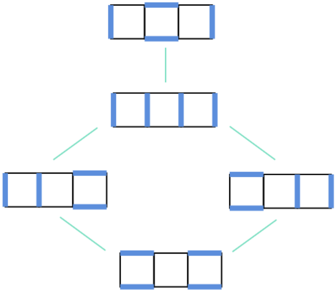


Modeling Subrepresentations Combinatorially

$$F_{(1,1,1)}(u_0, u_1, u_2) = 1 + u_0 + u_2 + u_0 u_2 + u_0 u_1 u_2$$

$$Q = 0 \longleftarrow 1 \longrightarrow 2$$

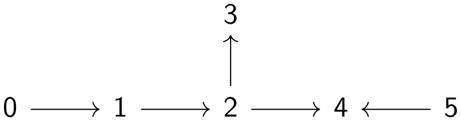




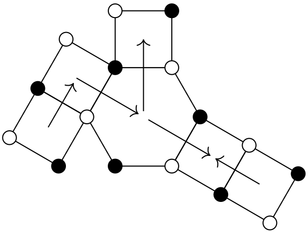
Defining the Base Graph

We are inspired by the type D_n case done by Musiker and Wright (2023).

1. Begin with a quiver of Dynkin type E_6.



2. Use a hexagon tile to represent the vertex 2, and fix a bipartite black and white coloring.
3. Attach square tiles for all other quiver vertices such that quiver arrows see "white on the right."



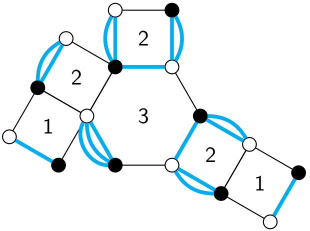
Mixed Dimer Configurations

Definition

A *mixed dimer configuration* for a dimension vector $\underline{d} \in \{0,1,2,3\}^6$ is a multiset of the edges of the base graph (called *dimers*) such that for all vertices v adjacent to a tile i,

- If $d_i = 3$, then v is contained in exactly 3 dimers.
- If $d_i = 2$, then v is contained in at least 2 dimers.
- If $d_i = 1$, then v is contained in at least 1 dimer.

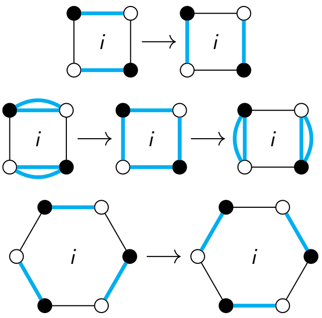
The following is a mixed dimer configuration for $\underline{d} = (1,2,3,2,2,1)$.



Flip Operations

Definition

A *flip* on a square or hexagon swaps dimers on black-to-white-clockwise edges for dimers on all white-to-black-clockwise edges.

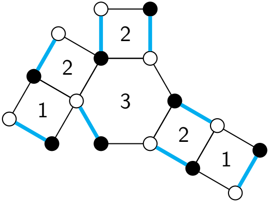


Observe that flips preserve the degree of every vertex of the base graph.

The Minimal Matching

Definition

Let G_i be the subgraph containing all tiles with label $\geq i$. The *minimal matching* is the multiset sum of dimers on black-to-white-clockwise edges along the boundaries of the G_i for $1 \leq i \leq 3$.



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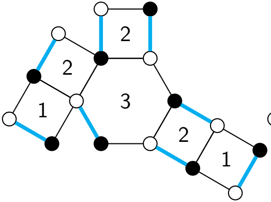

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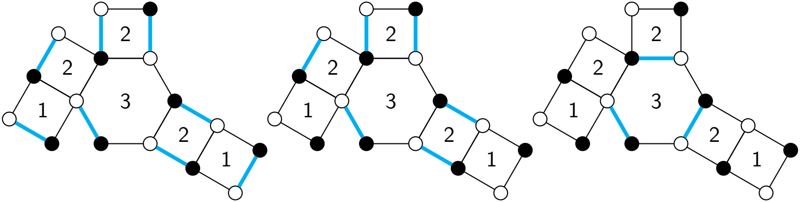

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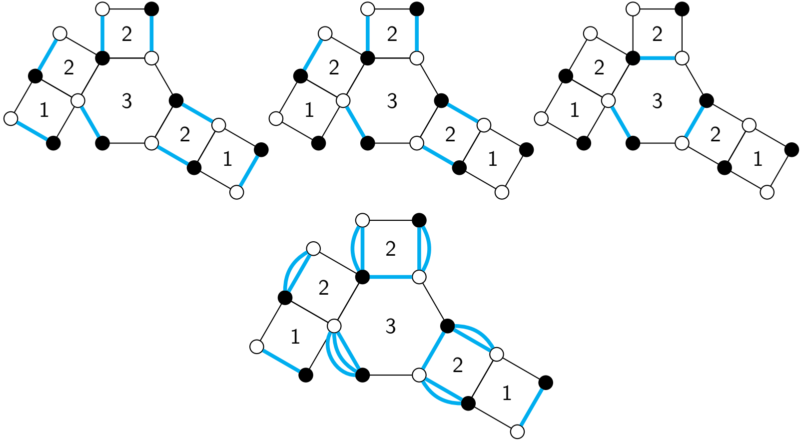
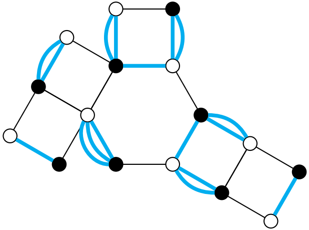


Poset of Mixed Dimer Configurations

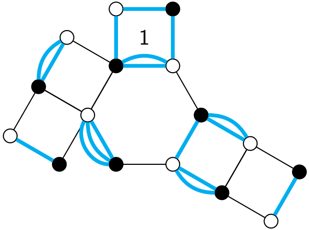
Definition

The *poset of mixed dimer configurations* contains all dimer configurations that can be obtained from the minimal matching by flips.

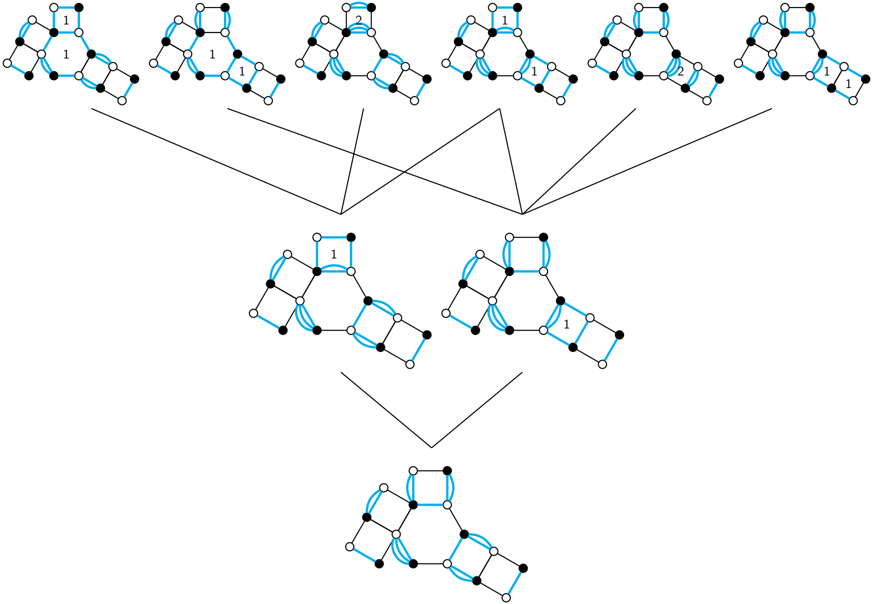
Correspondence between dimension vectors $\underline{e} = (e_0, e_1, \ldots, e_5)$ and mixed dimer configurations $D_{\underline{e}}$, where tile i has been flipped e_i times in $D_{\underline{e}}$:



$(0, 0, 0, 0, 0, 0)$



$(0, 0, 0, 1, 0, 0)$



$$F_{\underline{d}} = 1 + 2u_3 + 2u_4 + u_2u_3 + u_2u_4 + u_3^2 + 4u_3u_4 + u_4^2 + u_4u_5 + \cdots .$$

Sub-dimension Vector Conditions

Theorem (Tran '09)

For type D_n quivers Q, the term $u_0^{e_0} \cdots u_{n-1}^{e_{n-1}}$ appears in the F-polynomial for $\underline{d}$ if and only if

1. $0 \leq \underline{e} \leq \underline{d}$
2. $i \to j \implies e_i - e_j \leq \max(d_i - d_j, 0)$
3. *Q avoids having too many type D_n critical arrows $(d_i, e_i) \to (d_j, e_j)$.*

Type D_n critical arrows are $(1,1) \to (2,1)$ or $(2,1) \to (1,0)$.

E_6 Analog of Tran's Conditions

Theorem (AAC '24+)

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The type E_6 critical arrows are

Type D_n	$(1,1) \to (2,1)$ or $(2,1) \to (1,0)$
New	$(2,1) \to (3,1)$ or $(3,1) \to (2,0)$
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Forbidden Critical Arrow Combinations

We separate type E_6 critical arrows into three categories.

- $(1,1) {\color{blue}\rightarrow} (2,1)$ and $(2,1) {\color{blue}\rightarrow} (1,0)$
- $(2,1) {\color{orange}\rightarrow} (3,1)$ and $(3,2) {\color{orange}\rightarrow} (2,1)$
- $(2,2) {\color{teal}\rightarrow} (3,2)$ and $(3,1) {\color{teal}\rightarrow} (2,0)$

The forbidden combinations of critical arrows for $\underline{d} = (1,2,3,2,2,1)$:

$${\color{teal}\rightarrow}{\color{teal}\rightarrow}$$

$${\color{teal}\rightarrow}{\color{orange}\rightarrow}$$

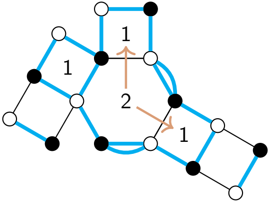
$${\color{teal}\rightarrow}{\color{blue}\rightarrow}$$

$${\color{blue}\rightarrow}{\color{blue}\rightarrow}{\color{orange}\rightarrow}$$

$${\color{blue}\rightarrow}{\color{orange}\rightarrow}{\color{orange}\rightarrow}$$

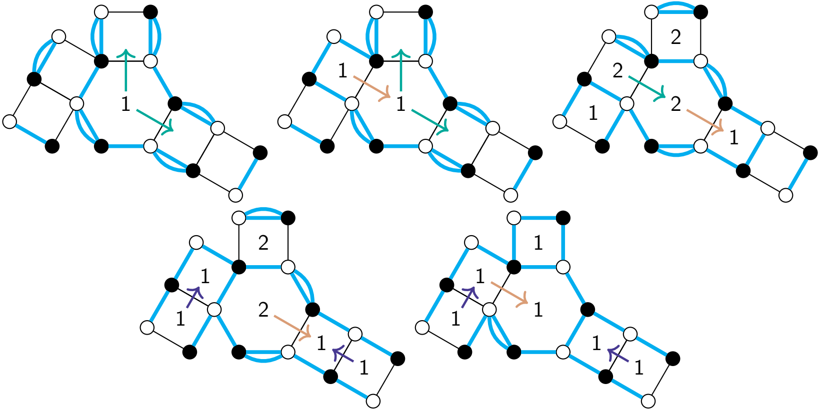
$${\color{orange}\rightarrow}{\color{orange}\rightarrow}{\color{orange}\rightarrow}$$

Critical Arrows Example



$(0, 1, 2, 1, 1, 0)$ appears in the F-polynomial, but $(0, 0, 2, 1, 1, 1)$ does not.

Forbidden Combinations of Critical Arrows



Critical Arrows and Mixed Dimers

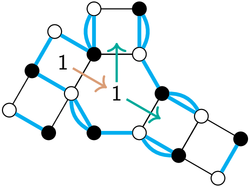
Proposition (Musiker and Wright '23)

The number of dimers in $D_{\underline{e}}$ on the edge between tiles i and j is

$$\max(d_i - d_j, 0) + (e_j - e_i).$$

Proposition (AAC '24+)

For a type E_6 critical arrow $i \to j$, there are no dimers along the edge between tiles i and j.

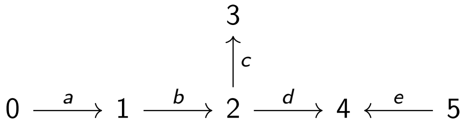


For the critical arrow $(3, 1) \to (2, 0)$,

$$\max(d_i - d_j, 0) + (e_j - e_i) = (3 - 2) + (0 - 1) = 0.$$

Representation Theoretic Justification for Critical Arrows

1. Let $\underline{d} = (1, 2, 3, 2, 2, 1)$ and $Q =$



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$$\begin{array}{ccccccccc} & & & & 3 & & & & \\ & & & & \uparrow c & & & & \\ 0 & \xrightarrow{a} & 1 & \xrightarrow{b} & 2 & \xrightarrow{d} & 4 & \xleftarrow{e} & 5 \end{array}$$

2. $M(\underline{d})$ can be visualized as the following labeled diagram of points and lines in the plane

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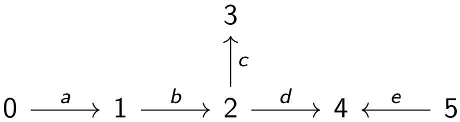

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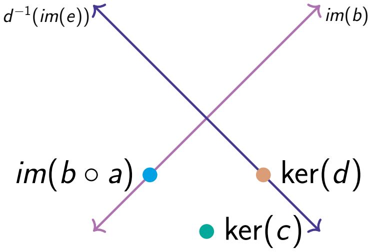
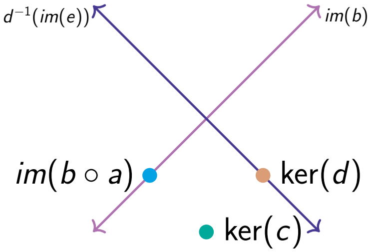
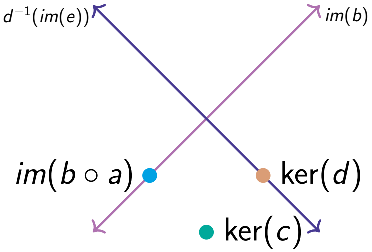


Representation Theoretic Justification for Critical Arrows



Q avoids having "too many" type E_6 critical arrows $(d_i, e_i) \to (d_j, e_j)^*$.

Representation Theoretic Justification for Critical Arrows



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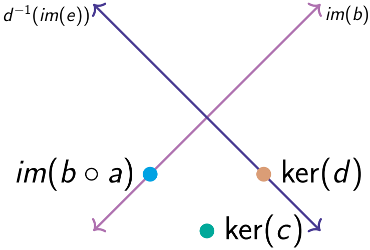
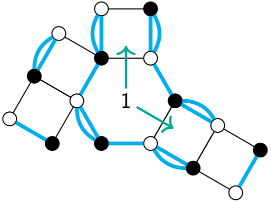
The forbidden critical arrow combinations correspond to the maps in $M(\underline{d})$ having "unusual" relationships with each other.

For example, the kernels of two maps may be equal, or the image of one map may contain the image of another.

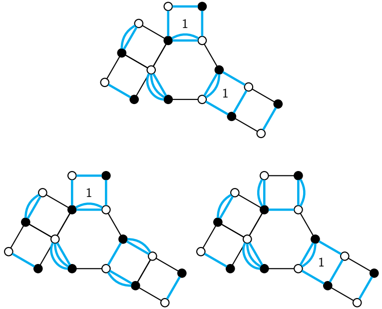
Representation Theoretic Justification for Critical Arrows



- $\ker(c) \neq \ker(d) \iff$ Two $(3,1) \to (2,0)$ arrows is forbidden



F-polynomial Coefficients

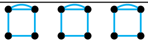
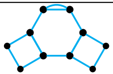
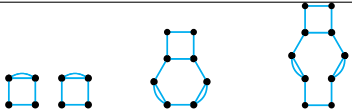
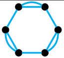
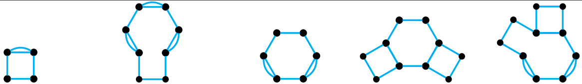


$$F_{\underline{d}} = 1 + \mathbf{2u_3} + \mathbf{2u_4} + u_2u_3 + u_2u_4 + u_3^2 + \mathbf{4u_3u_4} + u_4^2 + u_4u_5 + \cdots .$$

Coefficients Conjecture

Conjecture

Coefficients greater than 1 in the F-polynomial for $\underline{d} = (1, 2, 3, 2, 2, 1)$ correspond to the following cycles in the mixed dimer configuration.

Coefficient	Cycles
8	
5	
4	
3	
2	

Future Directions

- Analyze the $\underline{d} = (1, 2, 3, 1, 2, 1)$ case for type E_6 quivers
- Look into type E_7 and E_8 quivers
- Characterize the F-polynomial coefficients by dimer cycles
- Find a combinatorial description in terms of dimer paths for forbidden configurations

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